Persistence of Normally Hyperbolic Invariant Manifolds: the noncompact case

Jaap Eldering
Mathematical Institute
Utrecht University
J.Eldering@uu.nl

Introduction

In dynamical systems, Normally Hyperbolic Invariant Manifolds (NHIMs) are a generalization to hyperbolic fixed points. Instead of one invariant fixed point, a whole manifold is considered, that is invariant under the flow. These NHIMs are fundamental for studying singular perturbation problems and can also be studied to test e.g. global and long term behavior of solutions, or construct normal forms for bifurcation problems. NHIMs have two important properties in common with hyperbolic fixed points. First, they persist under small perturbations of the system. This provides insight into the original system, closer to a well understood system. Second, each NHIM has stable and unstable manifolds; moreover, these are invariant fibrations over the base NHIM, such that fibers are mapped onto fibers under the flow, that is, these fibrations induce normal form coordinates in which the ‘horizontal’ flow along the NHIM decouples from the ‘vertical’ directions.

Normally hyperbolic invariant manifolds can, in the compact case, exactly be characterized by the fact that they persist under any $C^0$-small perturbation of the system. Proofs of persistence can be found in [Fenichel1971], while [HPS1977] conversely proved that persistent manifolds are normally hyperbolic. Since these classic results, the theory has been extended, for example, to semiflows in Banach spaces [Eldering2009, BLZ2008], for applications to partial differential equations. Extensions have also been made to non-autonomous systems [Yi1993]. Most of these results still assume the invariant manifold to be compact, however. Other results require a trivial ambient space.

We adapt ideas from [Eldering2009, Conlon2014] and employ the Perron method to prove persistence of noncompact NHIMs in a general noncompact setting. Furthermore, the result includes $C^{0,α}$-smoothness, for any $k,α$ that satisfies the spectral gap condition (1). This is optimal and characterizes the finite smoothness.

Normal hyperbolicity and the spectral gap condition

A normal invariant manifold $M$ is normally hyperbolic if the flow $\varphi^t$ is contracting and/or expanding in the directions normal to $M$ at exponential rates bounded away from zero, and moreover the contraction/expansion is stronger than any contraction or expansion of the flow along $M$ itself. In the following, we will assume that there are only constructive normal directions.

The figure shows a horizontal manifold $M$ with two fixed points; one unstable and one stable on $M$. Note that the rightmost fixed point is contractive, but the contraction along $M$ is weaker than that in the normal direction.

Normal hyperbolicity can be formulated in terms of the tangent flow $D\varphi^t$ on $TM\oplus N$. For fixed $p_0 < \rho < p_1$ and $C \geq 0$ we require

\[ \forall m \in M, t \geq 0 : \| D\varphi^t(m) \| \leq C e^{\lambda t}, \]

\[ \forall m \in M, t \leq 0 : \| D\varphi^t(m) \| \leq C e^{\lambda t}. \]

That is, the flow normal to $M$ contracts with exponential rate $e^{\lambda t}$, while the reverse flow tangential to $M$ expands with a rate at most $e^{-\lambda t}$, hence the forward flow cannot contract with a rate faster than $e^{\lambda t}$.

The dichotomy condition $p_0 < \rho < p_1$ on the growth rates can be generalized to the spectral gap condition

\[ p_0 < \rho < \rho_1 \quad \text{with} \quad \rho_1 > 1, \]

The factor $\rho_1$ determines a supremum for the differentiability degree $C^{\alpha,\rho_1}$, $\rho = \rho_1 - \alpha$ of the persisting manifold under a generic perturbation. See the example of optimal smoothness why this is the case. It should be noted that normal hyperbolicity can be defined a bit more generally by bounding the growth ratio $p_0/p_1$ on solution curves, instead of their global ratios.

A statement of the theorem

We prove the following

**Theorem 1 (Persistence of Normally Hyperbolic Invariant Manifolds).** Let $\mathcal{Q}$ be a smooth Riemannian manifold of bounded geometry and $\rho \in C^{0,\alpha}, \rho \geq 1, 0 \leq \alpha < 1/2$, a vector field on $\mathcal{Q}$ with all derivatives bounded and uniformly $\alpha$-Hölder continuous. Let $M \subset \mathcal{Q}$ be an (immersed, connected, and complete submanifold of $\mathcal{Q}$, given by $C^{0,\alpha}$-bounded graphs in normal coordinates. Assume that $M$ is a NHIM for $\varphi$, such that the spectral gap condition $p_0 < (1+\epsilon)\rho_1$ is satisfied.

Then for any perturbed vector field $\varphi_{\epsilon} \in C^{0,\alpha}$ with all derivatives bounded and uniformly $\alpha$-Hölder continuous $\varphi_{\epsilon}$ as long as close to $\epsilon$ in $C^0$-norm, there is a unique $\epsilon$-invariant manifold $M_\epsilon$ that is close and differmorphic to $M$. Moreover, $M_\epsilon$ is normally hyperbolic and $C^{0,\alpha}$.

Let us make some remarks about this theorem.

- The invariant manifold $M$ is not assumed to be compact. Instead, the system and normal hyperbolicity conditions are assumed to be uniformly bounded in a neighborhood of $M$. Bounded geometry of $\mathcal{Q}$ is required to make sense of boundedness and uniform continuity of $\varphi$ and $\mathcal{Q}$ (in terms of its graph representation). This bounded geometry condition is trivially satisfied if $\mathcal{Q} = \mathbb{R}^n$.
- Non-autonomous systems can be studied straightforwardly by adding time as an explicit variable to the system. This leads to so-called ‘integral manifolds’.
- Unlike other results on noncompact NHIMs, we do not assume that $M$ has a global coordinate chart or trivial normal bundle [KLA97, BLZ2008], not do we restrict the ambient space $\mathcal{Q} = \mathbb{R}^n$. This allows application, for example, in classical mechanical systems with nontrivial configuration spaces.
- The $C^{0,\alpha}$-smoothness result is optimal, as can be seen from the example, one cannot generally expect better smoothness of $M$, even if the system $\varphi$ and the unperturbed manifold $M$ are $C^{1,\alpha}$.

A uniform ambient space: bounded geometry

The generalization of the theory of normally hyperbolic invariant manifolds to the noncompact setting requires replacing compactness by uniformity assumptions. An important conclusion to be drawn from this work is, that indeed this adage holds, but probably in a more strict way than one would naïvly realize. Uniform estimates are not only required for the vector field defining the system, but for the underlying ambient space $\mathcal{Q}$. This is, as long as we are able to define uniform continuity of vector fields. This uniformity is formulated in terms of bounded geometry. The space $(\mathcal{Q}, g)$ is of bounded geometry if it has injectivity radius globally bounded from below and the Riemannian curvature and all its derivatives are globally bounded [Eldering2010].

We do not prove that bounded geometry is a necessary condition for persistence of noncompact NHIMs, but we have several examples that show that problems can occur when bounded geometry is not satisfied. One simple example shows that when the injectivity radius is not bounded, the perturbed manifold need not be topologically equivalent to the unperturbed manifold.

We consider the cylinder $\mathcal{Q} = R \times S^1$ with metric

\[ g(\theta, \theta') = d\theta + (\cos(\epsilon) - 1) d\theta'^2, \]

and the system $1 > \epsilon > 0$. Then $\mathcal{Q} = \mathbb{R}^{Q}(\theta)$ is a NHIM. Note that the contraction in the normal direction of $\theta$ is automatic due to the contracting metric along solution curves $f(\theta) = \theta + \epsilon$. If we perturb this system a little by adding a small vertical component for $\epsilon$, then the perturbed manifold is given by a solution curve that winds around the cylinder.

References


