# Persistence of Normally Hyperbolic Invariant Manifolds: the noncompact case

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## Introduction

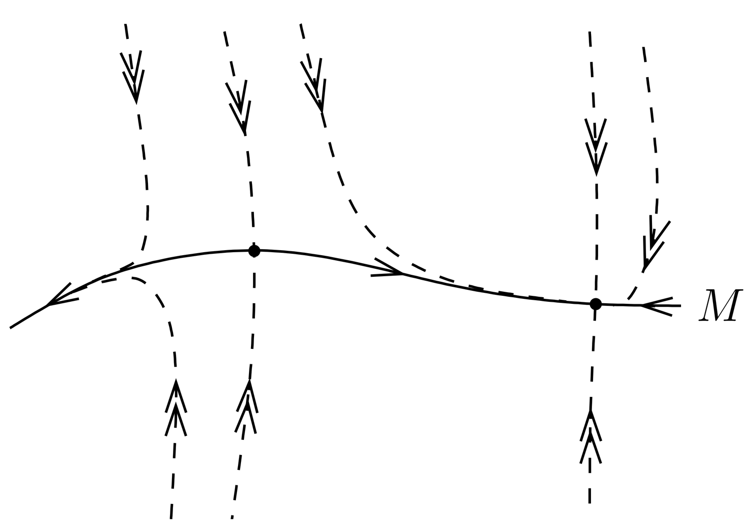
In dynamical systems, Normally Hyperbolic Invariant Manifolds (NHIMs) are a generalization to hyperbolic fixed points. Instead of one invariant fixed point, a whole manifold is considered, that is invariant under the flow. These NHIMs are a fundamental tool for studying singular perturbation problems and can also be used to study e.g. global and long term behavior of solutions, or construct normal form coordinates for bifurcation problems. NHIMs have two important properties in common with hyperbolic fixed points. First, they persist under small perturbations of the system. This provides insight in nontrivial systems close to a simpler, well understood system. Secondly, each NHIM has stable and unstable manifolds; moreover, these are invariant fibrations over the base NHIM, such that fibers are mapped onto fibers under the flow, that is, these fibrations induce normal form coordinates in which the 'horizontal' flow along the NHIM decouples from the 'vertical' directions.

Normally hyperbolic invariant manifolds can, in the compact case, exactly be characterized by the fact that they persist under any $C^1$-small perturbation of the system. Proofs of persistence can be found in [Fen71] and [HPS77], while [Mañ78] conversely proved that persistent manifolds are normally hyperbolic. Since these classic results, the theory has been extended, for example, to semi-flows in Banach spaces [Hen81; BLZ08], for applications to partial differential equations. Extensions have also been made to non-autonomous systems [Yi93]. Most of these results still assume the invariant manifold to be compact, however. Other results require a trivial ambient space.

We adapt ideas from [Hen81; VG87] and employ the Perron method to prove persistence of noncompact NHIMs in a general noncompact setting. Furthermore, the result includes $C^{k,\alpha}$-smoothness, for any $k+\alpha$ that satisfies the spectral gap condition (2). This is optimal and characterizes the finite smoothness.

### Normal hyperbolicity and the spectral gap condition

An invariant manifold $M$ is normally hyperbolic if the flow $\Phi^t$ is contracting and/or expanding in the directions normal to $M$ at exponential rates bounded away from zero, and moreover the contraction/expansion is stronger than any contraction or expansion of the flow along $M$ itself. In the following, we will assume that there are only contractive normal directions. The figure shows a horizontal manifold $M$ with two fixed points; one unstable and one stable along $M$. Note that the rightmost fixed point is contractive, but the contraction along $M$ is weaker than that in the normal direction.




Normal hyperbolicity can be formulated in terms of the tangent flow $\mathrm{D}\Phi^t$ on $\mathrm{T}M \oplus N$. For fixed $\rho_y < 0$, $\rho_x > \rho_y$, and $C > 0$ we require

$$\begin{aligned} &\forall\, m \in M,\, t \geq 0\colon \left\|\pi_N \circ \mathrm{D}\Phi^t(m)|_N\right\| \leq C\, e^{\rho_y t}, \\ &\forall\, m \in M,\, t \leq 0\colon \quad \left\|\mathrm{D}\Phi^t(m)|_{\mathrm{T}M}\right\| \leq C\, e^{\rho_x t}. \end{aligned} \tag{1}$$

That is, the flow normal to $M$ contracts with exponential rate $e^{\rho_y t}$, while the reverse flow tangential to $M$ expands with a rate of at most $e^{-\rho_x t}$, hence the forward flow cannot contract with a rate faster than $e^{\rho_x t}$.

This dichotomy condition $\rho_y < \rho_x$ on the growth rates can be generalized to the spectral gap condition

$$\rho_y < r\,\rho_x \quad \text{with} \quad r \geq 1. \tag{2}$$

The factor $r$ determines a supremum for the differentiability degree $C^{k,\alpha}$, $r = k+\alpha$ of the persisting manifold under a generic perturbation. See the example of optimal smoothness why this is the case. It should be noted that normal hyperbolicity can be defined a bit more generally by bounding the growth ratio $\rho_y/\rho_x$ on solution curves, instead of their global ratio.

## An example of optimal smoothness

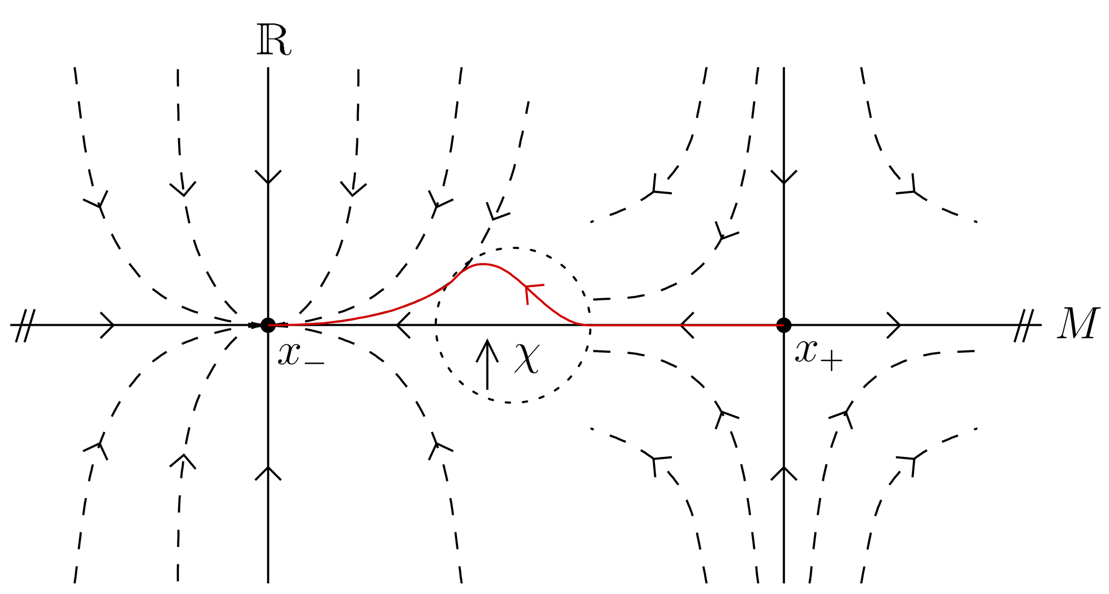
We present a simple example that shows that the spectral gap condition (2) determines an upper bound to the expected smoothness of the perturbed invariant manifold $\tilde{M}$, even if the original manifold $M$ and the system are $C^\infty$.

Let $M = S^1 \times \{0\}$ be the circle in the space $S^1 \times \mathbb{R}$. At $x_- = (0,0)$ and $x_+ = (\pi, 0)$, we create fixed points with a linear vector field in a neighborhood of these. Along $M$ these fixed point have stable and unstable eigenvalues $\lambda_- < 0 < \lambda_+$ respectively. In the vertical direction, we globally set $\dot{y} = \lambda_y \cdot y$ with a strong contractive eigenvalue $\lambda_y < \lambda_-$. This makes $M$ a normally hyperbolic invariant manifold with $\rho_x = \lambda_-$ and $\rho_y = \lambda_y$.

Next, we add a perturbation $\varepsilon\,\chi$ where $\chi$ is a $C^\infty$ function with support in a ball that intersects $M$ away from the fixed points $x_\pm$, and $\chi$ points upward. The points $x_\pm$ are still invariant, so they must also be on the perturbed manifold $\tilde{M}$. The local unstable manifold $W^U_{\mathrm{loc}}(x_+)$ must also be part of $\tilde{M}$, because it consists exactly of the only solution curves that do not diverge along $W^S_{\mathrm{loc}}(x_+)$ for $t \to -\infty$. The system is not modified right of $x_+$, so there $\tilde{M} = M$ is unchanged. When we follow the unstable manifold from $x_+$ to the left, it is lifted in the region where $\chi$ is nonzero as indicated by the red line in the figure. Once this curve leaves that region, it will follow a standard solution curve of the original linear vector field around $x_-$, that is

$$(x,y)(t) = (x_0\, e^{\lambda_- t},\, y_0\, e^{\lambda_y t}),$$

so it is described by a graph $y = C\,x^{\lambda_y/\lambda_-}$. Left of $x_-$, however, $\tilde{M}$ is described by the zero graph, so $\tilde{M} \in C^{k,\alpha}$ for $k+\alpha = \lambda_y/\lambda_-$, which is the supremum of what the spectral gap condition (2) prescribes. Note that this example holds for all $\varepsilon > 0$, so for arbitrarily small perturbations.




## Statement of the theorem

We prove the following

**Theorem 1 (Persistence of Normally Hyperbolic Invariant Manifolds).**
*Let $(Q,g)$ be a smooth Riemannian manifold of bounded geometry and $v \in C^{k,\alpha}$, $k \geq 1$, $0 \leq \alpha \leq 1$, a vector field on $Q$ with all derivatives bounded and uniformly $\alpha$-Hölder continuous. Let $M \subset Q$ be an (immersed), connected, and complete submanifold of $Q$, given by $C^{k,\alpha}$-bounded graphs in normal coordinates. Assume that $M$ is a NHIM for $v$, such that the spectral gap condition $\rho_y < (k+\alpha)\,\rho_x$ is satisfied.*

*Then for any perturbed vector field $\tilde{v} \in C^{k,\alpha}$ with all derivatives bounded and uniformly $\alpha$-Hölder continuous that is sufficiently close to $v$ in $C^1$-norm, there is a unique $\tilde{v}$-invariant manifold $\tilde{M}$ that is close and diffeomorphic to $M$. Moreover, $\tilde{M}$ is normally hyperbolic and $C^{k,\alpha}$.*

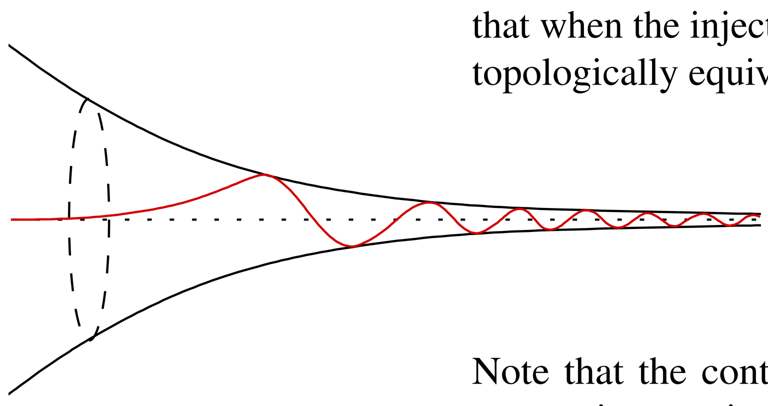
Let us make some remarks about this theorem.

- The invariant manifold $M$ is not assumed to be compact. Instead, the system and normal hyperbolicity conditions are assumed to be uniformly bounded in a neighborhood of $M$. Bounded geometry of $Q$ is required to make sense of boundedness and uniform continuity of $v$ and $M$ (in terms of its graph representation). This bounded geometry condition is trivially satisfied if $Q = \mathbb{R}^n$.
- Non-autonomous systems can be studied straightforwardly by adding time as an explicit variable to the system. This leads to so-called 'integral manifolds'.
- Unlike other results on noncompact NHIMs, we do not assume that $M$ has a global coordinate chart or trivial normal bundle [Hen81; Sak90], nor do we restrict the ambient space $Q = \mathbb{R}^n$ [BLZ08]. This allows application, for example, in classical mechanical systems with nontrivial configuration spaces.
- The $C^{k,\alpha}$-smoothness result is optimal: as can be seen from the example, one cannot generally expect better smoothness of $\tilde{M}$, even if the systems $v, \tilde{v}$ and the unperturbed manifold $M$ are $C^\infty$.

## A uniform ambient space: bounded geometry

The generalization of the theory of normally hyperbolic invariant manifolds to the noncompact setting requires replacing compactness by uniformity assumptions. An important conclusion to be drawn from this work is, that indeed this adage holds, but probably in a more strict way than one would naively realize. Uniform estimates are not only required for the vector field defining the system, but for the underlying ambient space $(Q,g)$ as well, already in order to be able to even define uniform continuity of vector fields. This uniformity is formulated in terms of bounded geometry. The space $(Q,g)$ is of bounded geometry if it has injectivity radius globally bounded from below and the Riemannian curvature and all its derivatives are globally bounded [Eic91].

We do not prove that bounded geometry is a necessary condition for persistence of noncompact NHIMs, but we have several examples that show that problems can occur when bounded geometry is not assumed. One simple example shows that when the injectivity radius is not bounded, the perturbed manifold need not be topologically equivalent to the unperturbed manifold.



We consider the cylinder $Q = \mathbb{R} \times S^1$ with metric

$$g(x,\theta) = \,\mathrm{d}x^2 + \exp(-x)\,\mathrm{d}\theta^2$$

and the system $\dot{x} = 1$, $\dot{\theta} = 0$. Then $M = \mathbb{R} \times \{0\}$ is a NHIM. Note that the contraction in the normal direction of $\theta$ is automatic due to the contracting metric along solution curves $x(t) = x_0 + t$. If we perturb this system a little by adding a small vertical component for $\dot{\theta}$, then the perturbed manifold is given by a solution curve that winds around the cylinder.

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