Mathemagics

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Internet
Internet
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Will the Internet stay connected?
Internet as a graph

• Servers/computers = vertices
• Connections = edges
• How will this graph look like?
• **Connected graph:**
  There is a path along edges from any vertex to any other vertex.

• Will the Internet stay connected under failures, overloads, attacks?
Mini-Internet

- A channel is available with probability $p$, $0 < p < 1$
- A channel is unavailable with probability $1 - p$
Probability to disconnect the network

\[ 3p(1-p)^2 + (1-p)^3 \]

- When \((1-p)\) is small, \((1-p) > 3p(1-p)^2 + (1-p)^3\)
- The network is more robust than one channel!
- **What about large networks?**
Erdös-Rényi random graph (1959)

- $n$ vertices
- An edge between two vertices exists with probability $p$
- Independently of other edges
- Take $p = p(n)$

**Theorem (Erdös-Rényi).**
- If $p > \ln(n)/n$, then with high probability the network is **connected**
- If $p < \ln(n)/n$, then with high probability the network is **disconnected**
- If $p = \ln(n)/n$, then the network is **disconnected with probability, which converges to $e^{-1}$**
Phase transition

Ice turning to water at 0°C
Phase transition

- **Theorem (Erdös-Rényi).**
  - If $p > \ln(n)/n$, then with high probability the network is connected.
  - If $p < \ln(n)/n$, then with high probability the network is disconnected.
  - If $p = \ln(n)/n$, then the network is disconnected with probability, which converges to $e^{-1}$.

- Critical probability $p = \ln(n)/n$.
- Decreases with $n$.
- Again, larger networks are more robust.
Example

- $n=100$, $\frac{\ln(n)}{n} \approx 0.046$

$p=0.04$  $p=0.05$
Magic revealed

• Most likely way to get the network disconnected: completely disconnect at least one of the vertices
  • It is more difficult to disconnect a group of vertices
• $P(\text{one vertex is disconnected}) = (1 - p(n))^{n-1}$
• Average number of disconnected vertices $= n \ (1 - p(n))^{n-1}$
• Substitute $p(n) = c \ ln(n)/n$
\[
\lim_{n \to \infty} n \left(1 - \frac{c \ln(n)}{n}\right)^{n-1} = \lim_{n \to \infty} ne^{-c\ln(n)} = \lim_{n \to \infty} n^{1-c}.
\]

- If \(c<1\) then the average number of disconnected vertices goes to infinity
- If \(c>1\) then the average number of disconnected vertices goes to zero
- The actual number of disconnected vertices is close to its average
- If \(c=1\), the number of disconnected vertices converges to a Poisson(1) distribution, and \(P(\text{no disconnected vertices})=e^{-1}\)
Back to the Internet

• Erdös-Rényi random graph is not a realistic model for the Internet

• Hubs, backbone, bandwidth

• A lot of research on robustness of the Internet

• However, even the simplest model given important insights:
  • Robustness of large network
  • Phase transition
Mathemagics forever!

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