The deviation matrix, Poisson's equation, and QBDs

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extensions with S. D., Dario Bini and Beatrice Meini

Poisson's equation

One formulation:

$$(I - P)\underline{x} = \underline{d} - z\underline{1}$$

where

- P is the transition kernel of a Markov process,
- $\underline{d}$ is a given function of the state space, and
- $\underline{x}$ and z are the unknowns.

- P is a stochastic matrix, $P \geq 0$, $P\underline{1} = \underline{1}$.
- finite or denumerable state space,
- irreducible, non-periodic, positive recurrent.

Found in many places: Markov reward processes, Central limit thm for M.C., perturbation analysis, ...
Focus varies: need a specific solution x, z, or all solutions of the equation

Outline

Finite State space

For finite M.C., z is no worries

$$(I - P)\underline{x} = \underline{d} - z\underline{1}$$

Markov chain is finite and irreducible, thus there exists a unique $\underline{\pi}$ such that

$$\underline{\pi}^{\mathrm{t}}(I - P) = \underline{0}, \qquad \underline{\pi}^{\mathrm{t}}\underline{1} = 1.$$

Premultiply by $\underline{\pi}^{\mathrm{t}}$ and get $z = \underline{\pi}^{\mathrm{t}}\underline{d}$

So, might as well have written

$$(I - P)\underline{x} = \underline{d} \qquad \text{with } \underline{\pi}^{\mathrm{t}}\underline{d} = 0.$$

Of course, $I - P$ is singular.

Generalized inverses

- Generalized inverse of A: $\quad AA^{+}A = A \qquad A^{+}AA^{+} = A^{+}$
- Group inverse $A^{\#}$: in addition, $AA^{\#} = A^{\#}A$ — unique when it exists

- Irreducible finite MC: $(I-P)^{\#}$ exists and is unique solution to

$$(I-P)(I-P)^{\#} = I - \underline{1}\underline{\pi}^{\mathrm{t}}, \qquad \underline{\pi}^{\mathrm{t}}(I-P)^{\#} = \underline{0}$$

- Fundamental matrix: $Z = (I - P + \underline{1}\underline{\pi}^{\mathrm{t}})^{-1} = (I-P)^{\#} + \underline{1}\underline{\pi}^{\mathrm{t}}$.
- $(I-P)^{\#}$: preserves the structure of $I-P$

- For algebraic/geometric properties of $A^{\#}$: Campbell and Meyer, *Generalized Inverses of Linear Transformations*, 1979 — SIAM 2008

Finite state space is straightforward

$$(I - P)\underline{x} = \underline{d} \qquad \text{with } \underline{\pi}^{\mathrm{t}}\underline{d} = 0$$

If P is of finite order, then $\underline{x}$ is unique, up to an additive constant

$$\underline{x} = (I - P)^{\#}\underline{d} + c\underline{1}$$

- $(I - P)^{\#}$ is the group inverse of $(I - P)$
- c is an arbitrary constant

- actually, $c = \underline{\pi}^{\mathrm{t}}\underline{x}$

Deviation matrix

Define $N_j(n)$ as the number of visits to j in $[0 \ \text{to } n-1]$.

$$(I-P)^{\#}_{ij} = \lim_{n\to\infty} (\mathrm{E}_i[N_J(n)] - \mathrm{E}_\pi[N_j(n)])$$
$$= \lim_{n\to\infty} (\mathrm{E}_i[N_J(n)] - n\pi_j)$$

Deviation matrix $\mathcal{D}$

$$\mathcal{D} = \sum_{s\geq 0} (P^s - \underline{1\pi}^{\mathrm{t}})$$

Thus,

$$(I-P)^{\#} = \mathcal{D}$$

$$(I-P)\mathcal{D} = I - \underline{1\pi}^{\mathrm{t}}, \qquad \underline{\pi}^{\mathrm{t}}\mathcal{D} = \underline{0}$$

Important as one may rely upon physical interpretation In addition to geometric and algebraic properties

Continuous-time M.C. — infinite state space

- Continuous-time M.C. with generator Q

$$-Q^{\#} = \mathcal{D} = \int_0^\infty (e^{Qs} - \underline{1}\pi^{\mathrm{t}})\,\mathrm{d}s$$

- Infinite state space: $(I - P)^{\#}$ or $Q^{\#}$ not well defined, but $\mathcal{D}$ is OK since its physical meaning is preserved.

- Pauline Coolen-Schrijner and Erik van Doorn, *The deviation matrix of a continuous-time Markov chain*, 2002.

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Infinite State space

Infinite state space

$$(I - P)\underline{x} = \underline{d} - z\underline{1}$$

If state space is denumerably infinite, situation is more involved.

Example (Makowski and Shwartz, 2002)

$$P = \begin{bmatrix} q & p & 0 & & \\ q & 0 & p & 0 & \\ 0 & q & 0 & p & \\ & 0 & q & 0 & \\ & & & & \ddots \end{bmatrix}$$

with $p + q = 1$. Take any $\underline{d}$. For any z real, there exists a solution $\underline{x}$.

If one looks for a specific x and computes a solution,
how does one know it's the right one?

Constructive solution

Assume $\underline{\pi}^{\mathrm{t}}|\underline{d}| < \infty$

Take j to be an arbitrary state, T its first return time

One solution is given by

$$z = \underline{\pi}^{\mathrm{t}}\underline{d}$$

$$x_i = \mathrm{E}[\sum_{0 \leq n < T} d_{\Phi_n} | \Phi_0 = i] - z\mathrm{E}[T | \Phi_0 = i].$$

Think of d_i as a reward per unit of time spent in state i.

- z is the asymptotic expected reward per unit of time.
- x_i is the expected difference if start from i, up to a constant.
- $x_j = 0$.

Very much like $\mathcal{D}$ but sum to T instead of ∞

From now on: $\underline{\pi}^{\mathrm{t}}\underline{d}$.

Constructive solution — Censoring

Subset of states A, T first return time to A,

$$P = \begin{bmatrix} P_{AA} & P_{AB} \\ P_{BA} & P_{BB} \end{bmatrix} \qquad N_B = \sum_{n \geq 0} P_{BB}^n$$

$$\gamma_i = \mathrm{E}[\sum_{0 \leq n < T} d_{\Phi_n} | \Phi_0 = i]$$

One solution is given by

$$\underline{x}_A = \underline{\gamma}_A + (P_{AA} + P_{AB} N_B P_{BA}) \underline{x}_A$$

$$\underline{x}_B = \underline{\gamma}_B + N_B P_{BA} \underline{x}_A$$

This is Censoring, or Schur complementation.

QBDs — a primer

QBDs

Markov chains on two-dimensional state space

$$(n, \varphi) : \quad n = 0, 1, 2, \ldots ; \quad \varphi = 1, 2, \ldots, M$$

Often,

- n is length of a queue, named the level.
  changes by one unit at most
- φ may be many different things, named the phase.
- here $\quad M < \infty$

Transition matrix

Block-structured transition matrix:

$$P = \begin{bmatrix} A_* & A_1 & 0 & 0 & \cdots \\ A_{-1} & A_0 & A_1 & 0 & \\ 0 & A_{-1} & A_0 & A_1 & \ddots \\ 0 & 0 & A_{-1} & A_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

Transition probabilities:

$(A_1)_{ij}$ probability to go up from (n, i) to $(n+1, j)$

Transition matrix

Block-structured transition matrix:

$$P = \begin{bmatrix} A_* & A_1 & 0 & 0 & \cdots \\ A_{-1} & A_0 & A_1 & 0 & \\ 0 & A_{-1} & A_0 & A_1 & \ddots \\ 0 & 0 & A_{-1} & A_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

Transition probabilities:

$(A_{-1})_{ij}$ probability to go down from (n, i) to $(n-1, j)$

Transition matrix

Block-structured transition matrix:

$$P = \begin{bmatrix} A_* & A_1 & 0 & 0 & \cdots \\ A_{-1} & A_0 & A_1 & 0 & \\ 0 & A_{-1} & A_0 & A_1 & \ddots \\ 0 & 0 & A_{-1} & A_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

Transition probabilities:

$(A_0)_{ij}$ probability to stay in level n, (n, i) to (n, j), $\quad n \neq 0$

Transition matrix

Block-structured transition matrix:

$$P = \begin{bmatrix} A_* & A_1 & 0 & 0 & \cdots \\ A_{-1} & A_0 & A_1 & 0 & \\ 0 & A_{-1} & A_0 & A_1 & \ddots \\ 0 & 0 & A_{-1} & A_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

Transition probabilities:

$(A_*)_{ij}$ probability to remain in level 0, $(0,i)$ to $(0,j)$

matrices for QBDs

Analysis makes extensive use of matrices

$$G_{ij} = \mathrm{P}[T < \infty, \Phi_T = (0, j) | \Phi_0 = (1, i)],$$

$$R_{ij} = \mathrm{E}[\sum_{0 \leq t < T} \mathbb{1}[\Phi_t = (1, j)] | \Phi_0 = (0, i)],$$

$$U = A_0 + A_1 G$$

T first return time to level 0

Deviation matrix for QBDs

Constructive solution for QBDs

$$(I - P)X = D \qquad \text{with } D = I - \underline{1}\pi^{\mathrm{t}}$$

$$X_A = \Gamma_A + (P_{AA} + P_{AB}N_B P_{BA})X_A$$
$$X_B = \Gamma_B + N_B P_{BA} X_A$$

A is level 0, B is collection of all levels ≥ 1.

$$P_{AA} + P_{AB}N_B P_{BA} = A_* + A_1 G = P_*$$

$$N_B P_{BA} = \begin{bmatrix} G \\ G^2 \\ \vdots \end{bmatrix}$$

$$X_0 = \Gamma_0 + P_* X_0 \qquad X_n = \Gamma_n + G^n X_0 \qquad \text{all } n \geq 1$$

Γ_n = accumulation during first passage time to level 0

Accumulation until level 0

$$(I - P)X = D \qquad \text{with } D = I - \underline{1}\pi^{\mathrm{t}}$$

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \end{bmatrix} + \begin{bmatrix} A_0 & A_1 & \\ A_{-1} & A_0 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \end{bmatrix}$$

$$= \sum_{n \geq 0} \begin{bmatrix} A_0 & A_1 & \\ A_{-1} & A_0 & \\ & & \ddots \end{bmatrix}^n \begin{bmatrix} D_1 \\ D_2 \\ \vdots \end{bmatrix}$$

Need

$$N = \sum_{n \geq 0} [\bullet]^n$$

Expected sojourn times

$N_{(n,i)(k,j)}$ is expected number of visits to (k, j) before level zero, starting from (n, i).

For $n > k$

$$n \underset{G}{\rightarrow} n-1 \underset{G}{\rightarrow} \cdots \underset{G}{\rightarrow} k$$

(a) go down $n - k$ levels from n to k and

(b) start counting

$$N_{nk} = G^{n-k} N_{kk}$$

Expected sojourn times

$N_{(n,i)(k,j)}$ is expected number of visits to (k,j) before level zero, starting from (n,i).

For $n < k$

$$n \overset{(k)}{\rightarrow} n \overset{(k)}{\rightarrow} n \cdots n \overset{(k)}{\rightarrow} n \quad \triangleleft$$

(a) count visits to level n,

(b) for each of these, count visits to level $k - n$ steps higher.

$$N_{nk} = N_{nn} R^{k-n}$$

Expected sojourn times

$N_{(n,i)(k,j)}$ is expected number of visits to (k,j) before level zero, starting from (n,i).

For $n = k$

$$n \quad \underset{G}{\overset{(n)}{\rightarrow}} \quad n-1 \quad \underset{G}{\overset{(n)}{\rightarrow}} \quad n-2 \quad \cdots \quad \underset{G}{\overset{(n)}{\rightarrow}} \quad 1 \quad \underset{G}{\overset{(n)}{\rightarrow}} \quad 0 \quad \lhd$$

(a) Starting from level n, trajectory $n \rightsquigarrow n-1 \rightsquigarrow \ldots \rightsquigarrow 1 \rightsquigarrow 0$

(b) A bit of calculation

$$N_{nn} = \sum_{0 \le \nu \le n-1} G^{\nu} (I - U)^{-1} R^{\nu}$$

Summary

Interesting pattern

$$N = \begin{bmatrix} N_{11} & N_{11}R & N_{11}R^2 & N_{11}R^3 & \dots \\ G\,N_{11} & N_{22} & N_{22}R & N_{22}R^2 & \\ G^2N_{11} & G\,N_{22} & N_{33} & N_{33}R & \\ G^3N_{11} & G^2N_{22} & G\,N_{33} & N_{44} & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$N_{11} = (I-U)^{-1}$$

$$N_{kk} = (I-U)^{-1} + G\,N_{k-1,k-1}R \qquad k \geq 2$$

Deviation matrix

$$\mathcal{D} = \begin{bmatrix} \mathcal{D}_0 \\ \mathcal{D}_1 \\ \mathcal{D}_2 \\ \vdots \end{bmatrix} \qquad \mathcal{D}_n = \begin{bmatrix} D_{n0} & D_{n1} & D_{n2} & \cdots \end{bmatrix}$$

$$\mathcal{D}_0 = (I - P_*)^{\#}\{\begin{bmatrix} I & 0 & 0 \ldots \end{bmatrix} - \underline{1\pi}^{\mathrm{t}} + \begin{bmatrix} 0 & A_1 & 0 & \ldots \end{bmatrix} \Gamma_1\} + \underline{1v}^{\mathrm{t}}$$

$$\begin{bmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \\ \vdots \end{bmatrix} = N \begin{bmatrix} 0 & I & 0 & \cdots \\ 0 & 0 & I & \\ \vdots & & & \ddots \end{bmatrix} - N\,\underline{1\pi}^{\mathrm{t}} + \begin{bmatrix} G \\ G^2 \\ \vdots \end{bmatrix} \mathcal{D}_0$$

Look for all the solutions

Current work with Dario B., Sarah D, and Beatrice M.

Matrix difference equations

$$(I - P)\underline{x} = \underline{d}$$

For QBDs:

$$(I - A_*)\underline{x}_0 - A_1\underline{x}_1 = \underline{d}_0 \qquad (1)$$

$$-A_{-1}\underline{x}_{n-1} + (I - A_0)\underline{x}_n - A_1\underline{x}_{n+1} = \underline{d}_n \qquad n \geq 1 \qquad (2)$$

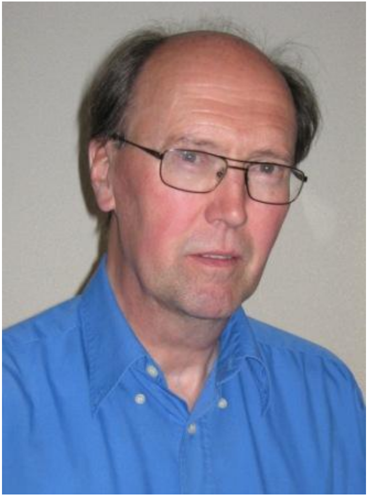
Standard thm: general solution of eqn (2) using spectral decomposition of

- matrix G
- Drazin inverse of matrix similar to R

Eqn (1) gives boundary conditions

Project:

- reconcile our solution with this formulation
- write general solution in terms of R and G directly



Best wishes, Erik, and I hope you have fun over the next umpteen years !