Stabilisation of Linear Time-Invariant Systems subject to Output Saturation

by

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Abstract

Nowadays we heavily rely on all kinds of control systems, ranging from boilers and thermostats to airplanes and GPS satellites. By means of measurement devices, useful information about a dynamical system can be obtained. Using the measurement data, a control law generating an input may be designed to meet specific demands of the closed-loop dynamics.

Output saturation, describing range limitations of measurement devices, has not received much attention yet. Since output saturation is present in many automatic control systems, controllers coping with this measurement limitation are desired. The main difficulty with output saturation is that the magnitude of a saturated measurement is unknown. To be able to control the system, the output must be steered towards the region where it does not saturate. If this can be achieved, well-known observation techniques can be applied locally.

We specialise to the class of controllable and observable linear time-invariant systems with saturated output. We show that, if such a system is stable, a controller global asymptotically stabilising the system can be designed by means of a quadratic Lyapunov function. LaSalle's invariance principle is necessary to prove this result. First continuous-time systems are treated. Since implementability is the major issue, we expand the results to discrete-time systems. Furthermore, we discuss how a continuous-time system can be discretised, and under which conditions an implementable controller design can be achieved.

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Chapter 1

Introduction

1.1 Motivation

The theory of automatic control proved to be of major importance for centuries. The ancient Greeks used an automatic control system to accurately determine time. Inventions from the industrial revolution, such as the steam engine, created new requirements for automatic control systems. Until then, the design of control systems was a combination of trial-and-error and intuition. Therefore, mathematical analysis of control systems was desired, and this was first used in the middle of the 19^{th} century. Since then, mathematics is the formal language of control theory. Nowadays we heavily rely on all kinds of control systems, ranging from boilers and thermostats to airplanes and GPS satellites. Mathematical modeling allows for proper analysis and control of dynamical systems. By means of measurement devices, useful information about a dynamical system can be obtained. Using the measurement data, a control law generating an input may be designed to meet specific demands of the closed-loop dynamics. The basic structure of an automatic control system is depicted in Figure 1.1.

Unfortunately, the basic structure as in Figure 1.1 is too simplistic for many control systems. Therefore, an extension of the model structure is required. It may occur that the input desired by the controller can not be reached, due to actuator limitations. This

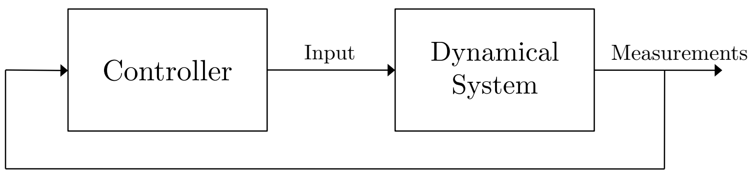


Figure 1.1: Structure of an automatic control system.

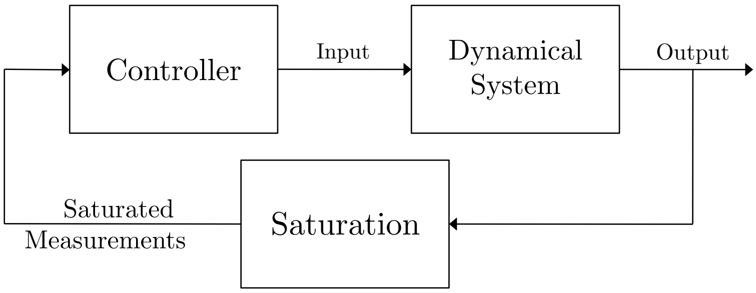


Figure 1.2: Structure of an automatic control system subject to output saturation.

phenomenon is called *input saturation*, and has been studied extensively for the last couple of years. In general, neglecting input saturation results in a behaviour that is far from optimal. Another possibility is to consider range limitations of measurement devices. If the measurement range is exceeded, and some minimum or maximum value is attained, we say that the measurement is *saturated*. We speak of *output saturation* in the latter case. Opposed to input saturation, output saturation in control systems has not received much attention yet. Since output saturation is present in many automatic control systems nowadays, controllers coping with this measurement limitation are desired. An automatic control system subject to output saturation can be viewed as in Figure 1.2. The main difficulty with output saturation is that the magnitude of a saturated measurement is unknown. To be able to control the system, the output must be steered toward the region where it does not saturate. If this can be achieved, well-known stabilisation techniques can be applied locally. Intuitively speaking, the output must be out of saturation long enough to guarantee output regulation. We will see that a solution to this problem is not that straightforward.

In 1996, Kreisselmeier proposed a stabilising controller for the entire class of continuous-time single-input-single-output (SISO) linear time-invariant (LTI) systems subject to output saturation, see [12]. This result was extended to multi-input-multi-output (MIMO) LTI systems in 2010, see [4]. Unfortunately, the suggested controller is very sensitive to measurement disturbances. Furthermore, it heavily relies on *continuous measurements*. Since most control systems depend on *sampled measurements*, the proposed controller is not implementable in general. Therefore, it is a milestone to design stabilising controllers for control systems with sampled saturated output. Obviously, sampled measurements give less information about some dynamical system than continuous measurements. A different approach is required, which we describe next.

1.2 Approach

In this thesis, we specialise to the class of *linear time-invariant* systems, and incorporate output saturation in the model structure. This is an obvious starting point, since LTI systems have many important applications in different fields, ranging from electrical engineering to biology and economics. While two LTI systems may be physically different, the beauty is that they may be mathematically *equivalent*. Considering the class of *neutrally stable* LTI systems first, we show that stabilising controllers using saturated measurements can be designed. Neutrally stable systems have the property that their trajectories neither converge to some fixed equilibrium point, nor diverge. This result holds in both continuous- and discrete-time, and can be extended to the class of *stable* LTI systems with output saturation.

1.3 Goal

The goal of this thesis is two folds. First of all, this work serves to give insight into systems subject to output saturation, by explaining the results obtained so far and describing the additional bottlenecks that are to be resolved. The main bottlenecks are *implementability* and rejection of *measurement noise*. After that, an elegant solution tackling those bottlenecks is proposed for the class of stable LTI systems with output saturation.

1.4 Structure of the Thesis

To start, some basic concepts regarding LTI systems are introduced in Chapter 2. The mathematical definition of a dynamical system is given. Then state space models are introduced, which serve as a mathematical framework for LTI systems. Controllability and observability are defined and intuitively explained. Finally, we explain the notion of stability and how LTI systems can be stabilised.

In Chapter 3, the saturation function is defined and LTI systems subject to output saturation are introduced. The solution for stabilisation of continuous-time LTI systems proposed by Kreisselmeier (see [12]) is discussed, along with its shortcomings that motivate further research. A procedure for discretisation is explained, and conditions are given for preservation of controllability and observability after discretisation of a continuous-time LTI system.

Specialising to the class of stable LTI systems subject to output saturation, a stabilising controller for both continuous-time and discrete-time systems is derived in Chapter 4. An observer and a feedback law can be designed independently. However, the separation principle fails and a more advanced approach is required to prove stability. We prove stability using Lyapunov functions and LaSalle's invariance principle. The resulting controllers have a certain immunity to measurement noise. The chapter ends with simulations verifying the theoretical results.

Chapter 5 provides conclusions and recommendations for future work.

Chapter 2

Basic Concepts of Linear Time-Invariant Systems

This chapter introduces some important definitions and theorems regarding *linear time-invariant* (LTI) systems, which are necessary for the sequel. Starting from the definition of a *dynamical system*, *linearity* and *time-invariance* are defined. Basically, we view a system in terms of *input*, *state* and *output*. Roughly speaking, the *state* of a dynamical system contains all the information that is necessary to determine its future evolution, without information about the past input. The concept of state is introduced, and this leads to so-called *state space representations*. State space models provide us with a beautiful general description of both continuous- and discrete-time LTI systems. Furthermore, state space models are very suitable for analysis. Looking at a state space representation, properties such as *controllability* and *observability* can immediately be verified. In words, a system is controllable if the input can be chosen such that the state can be steered to any trajectory within the possible behaviour. A system is observable if its state can be reconstructed from input and output measurements on some time interval. Finally the important concept of *stability* is described, which we need to analyse the effect of disturbances on the system dynamics.

2.1 Some Important Definitions

We start with the mathematical definition of a dynamical system.

Definition 1 (Dynamical System [14, 15])**.** *A* dynamical system Σ *is a triple* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ *where* $\mathbb{T} \subset \mathbb{R}$ *and* $\mathcal{B} \subset \{w : \mathbb{T} \to \mathbb{W}\}$.

Here $\mathbb{T}$ is the time axis, describing the set of time instants we consider. $\mathbb{W}$ is the set of possible outcomes, and $\mathcal{B}$ denotes the behaviour of the system. For continuous-time systems, usually $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}_+$, whereas for discrete-time systems $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{Z}_+$ are often useful. For our purposes, we consider $\mathbb{W} = \mathbb{R}^q$. We give an example of a dynamical system.

Example 1 (Cycling Paul)**.** *Paul likes to go cycling, and decides to model his cycling behaviour. Paul has a mass of m kilograms. Furthermore, it is assumed that his resistance is proportional to his speed, with proportionality coefficient k. Denoting the transversal component of the applied force at time t by $F(t)$, this yields the equation*

$$F(t) = m\frac{d^2}{dt^2}s(t) + k\frac{d}{dt}s(t), \tag{2.1}$$

where $s(t)$ is the total distance travelled up to time t. Paul starts at time zero with an initial distance of zero kilometres. A description of the behaviour for the dynamical system "cycling Paul" is

$$\mathcal{B} = \left\{ \begin{bmatrix} F \\ s \end{bmatrix} : \mathbb{R}_+ \to \mathbb{R}^2 \mid F(t) = m\frac{d^2}{dt^2}s(t) + k\frac{d}{dt}s(t),\ s(0) = 0 \right\}. \tag{2.2}$$

The dynamical system is described by $\Sigma = (\mathbb{R}_+, \mathbb{R}^2, \mathcal{B})$.

Now linearity and time-invariance of dynamical systems are defined.

Definition 2 (Linearity [14, 15])**.** *A system $\Sigma = (\mathbb{T}, \mathbb{R}^p, \mathcal{B})$ is* linear *if*

$$w_1, w_2 \in \mathcal{B} \Rightarrow \alpha w_1 + \beta w_2 \in \mathcal{B} \qquad \forall \alpha, \beta \in \mathbb{R} \tag{2.3}$$

If the response of a dynamical system does not change with time, that system is said to be time-invariant. Many dynamical systems possess this property.

Definition 3 (Time-Invariance [14, 15])**.** *Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$. A system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ is* time-invariant *if for every $\tau \in \mathbb{T}$ we have*

$$w \in \mathcal{B} \Rightarrow \sigma_\tau w \in \mathcal{B} \tag{2.4}$$

where σ_τ is the shift operator *defined as $(\sigma_\tau w)(t) = w(t - \tau)$.*

Systems that are both linear and time-invariant are LTI systems. Examples of LTI systems are mass-spring-damper systems and electrical circuits made up of capacitors, inductors and resistors. While mechanical- and electrical systems are physically different, they may be mathematically *equivalent*. This motivates the desire for a general mathematical model, and makes mathematics very powerful.

The next step is to define the concept of state. This concept simplifies the analysis of dynamical systems, since the state exactly contains the information that is critical for determination of the system evolution, without information about the past input.

Definition 4 (State [14]). *If a system with input u and output y is of the form*

$$y(t) = \mathcal{H}(x(t_0), u(\tau)\,|_{\tau\in[t_0,t]}, t) \qquad \forall t \in \mathbb{T},\ t \geq t_0 \tag{2.5}$$

for some map $\mathcal{H}$ and time axis $\mathbb{T} \subset \mathbb{R}$, then we say that x is a state *for the system.*

Throughout this report, the state is denoted by x, and is a function of time. In the next section state space models are introduced. We will see that LTI systems can be written in a very elegant way, using the notion of state.

2.2 State Space Models

For the class of (finite-dimensional) LTI systems, the behaviour can be described in a very convenient way. The input, state and output of a differential/difference LTI system are related through a set of first order linear differential/difference equations with constant coefficients. In general, systems may have multiple inputs and outputs. Those systems are called multi-input-multi-output (MIMO), as opposed to single-input-single-output (SISO) systems. We consider systems with input vector

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \tag{2.6}$$

and output vector

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}. \tag{2.7}$$

Moreover, the state is usually multi-dimensional. We denote the state vector $x \in \mathbb{R}^n$ by

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}. \tag{2.8}$$

Continuous- and discrete-time LTI systems are distinguished, since their mathematical properties are different. For both continuous- and discrete-time systems, the same results regarding controllability and observability hold. However, stability properties require to be stated separately.

2.2.1 Continuous-Time LTI Systems

Continuous-time LTI systems can often be written in the form

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases}, \qquad t \in \mathbb{R}, \tag{2.9}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are given. $\dot{x}$ denotes the derivative of x with respect to time. The first equation of (2.9) is called the *state equation*, because it describes the evolution of the state for some given input. The second equation is the *output equation*, which describes the measured quantities of the system.

The solution of the state equation is explicitly given by

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau, \tag{2.10}$$

for given input u on $[t_0, t)$ and initial state $x(t_0)$. In addition, the solution of the output equation is

$$y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau. \tag{2.11}$$

Example 2 (Cycling Paul continued)**.** *Paul realises that his cycling behaviour is linear and time-invariant, and decides to make a state space model with* $u(t) := F(t)$ *as input and* $y(t) := s(t)$ *as output. He defines the state vector* $x(t) := (s(t), \dot{s}(t))$*. A state space representation for our bike fanatic is*

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{k}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \end{aligned} \tag{2.12}$$

Now using equation (2.11) *for a given input results in* $y(t)$*. (Details on the calculation of* $y(t)$ *are omitted.) Suppose that the trip lasts* T *time units. Then* $y(T)$ *is the total cycling distance.*

2.2.2 Discrete-Time LTI Systems

In discrete-time, many LTI system can be described by the system of equations

$$\left\{ \begin{array}{lcl} x[k+1] & = & Ax[k] + Bu[k] \\ y[k] & = & Cx[k] \end{array} \right. , \qquad k \in \mathbb{Z}. \tag{2.13}$$

The dependence on discrete time is indicated by square brackets. The explicit solution of the state- and output equation are

$$x[k] = A^{k-k_0} x[k_0] + \sum_{j=k_0}^{k-1} A^{k-1-j} Bu[j] \tag{2.14}$$

and

$$y[k] = CA^{k-k_0} x[k_0] + \sum_{j=k_0}^{k-1} CA^{k-1-j} Bu[j] \tag{2.15}$$

respectively, for given input u on time instants $k_0, \ldots, k-1$ and initial state $x[k_0]$.

2.2.3 Controllability and Observability

Controllability of dynamical systems is an important issue in many applications. For example, consider a cruise controller for a car. The dynamics of the car can be modelled as a dynamical system, of which the input is the throttle opening and the output is the velocity. Given a reference velocity to the system, it is desired for the car to maintain that velocity by applying an input. If the system is controllable, such an input exists if the desired velocity is within the system behaviour.

We give the formal definition of controllability.

Definition 5 (Controllability [15]). *Let $\mathcal{B}$ be the behaviour of a time-invariant dynamical system. This system is called* controllable *if for any two trajectories $w_1, w_2 \in \mathcal{B}$ there exist a $t_1 \geq 0$ and a trajectory $w \in \mathcal{B}$ with the property*

$$w(t) = \left\{ \begin{array}{ll} w_1(t) & t \leq 0 \\ w_2(t) & t \geq t_1 \end{array} \right. \tag{2.16}$$

For both the systems (2.9) and (2.13), controllability can be determined by looking at the rank of the so-called *controllability matrix*

$$\mathcal{C} = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array} \right]. \tag{2.17}$$

Theorem 1 (Controllability [15])**.** *Consider the system* (2.9)*. The system is controllable if and only if its controllability matrix* $\mathcal{C}$ *has rank* n*.*

Obviously, controllability of (2.9) can be determined by only looking at the state equation. Occasionally, we talk about controllability of the pair (A, B). This is identical to controllability of the system (2.9). Theorem 1 also holds for the discrete-time system (2.13), see [14].

An important concept which is strongly related to controllability is that of observability. Loosely speaking, observability of a dynamical system means that its state can uniquely be determined by looking at the input and the output on some time interval. The point is that, for an observable system, the state does not have to be measured directly in order to recover it. In reality it is not always possible and often expensive to measure each state component independently.

We define observability in analogy with the definition of observability in [15].

Definition 6 (Observability)**.** *Let* $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathcal{B})$ *be a time-invariant dynamical system. Trajectories in* $\mathcal{B}$ *are partitioned as* (w_1, w_2) *with* $w_i : \mathbb{R} \to \mathbb{W}_i,\ i = 1, 2$*. We say that* w_2 *is* observable *from* w_1 *if* $(w_1, w_2),\ (w_1, w_2^{'}) \in \mathcal{B}$ *implies* $w_2 = w_2^{'}$*.*

If the state is to be observed from the input/output pair, one should make the substitution $w_1 = (u, y)$ and $w_2 = x$ in Definition 6. Looking at the *observability matrix* defined as

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \tag{2.18}$$

observability of the LTI system (2.9) can be investigated.

Theorem 2 (Observability [15])**.** *Consider the system* (2.9)*. The system is observable if and only if its observability matrix* $\mathcal{O}$ *has rank* n*.*

Observability of the system (2.9) depends on the pair (A, C). If $\mathcal{O}$ has rank n, the pair (A, C) is called observable. Theorem 2 can also be applied to the system (2.13), see [14].

Example 3 (Paul's Velocity)**.** *Consider the state equation of the "cycling Paul" model.*

$$\dot{x}(t) \quad = \quad \begin{bmatrix} 0 & 1 \\ 0 & -\frac{k}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t) \ . \tag{2.19}$$

First of all, assume that we continuously measure Paul's position, then $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$*. The corresponding observability matrix is*

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.20}$$

and has full rank. Hence the state is observable from input and output measurements. In other words, measuring the applied force and the position continuously, the velocity can be determined. Is it also possible to derive Paul his position by measuring his velocity? Notice that $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$ *in that case, thus the observability matrix is*

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{k}{m} \end{bmatrix}. \tag{2.21}$$

Obviously this matrix does not have full rank, so the state is not observable. This is intuitive, since only relative position can be determined by measuring velocity. An initial position is required to find the current position.

Notice the similarity between the conditions for controllability and observability of LTI systems. Observability and controllability are related through the following *duality*:

Theorem 3 (Duality)**.** (A, B) *is controllable if and only if* (A^T, B^T) *is observable.*

Theorem 3 is very useful for deriving observability theorems from controllability theorems, or vice versa.

2.2.4 Stability

The notion of stability of *autonomous* systems is discussed in this section. Automatic control systems are examples of autonomous systems, and are very important nowadays. First a general definition is given, and then some properties regarding stability of LTI systems are discussed.

Continuous-Time Systems

In continuous-time, a time-invariant autonomous system has the form

$$\dot{x} = f(x). \tag{2.22}$$

The assumption that f is *continuously differentiable* guarantees *existence* and *uniqueness* of solutions $x(t)$ for (2.22), for any initial condition $x(0) \in \mathbb{R}^n$, see [7]. The class of systems for which (2.22) holds includes systems where the input is a function of the state. Those systems are called *closed-loop* systems, and choosing the state feedback properly might result in desired behaviour of (2.22). To analyse the behaviour of autonomous systems, we need the definition of an *equilibrium point*.

Definition 7 (Equilibrium Point [10]). *Consider the system* (2.22). *We say that* $x^* \in \mathbb{R}^n$ *is an equilibrium point for the system if*

$$f(x^*) = 0. \tag{2.23}$$

If $x = x^*$, then the state will always remain in the equilibrium point x^*, under the assumption that (2.22) is not subject to any disturbances. Without loss of generality, we assume that $x = 0$ is an equilibrium point of (2.22). To classify stability of the equilibrium point $x = 0$, we state a definition.

Definition 8 (Stability of an Equilibrium Point [10]). *The equilibrium point* $x = 0$ *of* (2.22) *is*

- stable *if, for each* $\epsilon > 0$, *there is a* $\delta = \delta(\epsilon) > 0$ *such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \qquad \forall t \geq 0.$$

- unstable *if not stable.*
- asymptotically stable *if it is stable and* δ *can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0.$$

- globally asymptotically stable *if it is stable and*

$$\lim_{t\to\infty} x(t) = 0$$

  for all initial conditions.

Here $\|\cdot\|$ can be any desired *vector norm* which agrees with the definition (see Appendix A). In words, an equilibrium point is stable if a small disturbance has small effect on the evolution of the state. An equilibrium point is unstable if a small disturbance in the initial condition may result in huge differences as time progresses. Asymptotic stability of an equilibrium point implies that the state converges to the equilibrium point, provided that the disturbance is not too large. When global asymptotic stability holds, the state will always converge to the equilibrium point, no matter what the size of the disturbance is. Notice that a globally asymptotically stable equilibrium point is always unique.

We consider autonomous continuous-time LTI systems of the form

$$\dot{x} = Ax. \tag{2.24}$$

The origin is an equilibrium point of (2.24). However, it is not necessarily unique. Equilibria of autonomous LTI systems all have the same stability properties [15]. So for LTI systems, stability can be seen as a property of the system. Stability of (2.24) can be characterised in terms of the eigenvalues of A. To achieve this characterisation, we need the definition of a *semisimple* eigenvalue first.

Definition 9 (Semisimple Eigenvalue [15])**.** *Let $A \in \mathbb{R}^{n\times n}$ and let λ be an eigenvalue of A. We call λ a* semisimple *eigenvalue of A if the dimension of*

$$Ker(\lambda I - A) := \{v \in \mathbb{R}^n \mid (\lambda I - A)v = 0\} \tag{2.25}$$

is equal to the multiplicity of λ as a root of the characteristic polynomial of A.

The definition of a semisimple eigenvalue can be illustrated with an example.

Example 4. *Consider the system* (2.24) *and let*

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}. \tag{2.26}$$

Matrix A has eigenvalue λ with multiplicity 2*. For $\lambda I - A$ we have*

$$\lambda I - A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}. \tag{2.27}$$

A has only one linear independent eigenvector corresponding to λ, thus the dimension of $Ker(\lambda I - A)$ is 1*. This means that λ is not a semisimple eigenvalue of A.*

Now a theorem for stability of (2.24) is stated.

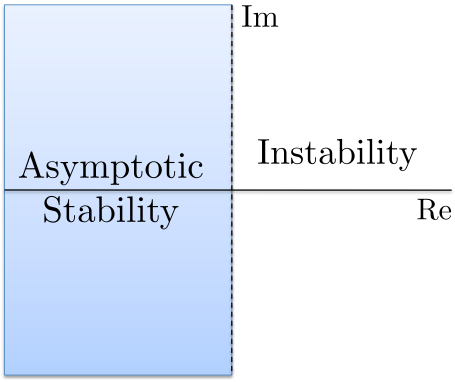


Figure 2.1: Stability regions for the autonomous system (2.24)

Theorem 4 (Stability of Continuous-Time LTI systems [15])**.** *The system* (2.24) *is:*

- asymptotically stable *if and only if the eigenvalues of A have negative real part.*
- stable *if and only if for each $\lambda \in \mathbb{C}$ that is an eigenvalue of A, either*
  1. *$Re(\lambda) < 0$, or*
  2. *$Re(\lambda) = 0$ and λ is a semisimple eigenvalue of A.*
- unstable *if and only if A has either an eigenvalue with positive real part or a nonsemisimple eigenvalue with zero real part.*

If (2.24) is asymptotically stable, we say that the matrix A is *Hurwitz*. This is exactly the case if all the eigenvalues of A are in the open left half complex plane. See Figure 2.1. Notice that for the system (2.24), the definitions of asymptotic stability and global asymptotic stability coincide.

Example 5. *Recall Example 4. We will intuitively explain why a zero eigenvalue must be semisimple for stability to hold. Consider the system*

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x. \tag{2.28}$$

The two eigenvalues of A are $\lambda_{1,2} = 0$. Using equation (2.10), we get

$$\begin{aligned} x(t) &= \exp\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t\right\} x(0) && (2.29)\\ &= \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t\right\} x(0) && (2.30)\\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0). && (2.31) \end{aligned}$$

The second step follows from the fact that $A^k = 0$ for all $k \geq 2$, and by applying the expansion $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$. Despite the fact that both eigenvalues are zero, the state grows linearly.

Discrete-Time Systems

Analogous to the continuous-time case, discrete-time autonomous systems are described by

$$x[k+1] = f(x[k]). \tag{2.32}$$

An equilibrium point for (2.32) is defined as follows.

Definition 10 (Equilibrium Point)**.** *Consider the system* (2.32)*. We say that x^* is an equilibrium point for the system if*

$$f(x^*) = x^*. \tag{2.33}$$

Notice that, compared to the continuous-time definition, the definition of an equilibrium point for discrete-time systems is different. Without loss of generality, we assume that $x = 0$ is an equilibrium point of (2.32). In analogy with Definition 8, we define stability for discrete-time systems.

Definition 11 (Stability of an Equilibrium Point)**.** *The equilibrium point $x = 0$ of* (2.32) *is*

- stable *if, for each $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that*

$$\|x[0]\| < \delta \Rightarrow \|x[k]\| < \epsilon, \qquad \forall k \geq 0.$$

- unstable *if not stable.*

- asymptotically stable *if it is stable and* δ *can be chosen such that*

$$\|x[0]\| < \delta \Rightarrow \lim_{k\to\infty} x[k] = 0.$$

- globally asymptotically stable *if it is stable and*

$$\lim_{k\to\infty} x[k] = 0$$

for all initial conditions.

For linear autonomous systems

$$x[k+1] = Ax[k], \tag{2.34}$$

results comparable to those in theorem 4 can be derived.

Theorem 5 (Stability of Discrete-Time LTI systems [15])**.** *The system* (2.34) *is:*

- asymptotically stable *if and only if the eigenvalues of* A *have modulus smaller than one.*
- stable *if and only if for each* $\lambda \in \mathbb{C}$ *that is an eigenvalue of* A*, either*
  1. $|\lambda| < 1$*, or*
  2. $|\lambda| = 1$ *and* λ *is a semisimple eigenvalue of* A*.*
- unstable *if and only if* A *has either an eigenvalue with modulus larger than one or a nonsemisimple eigenvalue with modulus one.*

If (2.34) is asymptotically stable, we say that the matrix A is *Schur*. It is not the left half complex plane, but the unit circle which characterises stability of a discrete-time system. Figure 2.2 illustrates this idea.

2.2.5 Stabilisation of LTI Systems

In this section we briefly discuss two types of stabilising feedback controllers for both continuous- and discrete-time LTI systems. The first type is a *static state feedback* and the second is a *dynamic output feedback*. For controllable LTI systems, a static feedback stabilising the system always exists, provided that the initial state is given. However, in most of the cases the state is unknown. The class of LTI systems that are both controllable and observable can always be stabilised by dynamic output feedback.

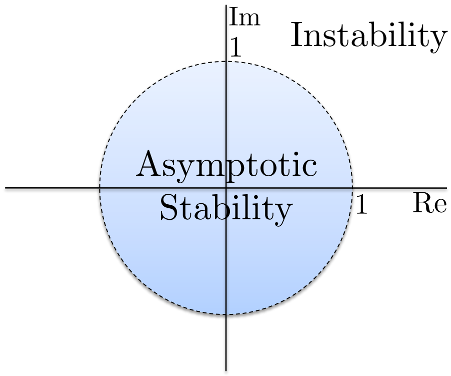


Figure 2.2: Stability regions for the autonomous system (2.34)

Static State Feedback

Consider the continuous-time state equation

$$\dot{x} = Ax + Bu. \tag{2.35}$$

A classical choice is to apply a state feedback of the form

$$u = Fx \tag{2.36}$$

to stabilise the system (2.35). $F \in \mathbb{R}^{m \times n}$ is a parameter matrix. With the control law (2.36), the state equation reduces to the autonomous system

$$\dot{x} = (A + BF)x. \tag{2.37}$$

Notice that $x = 0$ is an equilibrium point. As described in section 2.2.4, the eigenvalues of $A + BF$ determine the stability of the equilibrium point $x = 0$. A famous result is that F can always be chosen such that the matrix $A+BF$ has any desired eigenvalues, if (A, B) is controllable [15]. This means that there exists an F such that (2.37) is globally asymptotically stable.

The same result holds for controllable discrete-time LTI systems. However, for discrete-time systems we get something extra. Let us take a closer look at the autonomous discrete-time LTI system

$$x[k + 1] = (A + BF)x[k], \qquad k \in \mathbb{Z}. \tag{2.38}$$

If F is chosen such that all the eigenvalues of $A + BF$ are equal to *zero*, we get that $A + BF$ is a nilpotent matrix. This follows from the fact that a matrix is nilpotent if and only if all of its eigenvalues are zero [17]. This yields

$$x[k + n] = (A + BF)^n x[k] = 0. \tag{2.39}$$

The state of the system is zero within n steps. A controller achieving $x = 0$ within n steps is called a *dead beat* controller.

Dynamic Output Feedback

So far, we have seen that a stabilising controller exists, provided that the system is controllable and the initial state is known. Usually, the state of a system is unknown. In that case, we need a systematic way to obtain information about the state. There are several techniques providing estimates of the state by measuring some output variable $y(t)$. A commonly used technique is to define a dynamical system for the estimate of the state. Such a dynamical system is called a *state observer*. The objective of an observer is to make the error

$$e(t) := x(t) - \hat{x}(t) \tag{2.40}$$

smaller as time progresses. Here $\hat{x}(t)$ denotes the state estimate at time t. In this section an observer design is discussed, to provide us with an estimate of the state. If the error is small enough, the state estimate can be treated as the state itself. This allows us to steer the behaviour of the system, by defining some feedback using the state estimate.

For continuous-time LTI systems, a linear observer of the form

$$\dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{\text{copy plant}} + \underbrace{K(y - \hat{y})}_{\text{innovation}}, \qquad \hat{y} = C\hat{x} \tag{2.41}$$

is very popular. $K \in \mathbb{R}^{n \times p}$ is the so-called *observer gain*. The dynamics (2.41) are governed by a copy of the plant and an innovations term. The copy of the plant lets the state estimate undergo the same evolution as the real state. This is intuitive, since it serves as some sort of tracking when the estimate is good enough. The innovations term corrects for the measurement error, improving the estimate if K is chosen properly. Combining (2.35) and (2.41), dynamics for the error can be derived.

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= (Ax + Bu) - (A\hat{x} + Bu + K(Cx - C\hat{x})) \\ &= (A - KC)e \end{aligned} \tag{2.42}$$

If (A, C) is observable, K can always be chosen such that $A - KC$ has any desired eigenvalues [15].

A natural choice is to feed back the estimated state to the input of the system using the feedback law

$$u = F\hat{x}. \tag{2.43}$$

Doing so, we obtain a closed-loop system where the observer is part of the controller. The resulting closed-loop dynamics are given by

$$\begin{cases} \dot{x} &= (A+BF)x - BFe, \\ \dot{e} &= (A-KC)e, \\ e &= x - \hat{x}. \end{cases} \tag{2.44}$$

Writing the state- and the error equation in matrix form, we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A+BF & -BF \\ 0 & A-KC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}. \tag{2.45}$$

The eigenvalues of $A+BF$ and $A-KC$ together determine the stability of the closed-loop dynamics. In fact, the observer and the controller can be designed *independently*. This is known as the *separation principle* for LTI systems. We state a famous result:

Theorem 6 (Stabilising Dynamical Controller [14]). *If the system* (2.9) *is controllable and observable, then there exist matrices F and K such that $A+BF$ and $A-KC$ are Hurwitz. In that case,* $\lim_{t\to\infty} x(t) = 0$ *and* $\lim_{t\to\infty} \hat{x}(t) = 0$ *for all initial conditions $x(0)$ and $\hat{x}(0)$.*

We state without proof that similar results holds for discrete-time LTI systems.

Theorem 7 (Stabilising Dynamical Controller). *If the system* (2.13) *is controllable and observable, then there exist matrices F and K such that $A+BF$ and $A-KC$ are Schur. In that case,* $\lim_{k\to\infty} x[k] = 0$ *and* $\lim_{k\to\infty} \hat{x}[k] = 0$ *for all initial conditions $x[0]$ and $\hat{x}[0]$.*

For discrete-time LTI systems, a *dead beat* observer may be designed to make the error zero within n steps. In combination with a dead beat controller, the system is stabilised within a finite number of steps.

Now the basic concepts of LTI systems are introduced. In the next chapter, output saturation is incorporated, resulting in an overall system which is piecewise linear. The difficulties of designing a controller are explained, and a solution for continuous-time LTI systems is discussed.

Chapter 3

LTI Systems Subject to Output Saturation

A lot of research has been devoted to the problem of stabilising LTI systems subject to *input saturation*, modelling actuator limitations. However, stabilisation of LTI systems with *output saturation* has not received much attention yet. Output saturation may occur due to range limitations of some measurement device. Consider a camera on a robot for example, whose objective is to track an object. If the object to be tracked is in range, the camera can detect it and determine its position. If the object is not in range, the camera detects nothing, and its position can not be deduced. In many applications however, it is known on which side of the camera the object is located. In the latter case, we say that the measurement is *saturated*.

To incorporate output saturation in a dynamical system, a *saturation function* is defined in Section 3.1. Thereafter, Section 3.2 gives an overview of the difficulties we face when considering output saturation in state space models. Results regarding LTI systems subject to output saturation are discussed in Section 3.3. Until now, only stabilisation of continuous-time output saturated systems is considered in the literature. See for example [4, 11, 12]. In [12], a feedback compensator for a SISO LTI system subject to output saturation is presented, globally asymptotically stabilising the system. In Section 3.3, the design of this feedback compensator is explained in more detail. This gives us an idea of the difficulties we face when trying to implement the proposed controller. Motivated by the implementability issues, Section 3.4 explains how continuous-time state space models can be discretised. Furthermore, a condition is given for preservation of controllability and observability of a discretised system.

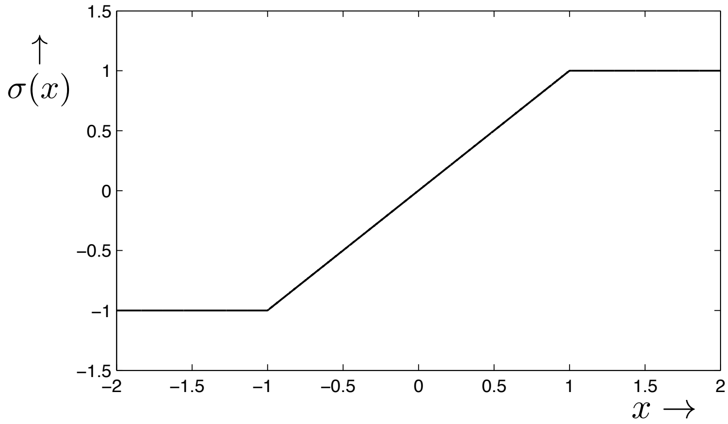


Figure 3.1: The saturation function (3.1)

3.1 The Saturation Function

To start this chapter, the saturation function is defined. We decide to use a piecewise linear saturation function, which assumes correct measurements in the range $[-1, 1]$. Any values above 1 or below -1 are mapped to 1 and -1 respectively.

Definition 12 (Scalar Saturation Function)**.** *The* scalar saturation function $\sigma : \mathbb{R} \to \mathbb{R}$ *is defined as*

$$\sigma(x) := \begin{cases} x & \text{if} \quad |x| < 1 \\ -1 & \text{if} \quad x \leq -1 \\ 1 & \text{if} \quad x \geq 1 \end{cases} \tag{3.1}$$

This function can be shifted and scaled as desired, without influencing the generality. See Figure 3.1 for an illustration of the saturation function (3.1). This function is piecewise linear, which allows us to use LTI systems theory locally. However, an LTI system subject to output saturation is only *piecewise* linear. We will see that the solution for stabilising a system with saturated outputs is not that straightforward.

As we consider MIMO LTI systems in the sequel, a *vector saturation function* is desired to model saturation in all of the measurements. The vector saturation function is defined element-wise, making use of the scalar saturation function.

Definition 13 (Vector Saturation Function)**.** *Let* $x = [x_1, \ldots, x_n]^T$*. The* vector saturation function $\boldsymbol{\sigma}(.) : \mathbb{R}^n \to \mathbb{R}^n$ *is defined element-wise as*

$$\boldsymbol{\sigma}(x) := \begin{bmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_n) \end{bmatrix}. \tag{3.2}$$

The saturation function (3.2) is directly applicable to systems using measurement devices with limited range. Our choice for a saturation function is a logical one, since many physical saturation elements can be closely approximated by a piecewise linear function of the form (3.2). If a measurement device provides correct measurements only within some specific range, the saturation function (3.2) can be used as a generalisation.

3.2 Introduction to Output Saturated LTI Systems

To introduce the difficulties arising in a system with saturated output, consider a continuous-time SISO LTI system of the form

$$\begin{cases} \dot{x} &= Ax + bu \\ z &= cx \end{cases} \tag{3.3}$$

with $x \in \mathbb{R}^n$. We explicitly note that (3.3) is allowed to be *unstable*. Furthermore, we assume that the measured output y is a saturated version of z:

$$y = \sigma(z). \tag{3.4}$$

The problem of stabilising the overall system (3.3)-(3.4) is trivial if the state is given, since controllability of (A, b) guarantees stabilisability of the state by static state feedback. The saturation element plays no role then. So suppose that the initial state is unknown. Our only hope to recover the state is by measuring the output y, and use this together with information about the input u. Assuming observability of the pair (A, c), the state can be observed as long as y is not saturated. In that case, the overall system behaves locally as an LTI system, and the techniques described in Chapter 2 can be applied. An observer design can be done to improve the state estimate. However, no state observation is possible when y is saturated.

A general question of interest is under which conditions a saturated output can be desaturated. If (A, b) is controllable, the existence of an input desaturating the output

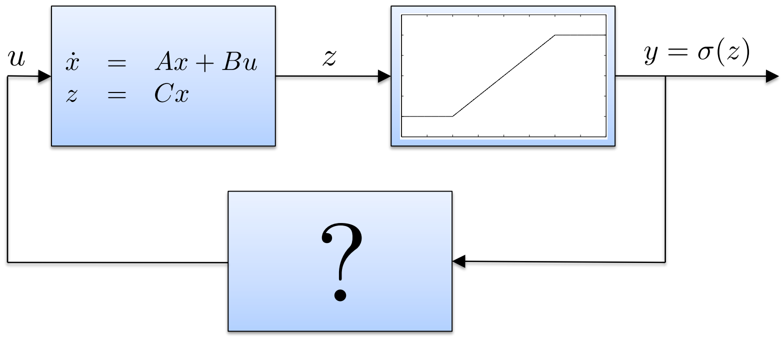


Figure 3.2: Schematic overview of the general problem.

is guaranteed. To see this, note that

$$z(t) = ce^{At}x(0) + \int_0^t ce^{A(t-\tau)}bu(\tau)d\tau. \tag{3.5}$$

For any $x(0) \in \mathbb{R}^n$, the input can be chosen such that the integral term dominates over $ce^{At}x(0)$. Hence the state can be observed. How to choose the input is another problem of interest. An input which is too small not necessarily lets the output cross zero. We are forced to choose a diverging input in case of an unstable system, resulting in an output which only desaturates on a very short time interval. To recover the state, a very fast observer is required in the latter case. This means that the observer is hardly implementable and very sensitive to measurement noise in practice.

We are interested in a controller stabilising the system (3.3)-(3.4) by output feedback. Or, stated more generally, stabilise the MIMO variant of (3.3)-(3.4). See Figure 3.2 for a schematic overview of the general problem. In [12], Kreisselmeier proposes a solution for the SISO case.

3.3 Kreisselmeier's Solution

In this section Kreisselmeier's solution to the stabilisation of the overall system (3.3)-(3.4) is explained. Controllability and observability of (3.3) are assumed. See [12] for the original publication of Kreisselmeier.

3.3.1 Solution Concept

As discussed in the previous section, controllability and observability of (3.3) imply controllability and observability of the overall system. Provided that (A, b) is controllable, the existence of an input desaturating the output is clear. Without knowledge about the initial state, Kreisselmeier constructs an input that eventually dominates over the potential growth rate of the system. To achieve this, time intervals of length T are introduced, where $T > 0$ is a parameter. To explain Kreisselmeier's method, first of all his algorithm described in [12] is given. Thereafter, the algorithm is explained in detail.

The input applied on the k^{th} interval is given by

$$u(kT + \tau) = -b^T e^{-A^T \tau} U_k, \qquad \tau \in [0, T), \qquad k \in \mathbb{Z}_+. \tag{3.6}$$

Here U_k explicitly depends on the interval length T and on k, and is defined as

$$U_k = \begin{cases} \alpha^k B^{-1}(T)[\alpha h - e^{AT} h] y(kT) & \text{if} \quad \theta_k = 0, \\ B^{-1}(T) e^{AT}[e^{AT} D_k^{-1} \xi_k - B(T) U_{k-1}] & \text{if} \quad \theta_k > 0. \end{cases} \tag{3.7}$$

In (3.7), $h \in \mathbb{R}^n$ is any vector such that $ch > 0$, further $\alpha := \exp\{2T \cdot \|A\|\}$, and

$$\begin{aligned} \theta_0 &:= 0; \ \theta_k := \int_{kT-T}^{kT} [1 - |y(t)|] dt, \qquad k > 0, & (3.8) \\ B(\tau) &:= \int_0^\tau e^{A(\tau - s)} b b^T e^{-A^T s} ds, & (3.9) \\ D_k &:= \int_0^T e^{A^T \tau} c^T c e^{A\tau} [1 - |y(kT - T + \tau)|] d\tau, & (3.10) \\ \xi_k &:= \int_0^T e^{A^T \tau} c^T [y(kT - T + \tau) + cB(\tau) U_{k-1}][1 - |y(kT - T + \tau)|] d\tau. & (3.11) \end{aligned}$$

For the definition of the matrix norm, appearing in the definition of α, see Appendix A.

First of all, we focus on (3.7). Consider the interval $[kT - T, kT)$. If only saturated measurements are available, we have that $|y(t)| = 1$ for $t \in [kT - T, kT)$. Looking at (3.8), this gives that $\theta_k = 0$. On the other hand, if observations are available in the interval $[kT - T, kT)$, then $\theta_k > 0$ and a different input is applied. Therefore, depending on whether observations are available in the interval $[kT - T, kT)$, a different input is applied in the next interval $[kT, kT + T)$.

Consider the case that no observations are available in the interval $[kT - T, kT)$. Looking at (3.7), the factor α^k lets the size of the input grow exponentially. Especially the usage of the norm of A makes the growth enormous. For a large initial state, this results in a huge input, and hence the output graph is very steep at the point of desaturation (we refer to section 3.3.2 for details). To determine the state from a tiny interval, a very fast observer is required. However, in the continuous-time case, an arbitrary small interval is sufficient for determination of the state from input and output measurements. The structure of (3.10) reveals that D_k is very small if the output is desaturated for a very short amount of time. The inverse of D_k, appearing in the input, is therefore huge.

The inverse of both $B(T)$ and D_k appear in the input. To clarify that those inverses exist, we introduce the *controllability-* and *observability Gramians*

$$W_c(t) = \int_0^t e^{-A\tau} bb^T e^{-A^T\tau} d\tau, \tag{3.12}$$

and

$$W_o(t) = \int_0^t e^{-A^T\tau} c^T c e^{-A\tau} d\tau \tag{3.13}$$

respectively. A well known result is that the controllability/observability Gramian of a controllable/observable LTI system (3.3) is invertible for any $t > 0$ [15]. Since controllability and observability of (3.3) is assumed, both Gramians are invertible. Looking at (3.9), this yields that $B(T)$ is invertible. Furthermore, since D_k is only utilised in the input when $\theta_k > 0$, and $z(t)$ is continuous, the inverse of D_k exists when necessary.

In fact, Kreisselmeier proposes a *nonlinear dead beat controller*. Provided that $\theta_k > 0$ for some k, the state is zero within T time units. To see this, we use the state equation of (3.3) together with (3.6) to write

$$\dot{x}(kT+\tau) = Ax(kT+\tau) - bb^T e^{-A^T\tau} U_k, \qquad \tau \in [0,T]. \tag{3.14}$$

Solving (3.14) explicitly gives

$$\begin{aligned} x(kT+\tau) &= e^{A\tau}x(kT) - \left(\int_0^\tau e^{A(\tau-s)} bb^T e^{-A^T s} ds\right) U_k \\ &= e^{A\tau}x(kT) - B(\tau)U_k, \end{aligned} \tag{3.15}$$

where we used that U_k is independent of τ. An explicit equation for z then becomes

$$z(kT-T+\tau) = ce^{A\tau}x(kT-T) - cB(\tau)U_k, \qquad \tau \in [0,T]. \tag{3.16}$$

If $\theta_k > 0$, the output is observed in the interval $[kT-T, kT)$. Noting that $y(t) = z(t)$ whenever $1-|y(t)| \neq 0$ and multiplying (3.16) on both sides by $[1-|y(kT-T+\tau)|]e^{A^T\tau}c^T$, we obtain

$$\begin{aligned} &[1-|y(kT-T+\tau)|]e^{A^T\tau}c^T c e^{A\tau} x(kT-T) \\ &= [1-|y(kT-T+\tau)|]e^{A^T\tau}c^T \left[y(kT-T+\tau) + cB(\tau)U_k\right]. \end{aligned} \tag{3.17}$$

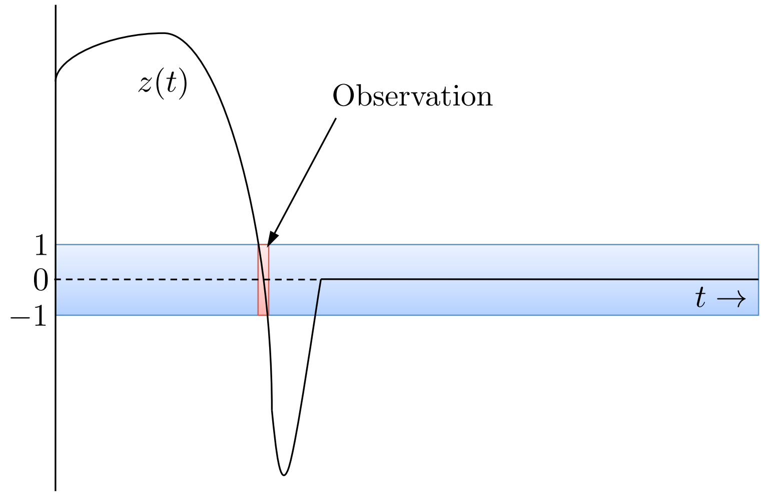


Figure 3.3: Illustration of Kreisselmeier's dead beat controller.

Looking at (3.10) and (3.11), integration of (3.17) gives that $D_k x(kT-T) = \xi_k$. Since $\theta_k > 0$, the invertibility of D_k together with (3.7) results in

$$\begin{aligned} U_k &= B^{-1}(T)e^{AT}[e^{AT}x(kT-T) - B(T)U_{k-1}] \\ &= B^{-1}(T)e^{AT}x(kT). \end{aligned} \tag{3.18}$$

In the last step we used equation (3.15). Substituting the resulting U_k into (3.15) again yields $x(kT+T) = 0$. By induction, $x((k+i)T) = 0$ for all $i \geq 1$. Figure 3.3 depicts how Kreisselmeier's controller works. For the proof of global asymptotic stability of the closed-loop system, we refer to [12].

3.3.2 Implementability Issues

As we already stated briefly, Kreisselmeier's dead beat controller assumes a very fast observation. In continuous-time this is possible, provided that no noise is present in the system, because the state of an observable system can be recovered by continuous measurement on an arbitrary small interval. However, nowadays the majority of industrial controllers is digital, so *discrete-time*. Discretisation of continuous-time systems is necessary to allow for implementation. Furthermore, it is plausible to assume *sampled measurements*. Is the controller (3.6)-(3.11) implementable? Or, how does the controller cope with system- and/or measurement disturbances? Implementability and disturbance rejection are important topics to guarantee applicability of controllers. In this section, attention is given to those questions of interest. First we study implementability of the controller (3.6)-(3.11) in detail. Later, we briefly explain why Kreisselmeier's controller is sensitive to measurement disturbances.

Assume that a measurement is done every S time units. So S is the *sampling period*. This yields an array of measurements

$$y(0), y(S), \ldots, y(kS), \ldots \tag{3.19}$$

Applying Kreisselmeier's algorithm, the first thing to notice is that S must be small enough to guarantee measurements $|y(kS)| < 1, \quad k \in \mathbb{Z}_+$. The choice of a proper sampling period depends explicitly on the system dynamics and the initial state. Namely, the larger the real part of the eigenvalues of A and the initial state $x(0)$ are, the larger the input (3.6) will grow before desaturation of the output. The consequence is that the output desaturates only on a very small time interval.

To clarify the above argumentation, let $z(t)$ be saturated on $[0, \bar{k}T]$. Then $|y(t)| = |y(0)| = 1$ for $t \in [0, \bar{k}T]$. Combining (3.7) with (3.15), we get

$$\begin{aligned} x(kT+T) &= e^{AT}x(kT) - \alpha^k[\alpha h - e^{AT}h]y(kT) \\ &= e^{AT}(x(kT) + \alpha^k hy(0)) - \alpha^{k+1}hy(0), \qquad k \in [0, \bar{k}). \end{aligned} \tag{3.20}$$

This equation can be solved explicitly as

$$x(kT) = e^{kAT}(x(0) + hy(0)) - \alpha^k hy(0), \qquad k \in [1, \bar{k}]. \tag{3.21}$$

Hence,

$$z(kT) = ce^{kAT}(x(0) + hy(0)) - c\alpha^k hy(0), \qquad k \in [1, \bar{k}]. \tag{3.22}$$

If the real part of the eigenvalues of A and/or the initial state are large, it takes a long time for the input to compensate the growth of the first term on the right hand side of (3.22).

To investigate the transition speed from $y = 1$ to $y = -1$ (or vice versa) of an unstable system with large initial state, we look at the difference

$$z(kT+T) - z(kT) \tag{3.23}$$

for large k. Using the relation $z = cx$, and substituting (3.21) in (3.23) gives

$$z(kT+T) - z(kT) = \left[e^{AT} - I\right] e^{kAT}(x(0) + hy(0)) - (\alpha - 1)\alpha^k hy(0). \tag{3.24}$$

From the definition of α, we see that the rightmost term dominates for large k. The larger the initial condition $x(0)$ and/or the eigenvalues of A are, the faster the transition from $y = \pm 1$ to $y = \mp 1$ is.

Given some sampling period S, it is always possible to define an initial state $x(0)$ such that no non-saturated output measurements become available when crossing the non-saturated region for the first time. Hence, the system is not globally asymptotically

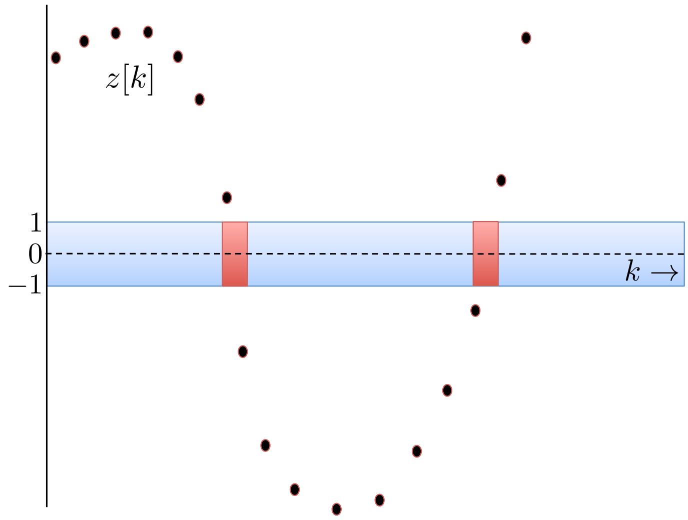


Figure 3.4: Loss of global asymptotic stability when the output is sampled.

stabilisable within T time units after crossing the non-saturated region. Despite of the input being such that $z(t)$ will cross the non-saturated region again and again, we cannot guarantee that measurements will eventually become available. This is depicted in figure 3.4.

To verify the implementability issues, we discretise Kreisselmeier's algorithm and simulate the response of an unstable system in closed-loop with Kreisselmeier's controller. For details on the algorithm, we refer to Appendix E. As a typical example of an unstable system that is both controllable and observable, we take

$$\left\{ \begin{array}{lcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & \sigma(Cx(t)) \end{array} \right. , \qquad t \in \mathbb{R}. \tag{3.25}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{3.26}$$

First of all we select the parameter h and the initial condition as

$$h := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x(0) := \begin{bmatrix} 5 \\ 5 \end{bmatrix}. \tag{3.27}$$

We choose $T = 10^{-3}$, and define a sampling period of $10^{-4}T$. The total simulation time is set to $99T$. The result is plotted in Figure 3.5, and agrees with our expectations. The applied input leads to a transition speed that is too fast to obtain measurements that are not saturated. Consequently, the input grows unbounded and the output diverges.

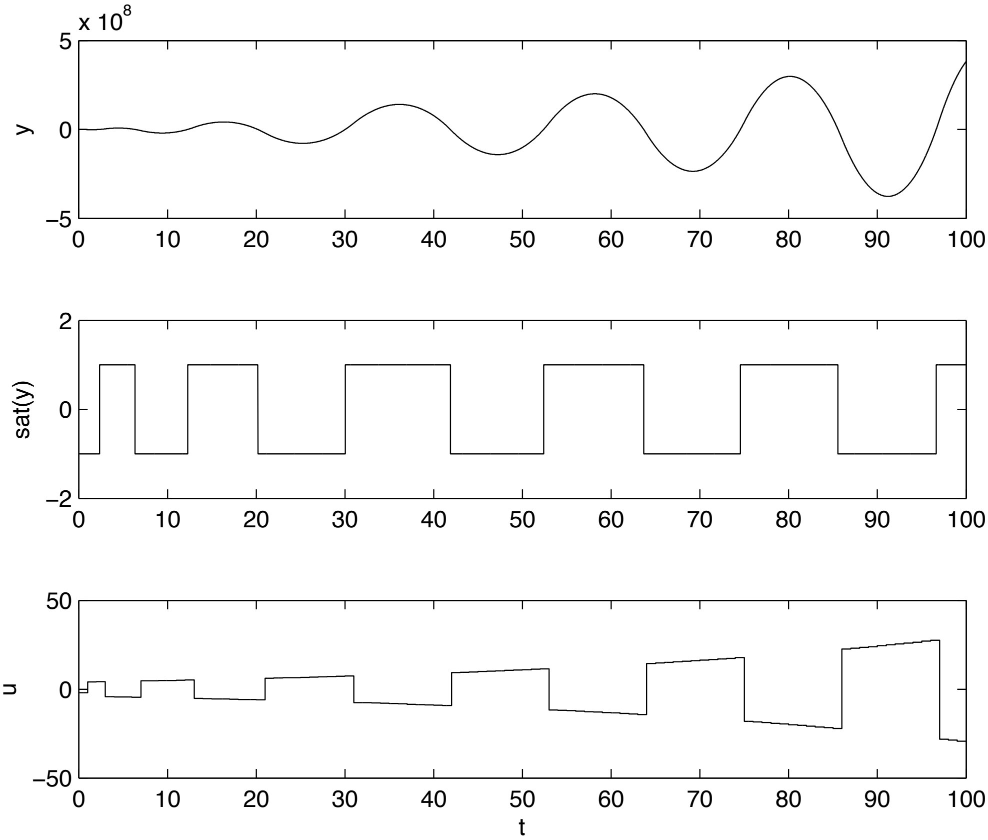


Figure 3.5: Response of the actual output, saturated output and input of the system (3.25) to Kreisselmeier's algorithm with initial conditions (3.27).

To illustrate what can happen when nonsaturated measurements become available, we simulate (3.25) with the following parameters:

$$h := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x(0) := \begin{bmatrix} 0.9 \\ 0 \end{bmatrix}, \tag{3.28}$$

$T = 10^{-2}$, sampling period $10^{-2}T$ and total simulation time $149T$. The first two measurements are not saturated. Despite of that, Figure 3.6 shows diverging behaviour of both the input and the output. This is probably due to numerical errors, because a small numerical error is amplified by the agressive behaviour of the dead beat observer. Setting

$$h := \begin{bmatrix} 100 \\ 0 \end{bmatrix}, \tag{3.29}$$

it even occurs that the entries in matrix D_k become infinitesimal small, such that matrix inversion is no longer possible.

Disturbance Rejection

In practice, small measurement- and system errors are always present in a dynamical system. Hence, disturbance rejection of a closed-loop system is desired. Kreisselmeier's controller is sensitive to measurement disturbances, especially when the initial state is large [4]. To see this, notice that measurement sensitivity is only important when z is not saturated. Assume that the initial state is large, resulting in a fast transition from $y = \pm 1$ to $y = \mp 1$. The entries of matrix D_k (see (3.10)) are very small then, because z is not saturated only on a very small interval. This results in a nearly singular matrix D_k, whose inverse is large. Now look at the control (3.7) for $\theta_k > 0$:

$$U_k = B^{-1}(T)e^{AT}[e^{AT}D_k^{-1}\xi_k - B(T)U_{k-1}], \tag{3.30}$$

and note the product $D_k^{-1}\xi_k$. The factor

$$y(kT - T + \tau) + cB(\tau)U_{k-1} \tag{3.31}$$

in ξ_k (see (3.11)) may contain measurement noise, which is amplified by the large inverse of D_k. Thus, a small measurement error might result in large errors in the applied control. Consequently, the dead-beat control structure is ruined in the presence of measurement errors.

To make a first step towards an implementable and noise rejecting controller, we propose to discretise the continuous-time state space model. This is done in the next section.

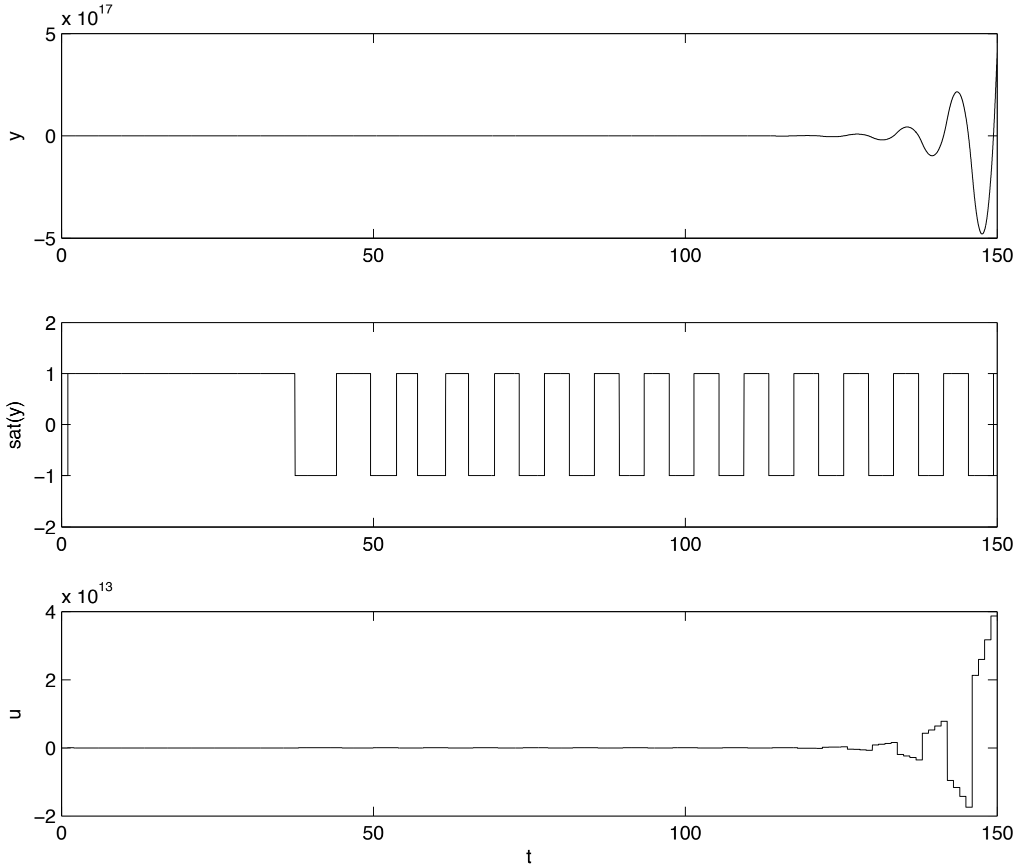


Figure 3.6: Response of the actual output, saturated output and input of the system (3.25) to Kreisselmeier's algorithm with initial conditions (3.28).

3.4 Discretisation of State Space Models

Most industrial measurement devices measure some quantity in a discrete fashion, instead of measuring continuously. Sampled measurements motivate the discretisation of continuous-time state space models. As we have seen in the previous section, a discrete-time LTI system subject to output saturation is more difficult to control than its continuous-time counterpart. Figure 3.4 makes this clear. For an n-dimensional discrete-time LTI system, at least n subsequent non-saturated measurements are required to observe the state dead beat, while an arbitrary small interval is satisfactory for a continuous-time LTI system.

There are many ways to cope with discrete-time input/output in a dynamical system which is continuous in nature. Despite of some dynamical system being continuous, we decide to look at its behaviour only at discrete points in time. Obviously, this gives us less information about the system. However, under certain conditions important properties of a continuous-time system are retained after discretisation. Furthermore, the assumption of a piecewise constant input is quite a natural one, and can conveniently be incorporated in the structure of a discrete-time system. An advantage of discrete-time models is that they allow for *dead beat* observation/control using a linear observer/-controller. For a controllable/observable discrete-time state space model, a dead beat controller/observer design can be done, such that the desired reference is reached in finite time. The disadvantage is that dead beat controllers and observers are sensitive to disturbances. This is an immediate consequence of the fact that dead beat controllers and observers are aggressive, in the sense that the control- and observer gains are large.

For the derivation of a discretised model, we use [1]. Consider the continuous-time state space representation

$$\Sigma_c : \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ z(t) & = & Cx(t) \end{array} \right. , \qquad t \in \mathbb{R}, \tag{3.32}$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$. We can discretise Σ_c, and look at its behaviour only at discrete points in time. For some sampling period S, we define $z[k] := z(kS)$, such that the output equation becomes

$$z[k] = Cx[k]. \tag{3.33}$$

The smaller the sampling period, the more information the samples give us about the continuous-time system. Assuming a constant input during each sampling interval,

$$u(kS + \tau) = u[k], \qquad \tau \in [0, S), \tag{3.34}$$

and equation (2.10), we obtain

$$x(kS + S) = e^{AS}x(kS) + \left(\int_0^S e^{A(S-\tau)} B d\tau \right) u(kS). \tag{3.35}$$

This can conveniently be expressed as

$$x[k+1] = A_d x[k] + B_d u[k], \tag{3.36}$$

where

$$A_d = e^{AS} \quad \text{and} \quad B_d = \int_0^S e^{A(S-\tau)} B d\tau. \tag{3.37}$$

Note that A_d and B_d explicitly depend on the sampling period S. The discretised model becomes

$$\Sigma_d : \begin{cases} x[k+1] &= A_d x[k] + B_d u[k] \\ z[k] &= Cx[k] \end{cases}, \quad k \in \mathbb{Z}. \tag{3.38}$$

An interesting question that arises is whether controllability and observability are preserved after discretisation of the model Σ_c. Intuitively, the sampling frequency plays a major role in this. We have to keep in mind that the underlying dynamics are continuous, and that discretisation adds more restrictions to the input and the output. Obviously, the smaller S is, the better the discretised model (3.38) approximates the continuous-time dynamics Σ_c. From this fact, it is quite intuitive that controllability and observability are only retained under some extra conditions on the dynamics. Those conditions are derived by Kalman, Ho and Narendra [9].

Theorem 8 (Kalman, Ho, Narendra [9])**.** *Suppose the system Σ_c from* (3.32) *is controllable. Restrict $u(t)$ to be given by* (3.34)*, i.e., piecewise constant on intervals of constant length S. Let $\lambda_i(A)$ denote the i^{th} eigenvalue of matrix A. A sufficient condition for controllability is that*

$$Im[\lambda_i(A) - \lambda_j(A)] \neq \frac{2\pi q}{S} \text{ whenever } Re[\lambda_i(A) - \lambda_j(A)] = 0, \quad q = \pm 1, \pm 2, \ldots \tag{3.39}$$

If Σ_c is single input, i.e. $u \in \mathbb{R}$, this condition is necessary as well.

Consequently, if Σ_c is controllable and condition (3.39) holds, then Σ_d is controllable. When discretising a state space model, controllability can always be preserved by choosing S small enough. However, depending on the matrix A, a large sampling period might be satisfactory. A similar result holds for *observability*: If condition (3.39) holds and the pair (A, C) is observable, then the pair (A_d, C) is observable. For an elegant and intuitive proof, we refer to [1].

We proceed with an example.

Example 6 (Discretisation of a State Space Model)**.** *Consider Σ_c with*

$$A := \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C := \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{3.40}$$

This system is both controllable and observable. We want to discretise this continuous-time system with sampling period S. We define the corresponding sampling frequency as $\omega_S := 2\pi/S$. Calculation of e^{AS} can be done by diagonalising A, and gives

$$A_d = \begin{bmatrix} \cos(\omega S) & \sin(\omega S) \\ -\sin(\omega S) & \cos(\omega S) \end{bmatrix}. \tag{3.41}$$

Using (3.37)*, for B_d we obtain*

$$\begin{aligned} B_d &= \int_0^S \begin{bmatrix} \cos(\omega(S-\tau)) & \sin(\omega(S-\tau)) \\ -\sin(\omega(S-\tau)) & \cos(\omega(S-\tau)) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \\ &= \int_0^S \begin{bmatrix} \sin(\omega(S-\tau)) \\ \cos(\omega(S-\tau)) \end{bmatrix} d\tau \\ &= \begin{bmatrix} (1-\cos(\omega S))/\omega \\ \sin(\omega S)/\omega \end{bmatrix}. \end{aligned} \tag{3.42}$$

The discretisation is done now. We are interested in a sampling period S which assures that the discretised system is controllable and observable. To this extent, we calculate the eigenvalues of A as $\lambda_{1,2} = \pm i\omega$. Looking at Theorem 8, we get that

$$Re[\lambda_1(A) - \lambda_2(A)] = 0, \tag{3.43}$$

so we must have

$$Im[\lambda_1(A) - \lambda_2(A)] = Im[i\omega - (-i\omega)] = 2\omega \neq \frac{2\pi q}{S}, \quad q = \pm 1, \pm 2, \ldots \tag{3.44}$$

Expressing S in terms of ω_S makes clear that

$$2\omega \neq q\omega_S, \quad q = \pm 1, \pm 2, \ldots \tag{3.45}$$

must hold. Choosing $\omega_S > 2\omega$ is a possible solution. Controllability and observability are preserved in that case. Note however, that choosing the sampling frequency as an irrational multiple of ω also works.

In summary, LTI systems subject to output saturation are introduced, and the main difficulties are discussed. Kreisselmeier's solution is described, along with its shortcomings for LTI systems with discrete-time measurements. A procedure for discretisation is given, serving as a first step towards an implementable and noise rejecting controller. Controllability and observability of a continuous-time system are preserved after discretisation, provided that the Kalman, Ho, Narendra condition holds. In the next chapter, we propose a stabilising controller for the class of stable LTI systems.

Chapter 4

Stabilisation of Stable LTI Systems with Saturated Output

An important class of systems are the so-called *neutrally stable systems*. Those systems possess the property that their trajectories neither converge to some fixed equilibrium point, nor diverge. For example, consider an industrial robot arm which is accidentally hit. The position and orientation of the robot arm change, and will remain changed if no action is taken. To avoid errors, automatic repositioning of the robot arm is desired. This is an example of a neutrally stable system that needs to be stabilised. The basic idea is illustrated in Figure 4.1, which displays a cone on a flat surface. The left cone will go back to its initial equilibrium point after a small perturbation, implying asymptotic stability. A small perturbation of the middle cone will let it roll, and it may come to rest in a different position and orientation. An equilibrium point with this property is called neutrally stable. Lastly, the equilibrium point of the right cone is unstable, since a small disturbance lets the cone diverge from the equilibrium point. (Figure 4.1 is copied from [2].)

For both continuous- and discrete-time neutrally stable MIMO LTI systems subject to output saturation, we show how to find a stabilising controller. Continuous-time systems are our starting point, and give us the possibility to introduce some important concepts. *Lyapunov functions* and *invariant sets* are introduced, which we need to prove stability. This is done in Section 4.1. In Section 4.2 we expand the theory to the class of discrete-time LTI systems. The same technique as for the continuous-time case is used. However, a scaling factor is necessary to find Lyapunov functions and prove stability. For both continuous- and discrete-time systems, *global asymptotic stability* of the resulting closed-loop system is proved. Lastly, we prove that the class of systems stabilised by the proposed controllers can be extended to the class of *stable LTI systems*.

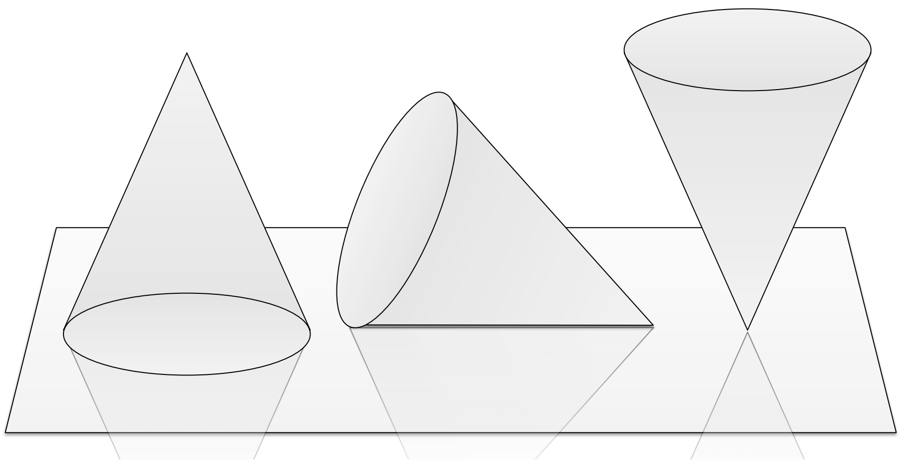


Figure 4.1: Asymptotic stability (left), neutral stability (centre), and instability (right) of a cone on a flat surface.

4.1 Solution for Deterministic Continuous-Time Systems

In this section we propose a solution for the stabilisation of the class of neutrally stable continuous-time LTI system with saturated output measurements. First the system is defined, and an introduction to the solution concept is given. Then an observer design to provide a state estimate is discussed. A feedback utilising this estimate is proposed, such that the closed-loop system is globally asymptotically stable. Finally, the class of stabilised systems is extended. It is shown that *stable* LTI systems subject to output saturation are stabilised by making use of the controller design for neutrally stable systems.

4.1.1 System Definition

Consider the dynamics

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & \boldsymbol{\sigma}(Cx(t)) \end{array} \right. , \quad t \in \mathbb{R}, \tag{4.1}$$

with $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. For the definition of the saturation function $\boldsymbol{\sigma}(.)$, we refer to Definition 13 on page 23. We assume that (4.1) is *neutrally stable* (for $u(t) = 0$). If no input is applied, the trajectories of such a system neither grow nor decay, but have a constant amplitude. All states are thus a mixture of constant and oscillating signals. This implies that every eigenvalue λ of A is on the imaginary axis. In other words, the sum of an eigenvalue and its complex conjugate is zero:

$$\lambda + \bar{\lambda} = 0. \tag{4.2}$$

It can be shown that there exists a basis such that

$$A + A^* = 0, \quad A \in \mathbb{C}^{n\times n}. \tag{4.3}$$

For every complex skew-symmetric matrix A, there exists a similarity transformation which reduces A to a real-valued skew-symmetric matrix [18]. We assume that A is in this form, thus

$$A + A^T = 0, \quad A \in \mathbb{R}^{n\times n}. \tag{4.4}$$

Stability properties are independent of the basis choice [5]. Therefore, we can assume without loss of generality that the basis choice is such that (4.4) holds.

Example 7. *The autonomous system*

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{bmatrix} x \tag{4.5}$$

is an example of a neutrally stable system. It is clear that $A = -A^T$. The first element of the state remains constant. The second and third element of the state oscillate with a constant amplitude.

We assume that (A, B) is controllable and (A, C) is observable. Notice that the state can only be observed if $|(Cx)_i| < 1$ for at least one i. Due to controllability, there always exists an input such that $|(Cx)_i| < 1$ is achieved for all i, implying observability of (4.1). The problem is however, that it is often unknown how to choose an input to achieve this.

4.1.2 Introduction of the Solution Concept

To introduce our strategy, we assume that the initial condition $x(0)$ is known. Then, given the input on the interval $[0, t)$, the state is known for all $t \geq 0$. This means that the output does not play any role, since it is not needed to obtain information about the state of the system. Therefore, the state equation of (4.1) is all we have to consider, which is linear and time-invariant. We propose a state feedback of the form

$$u = Fx \tag{4.6}$$

to stabilise the system (4.1). $F \in \mathbb{R}^{m\times n}$ is a parameter matrix. With the control law (4.6), the state equation reduces to the autonomous system

$$\dot{x} = (A + BF)x. \tag{4.7}$$

Notice that $x = 0$ is an equilibrium point. As described in Section 2.2.4, the eigenvalues of $A + BF$ determine the stability of the equilibrium point $x = 0$. Because of the controllability assumption, F can always be chosen such that the matrix $A + BF$ is *Hurwitz*.

Even stronger, F can be chosen such that $A + BF$ has any desired eigenvalues. Usually, the determination of a desired F requires quite some work. We will show that for the class of neutrally stable systems, there exists an F that can be directly derived from the dynamics, such that $x = 0$ is a *globally asymptotically stable* equilibrium point.

We need Lyapunov's stability theorem to proceed.

Theorem 9 (Lyapunov Stability Continuous-Time Systems [10])**.** *Let $x = 0$ be an equilibrium point for the autonomous system*

$$\dot{x} = f(x), \tag{4.8}$$

and $D \subset \mathbb{R}^n$ be a domain (open set) containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function, such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D \setminus \{0\}, \tag{4.9}$$

$$\dot{V}(x) \leq 0 \text{ in } D. \tag{4.10}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D \setminus \{0\}, \tag{4.11}$$

then $x = 0$ is asymptotically stable.

We say that a function $V(x)$ is a *candidate* Lyapunov function if (4.9) holds. A candidate Lyapunov function is always the starting point when trying to prove Lyapunov stability of an equilibrium point. When a candidate is chosen, the time derivative is computed by using the state equation, and conditions (4.10) and (4.11) can be checked. Unfortunately, there is no straightforward method for the construction of Lyapunov functions. Finding a good Lyapunov function is a matter of experience together with trial and error. Figure 4.2 intuitively explains Theorem 9. It is clear from the figure that $V(x) \geq 0$, and $V(x) = 0$ only for $x = 0$. If condition (4.10) holds, V does not grow, so neither will x. The arrows indicate that the state converges to zero if $\dot{V}(x) < 0$.

For the system (4.7), we define the candidate Lyapunov function

$$V_1(x) = x^T x. \tag{4.12}$$

This function is strictly positive for all $x \neq 0$, and $V_1(0) = 0$, so (4.9) holds. Taking the

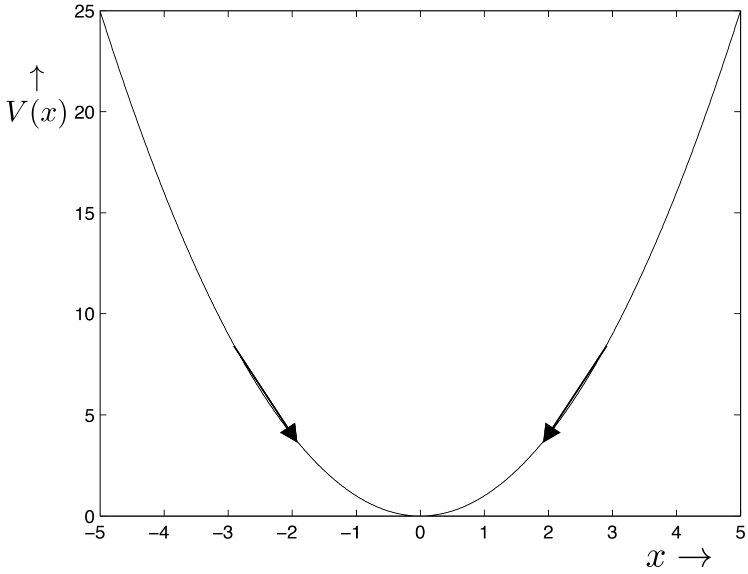


Figure 4.2: A typical Lyapunov function.

time derivative, we get

$$\begin{aligned}\dot{V}_1(x) &= x^T(A+BF)x + x^T(A+BF)^T x \\ &= x^T(A+A^T)x + x^T BFx + x^T F^T B^T x \\ &= 2x^T BFx. \end{aligned} \tag{4.13}$$

In the last step we used that $A + A^T = 0$. Selecting

$$F = -B^T, \tag{4.14}$$

we have

$$\dot{V}_1(x) = -2x^T BB^T x \leq 0. \tag{4.15}$$

$\dot{V}_1$ is negative semidefinite, implying stability of the autonomous system

$$\dot{x} = (A - BB^T)x. \tag{4.16}$$

However, we cannot say anything about *asymptotic stability* yet. This is where *LaSalle's Invariance Principle* comes in. This principle uses the notion of *positively invariant* sets, which we will introduce now. A set M is called *positively invariant* with respect to some system if

$$x(0) \in M \Rightarrow x(t) \in M, \qquad \forall t \geq 0. \tag{4.17}$$

In words, some set being positively invariant means that starting in that set, the state will always remain in that particular set. We state LaSalle's theorem (see Appendix C):

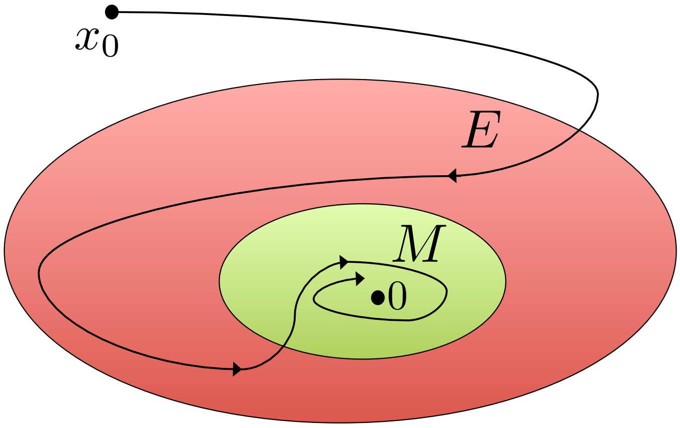


Figure 4.3: LaSalle's Invariance Principle.

Theorem 10 (LaSalle's Theorem Continuous-Time Systems)**.** *If there exists a continuously differentiable function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *satisfying*

1. $V(x) \geq 0$ *for all* $x \in \mathbb{R}^n$ *and* $V(0) = 0$*,*
2. $\dot{V}(x) \leq 0$ *for all* $x \in \mathbb{R}^n$*,*
3. *Let* M *be the largest positively invariant set contained in*

$$E = \{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\},$$

   and let $0 \in M$*.* 0 *is asymptotically stable for the dynamics restricted to* M*.*
4. *All solutions of the system* $\dot{x} = f(x)$ *are bounded,*

then the origin is globally asymptotically stable*.*

Figure 4.3 illustrates LaSalle's Invariance Principle, which is now intuitively explained. Starting from an initial state x_0, the state of a stable system $\dot{x} = f(x)$ approaches the set where $\dot{V}(x) = 0$, which is denoted by E. Given that the state is in E, at some point in time the state at least *converges* to the largest positively invariant set in E, denoted by M. The power of LaSalle's theorem is that global asymptotic stability can be determined by considering only the points in the limit set M. Once M is reached, the state will always remain in M by definition. If $x = 0$ is an asymptotically stable equilibrium point inside M, obviously the state converges to zero.

We are now ready to prove global asymptotic stability of the equilibrium point $x = 0$, for the autonomous system (4.16).

Theorem 11. *Assume that (A, B) is controllable and $A + A^T = 0$. Then $x = 0$ is a globally asymptotically stable equilibrium point of the autonomous system*

$$\dot{x} = (A - BB^T)x. \tag{4.18}$$

Proof. We define $V(x) = x^T x$, and check that all conditions of Theorem 10 hold. For the first condition, observe that $V(x) \geq 0$ and $V(0) = 0$. The second condition is clear from

$$\dot{V}(x) = -2x^T BB^T x \leq 0. \tag{4.19}$$

Moreover, V is continuously differentiable. To prove the third condition, we define

$$E = \{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\} = \{x \in \mathbb{R}^n \mid B^T x = 0\}. \tag{4.20}$$

We denote the largest positive invariant set contained in E by M. Using the definition of E, the dynamics restricted to M reduce to

$$\dot{x}(t) = Ax(t), \qquad \forall t \geq 0. \tag{4.21}$$

This yields the explicit expression

$$x(t) = e^{At}x(0), \qquad \forall t \geq 0. \tag{4.22}$$

From the definition of E, it is obvious that

$$B^T e^{At}x(0) = 0, \qquad \forall t \geq 0. \tag{4.23}$$

Taking derivatives with respect to time on both sides of (4.23), we obtain

$$\begin{bmatrix} B^T e^{At} \\ B^T A e^{At} \\ \vdots \\ B^T A^{n-1} e^{At} \end{bmatrix} x(0) = 0, \qquad \forall t \geq 0. \tag{4.24}$$

Since this equation holds for $t = 0$, we get

$$\begin{bmatrix} B^T \\ B^T A \\ \vdots \\ B^T A^{n-1} \end{bmatrix} x(0) = 0. \tag{4.25}$$

Using the relation $A + A^T = 0$ gives

$$\begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^{n-1})^T \end{bmatrix} x(0) = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}^T x(0) = 0. \tag{4.26}$$

Controllability of (A, B) implies full rank of the controllability matrix, and $x(0) = 0$ follows. Therefore, $x = 0$ is an asymptotically stable equilibrium point for the dynamics restricted to M. Stability of the autonomous system (4.18) implies that all solutions are bounded, implying the last condition of Theorem 10. Hence the origin is globally asymptotically stable. □

4.1.3 State Observation

Until now, we proved that a stabilising controller exists, provided that the initial state is known and the system (4.1) is controllable. Usually, the initial state of a system is unknown. In that case a systematic way to obtain information about the state is desired. As we have seen in section 2.2.5, a commonly used technique is to define a dynamical system for the estimate of the state, which is called a *state observer*. Denoting the estimate of the state at time t by $\hat{x}(t)$, the objective of an observer is to make the error

$$e(t) := x(t) - \hat{x}(t) \tag{4.27}$$

smaller as time progresses. In this section an observer design is proposed for the system (4.1). Thus, output saturation is taken into account. We propose to use a slight modification of a linear observer, which provides us with an estimate of the state.

We propose an observer of the form

$$\dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{\text{copy plant}} + K \underbrace{(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}))}_{\text{measurement error}}, \tag{4.28}$$

incorporating the vector saturation function in the dynamics. $K \in \mathbb{R}^{n \times p}$ is the so-called *observer gain*, and is to be determined. The dynamics (4.28) are governed by a copy of the plant and an innovations term. The copy of the plant lets the state estimate undergo the same evolution as the real state. This is intuitive, since it serves as some sort of tracking when the estimate is good enough. The innovations term corrects for the observation error, improving the estimate if K is chosen properly.

Compare (4.28) with (2.41) and recall (2.42). If both Cx and $C\hat{x}$ are not saturated, the observer dynamics are *linear*, hence the error dynamics are given by

$$\dot{e} = (A - KC)e. \tag{4.29}$$

Selecting $K = C^T$, it might come as no surprise that $e = 0$ is a globally asymptotically stable equilibrium point of (4.29), provided that (A, C) is an observable pair. We formalise this in a theorem.

Theorem 12. *Assume that (A, C) is observable and $A + A^T = 0$. Then $e = 0$ is a globally asymptotically stable equilibrium point of the autonomous system*

$$\dot{e} = (A - C^T C)e. \tag{4.30}$$

Proof. The theorem can be proved using the duality

$$(A, C) \text{ observable } \Leftrightarrow (A^T, C^T) \text{ controllable.} \tag{4.31}$$

We apply theorem 11 to the controllable pair (A^T, C^T). Provided that $A + A^T = 0$ and (A, C) observable, this gives that the matrix $A^T - C^T C$ is *Hurwitz.* Since the stability properties of a matrix do not change by taking the transpose, we get that

$$(A^T - C^T C)^T = (A - C^T C) \tag{4.32}$$

is also Hurwitz. Hence, the system

$$\dot{e} = (A - C^T C)e \tag{4.33}$$

is globally asymptotically stable. □

Elaborating on this argument, we select $K = C^T$ for the observer (4.28):

$$\dot{\hat{x}} = A\hat{x} + Bu + C^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})). \tag{4.34}$$

It is clear that this observer behaves as desired, as long as Cx and $C\hat{x}$ are not saturated. Now we discuss the behaviour of (4.34) when Cx and/or $C\hat{x}$ are saturated, which is best explained by looking at each element of $\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})$ independently. The following two cases are distinguished:

- $(Cx)_i$ and $(C\hat{x})_i$ are saturated and have the *same sign.* In this case we have that $(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}))_i = 0$, so the i^{th} output component has no effect on the correction. This is intuitive, because it is unclear how the i^{th} component should correct for the estimate. If all state elements are saturated with the same sign, the observer actually turns into a *predictor*, letting the state estimate undergo the same evolution as the state.
- At least one of the terms $(Cx)_i$ and $(C\hat{x})_i$ is saturated, and $(Cx)_i \neq (C\hat{x})_i$. The beauty of the observer (4.34) is that the i^{th} output component causes a correction now. This is intuitive, since we know in which direction to steer the estimate. We have to check whether a correction is done in the right direction. This is the case if

$$\text{sgn}\{(Ce)_i\} = \text{sgn}\{\sigma((Cx)_i) - \sigma((C\hat{x})_i)\}. \tag{4.35}$$

  Without loss of generality we assume that $(Cx)_i \geq 1$ and $(C\hat{x})_i < 1$. Then $(Cx)_i - (C\hat{x})_i > 0$. Furthermore, $\sigma((Cx)_i) - \sigma((C\hat{x})_i) > 0$. Hence, (4.35) is true.

As we verified, the sign of the measurement error term in the observer is correct in all possible situations. We are interested in the behaviour of the observation error corresponding to the observer (4.34). Combining the state equation of (4.1) with (4.34), the error dynamics

$$\dot{e} = Ae - C^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})) \tag{4.36}$$

are derived. Stability of the error dynamics is desired. To investigate stability, we define the candidate Lyapunov function

$$V_2(e) = e^T e. \tag{4.37}$$

This function is strictly positive for all $e \neq 0$ and $V_2(0) = 0$. Taking the time derivative of V_2 we obtain

$$\begin{aligned} \dot{V}_2(e) &= e^T[Ae - C^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}))] + [Ae - C^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}))]^T e \\ &= e^T(A + A^T)e - e^T C^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})) - (\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}))^T Ce \\ &= -2e^T C^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})) \\ &= -2\sum_{i=1}^{p}(Ce)_i(\sigma(Cx) - \sigma(C\hat{x}))_i \\ &\leq 0 \end{aligned} \tag{4.38}$$

To see why the last step follows, note that either $(\sigma(Cx) - \sigma(C\hat{x}))_i = 0$ or (4.35) holds. V_2 is continuously differentiable, so the dynamics (4.36) are *stable* according to theorem 9. Hence, the error can not increase with time. The error does not converge to zero asymptotically. To explain this, note that the error dynamics become

$$\dot{e} = Ae \tag{4.39}$$

if $\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}) = 0$. Neutral stability implies that (4.39) is just stable.

Figure 4.4 visualises the observer design. The saturated measurements together with the input are used to provide a state estimate. To attain global asymptotic stability of the equilibrium point $e = 0$, an *estimated state feedback* desaturating the output is proposed in the next section. This results in a closed-loop system where the observer (4.34) is part of a controller. As we will show, the designed controller globally asymptotically stabilises the system.

4.1.4 Global Asymptotic Stability of the Closed-Loop System

In this section, the feedback law and the state observer derived in the previous sections are combined. Given some initial state and initial state estimate, the observer updates the state estimate. A feedback utilising the estimated state generates an input for the

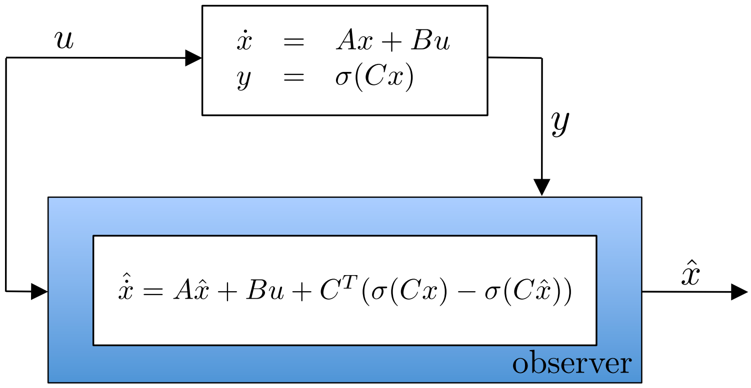


Figure 4.4: Visualisation of the state observer (4.34)

plant. See Figure 4.5. After defining the closed-loop system, an intuitive explanation to our approach is given. We conclude with a proof of global asymptotic stability for the closed-loop system.

For the system (4.1), we restate the observer

$$\dot{\hat{x}} = A\hat{x} + Bu + C^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})), \tag{4.40}$$

and apply the feedback control

$$u = -B^T\hat{x}. \tag{4.41}$$

Then the closed-loop dynamics are described by

$$\dot{e} = Ae - C^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})), \tag{4.42}$$

$$\dot{\hat{x}} = (A - BB^T)\hat{x} + C^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})), \tag{4.43}$$

$$e = x - \hat{x}. \tag{4.44}$$

The idea is to define a Lyapunov function for the closed-loop dynamics, and use LaSalle's theorem to prove global asymptotic stability.

First an intuitive explanation to our approach is given, by looking at the error dynamics (4.42) and the dynamics for the state estimate (4.43) *independently*. As we have shown in Section 4.1.3, the error dynamics are *stable*. To see whether (4.43) is stable, we propose the candidate Lyapunov function $V_3(\hat{x}) = \hat{x}^T\hat{x}$. With some algebra, for the time derivative we obtain

$$\dot{V}_3(\hat{x}) = -2\hat{x}^T BB^T\hat{x} + 2\hat{x}^T C^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})). \tag{4.45}$$

The time derivative of V_3 is not negative semidefinite, thus the *separation principle* fails here. To tackle this problem, a different approach is required to prove stability of

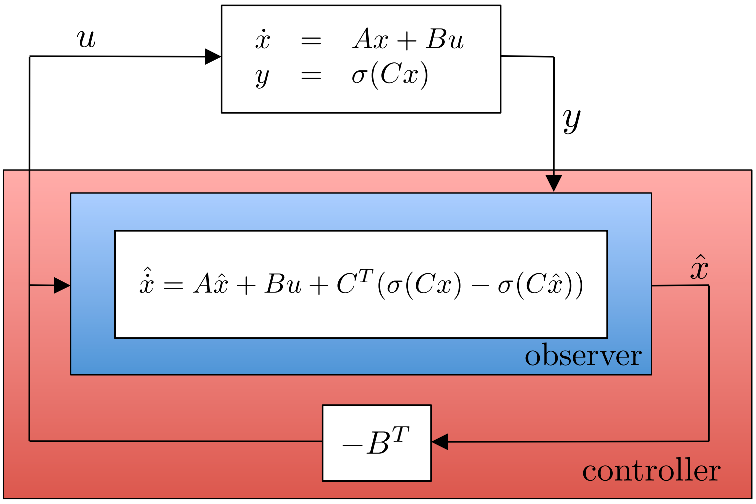


Figure 4.5: The closed-loop system (4.42)-(4.44)

the closed-loop system. Despite of that, global asymptotic stability can be intuitively explained. To see what happens, consider the case that $\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}) = 0$ and $e \neq 0$. Looking at (4.42), we see that the error remains constant in magnitude then. However, looking at (4.43), the cross terms vanish. Hence the state estimate converges to zero monotonically, due to Theorem 11. This leads to desaturation of the elements of $C\hat{x}$ at some point in time. Two possible situations may occur:

- $\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})$ becomes nonzero. This lets the norm of the error decrease monotonically because $\dot{V}_2$ is strictly negative. This gives that $\boldsymbol{\sigma}(C\hat{x}) \to \boldsymbol{\sigma}(Cx)$. Looking at (4.43), the state estimate converges to zero. (4.44) yields that $x(t) \to 0$ then.
- $\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})$ remains zero. Desaturation of the elements of $C\hat{x}$ lets the elements of Cx desaturate. Thus, the observer dynamics become linear, and the error dynamics locally behave according to (4.30). Theorem 12 guarantees that the norm of the error decreases monotonically, such that $\boldsymbol{\sigma}(C\hat{x}) \to \boldsymbol{\sigma}(Cx)$. The dynamics (4.43) let the state estimate converge to zero. Consequently, $x(t) \to 0$.

Intuitively, it is clear that this closed loop system is globally asymptotically stable. This turns out to be correct. We state a theorem:

Theorem 13. *Let* (4.1) *be a neutrally stable system, i.e. there is a basis choice such that* $A + A^T = 0$. *Furthermore, assume that* (A, B) *is controllable and* (A, C) *is observable. Then* $(e, \hat{x}) = (0, 0)$ *is a globally asymptotically stable equilibrium point of the closed-loop system* (4.42)-(4.44).

Proof. To prove that our intuition is correct, we define the function

$$V_4(e,\hat{x}) := e^T e \tag{4.46}$$

for the closed loop dynamics. When $(e,\hat{x}) = (0,0)$, then $x = 0$ automatically follows. For V_4 we have

$$V_4(0,0) = 0 \text{ and } V_4(e,\hat{x}) \geq 0 \text{ for } (e,\hat{x}) \neq (0,0). \tag{4.47}$$

Notice that $V_4(e,\hat{x})$ is positive *semidefinite*, so V_4 is not a candidate Lyapunov function. However, V_4 is continuously differentiable, and according to the first condition of Theorem 10, $V_4 \geq 0$ is sufficient. Since the time derivative of V_4 is negative semidefinite, the second condition is also fulfilled. It remains to prove that the third and the fourth condition of Theorem 10 hold.

To prove the third condition, we define E as the set of all points where $\dot{V}_4 = 0$:

$$E = \{(e,\hat{x}) \mid \boldsymbol{\sigma}(Cx) = \boldsymbol{\sigma}(C\hat{x}),\ e \in \mathbb{R}^n,\ \hat{x} \in \mathbb{R}^n\}. \tag{4.48}$$

For the closed-loop dynamics restricted to E we obtain

$$\begin{aligned} \dot{e} &= Ae && (4.49)\\ \dot{\hat{x}} &= (A - BB^T)\hat{x} && (4.50)\\ e &= x - \hat{x}. && (4.51) \end{aligned}$$

The state estimate converges to zero by Theorem 11. Then Cx and $C\hat{x}$ simultaneously desaturate, giving that there exists a T such that

$$Ce(t) = 0 \qquad \forall t > T. \tag{4.52}$$

For the error dynamics we get

$$\dot{e}(t) = Ae(t) = (A - C^T C)e(t) \qquad \forall t > T. \tag{4.53}$$

$A - C^T C$ is Hurwitz by Theorem 12, hence the error converges to zero. Consequently

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e(t) + \lim_{t\to\infty} \hat{x}(t) = 0, \tag{4.54}$$

so $x = 0$ is an asymptotically stable equilibrium point for the dynamics restricted to E.

Now we prove the fourth condition. The error remains bounded because it is stable. To see why the state estimate remains bounded, look at (4.43). Using that $A - BB^T$ is Hurwitz together with the fact that $C^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}))$ is bounded, we have that $\hat{x}$ remains bounded. ☐

Rejection of Measurement Noise

An important feature of our observer design (4.34) is that it provides a certain immunity to measurement noise. This is explained briefly now. Notice that measurement noise can only influence the saturated measurement when the output is small. Therefore, the observer behaves as a linear observer when measurement noise affects the closed-loop dynamics. Since only integrations need to be performed in constructing the state estimate, measurement noise is averaged over time. Thus, to some extent, the observer is immune to noise.

4.1.5 Extension to Stabilisation of Stable LTI Systems

A controller design for stabilisation of continuous-time *neutrally stable* LTI systems is finished now. For any initial state and any initial state estimate, global asymptotic stability of the desired equilibrium point is attained. The class of systems stabilised by the controller described in Section 4.1.4 can actually be extended. Note that the class of neutrally stable systems (i.e. $A + A^T = 0$ in some basis) does not include systems with some states converging to zero asymptotically. In this section we show that the class of *stable* LTI systems subject to output saturation is stabilised by the controller described in Section 4.1.4. We prove that for a stable matrix A there exists a basis choice such that $A + A^T \leq 0$. Recalling the Lyapunov functions (4.13) and (4.38), it is seen that their derivatives with respect to time remain negative semidefinite then. Hence, stability properties of the closed-loop system (4.42)-(4.44) do not change.

Every square matrix can be brought into *Jordan normal form* by a nonsingular transformation matrix S, see Appendix B. Therefore

$$A = SJS^{-1}, \tag{4.55}$$

where J is a so-called Jordan matrix. Since every Jordan block only contains one eigenvalue (up to multiplicity), the A matrix of a *stable* system can be split up in two independent parts:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \tag{4.56}$$

where A_1 is *Hurwitz* and A_2 is *neutrally stable*. All eigenvalues of A_1 have a negative real part, and all eigenvalues of A_2 lie on the imaginary axis. It is explicitly noted that A_1 is allowed to have Jordan blocks of dimension larger than 1×1.

We use the following fact from [6]: If A_1 is Hurwitz, there exists a $P = P^T > 0$ such that the Lyapunov inequality

$$A_1^T P + PA_1 < 0 \tag{4.57}$$

holds. A positive definite symmetric matrix has a unique positive definite symmetric square root [3]. Therefore, P has a unique square root $P^{1/2}$. Since a positive definite matrix is invertible, $P^{-1/2}$ exists. Using this fact and elaborating on equation (4.57), we get

$$\begin{aligned} P^{-1/2}A_1^T P P^{-1/2} + P^{-1/2} P A_1 P^{-1/2} &= P^{-1/2}A_1^T P^{1/2} + P^{1/2}A_1 P^{-1/2} \\ &= (P^{1/2}A_1 P^{-1/2})^T + P^{1/2}A_1 P^{-1/2} \\ &= \tilde{A}_1 + \tilde{A}_1^T \\ &< 0. \end{aligned}$$

In the last step we used that $P^{1/2}$ and $P^{-1/2}$ are symmetric. This yields the similarity transformation

$$\begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} P^{-1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} P^{1/2} & 0 \\ 0 & I \end{bmatrix}, \tag{4.58}$$

which proves that there exists a basis such that $A + A^T \leq 0$.

In summary, a controller design for stable continuous-time LTI systems subject to output saturation is done. This controller globally asymptotically stabilises the system. The elegance of the controller is that system parameters are used in the gains of the observer and the estimated state feedback. A similar theory for discrete-time LTI systems is desired, to get rid of implementability issues. The next section describes how a controller design for a stable discrete-time LTI system subject to output saturation can be done.

4.2 Solution for Deterministic Discrete-Time Systems

Stabilisation of discrete-time neutrally stable LTI systems with saturated output is considered in this section. We convert the theory for continuous-time systems to discrete-time systems. The demand for implementable controllers makes this an important expansion. A similar technique as in the continuous-time case is exploited. After defining the system, the definition of *discrete-time Lyapunov functions* is given. We proceed with an introductory example to derive a feedback law, by assuming that the initial state is given. Then an observer is designed and placed in closed-loop with a linear feedback controller. For construction of both the feedback law and the observer, a *scaling factor* must be included to attain stability. Global asymptotic stability of the closed-loop system is proved. The section concludes with an extension of the results to the class of *stable* discrete-time LTI systems subject to output saturation.

4.2.1 System Definition

The system is assumed to behave according to

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k \\ y_k &= \boldsymbol{\sigma}(Cx_k) \end{cases}, \qquad k \in \mathbb{Z}. \tag{4.59}$$

The input $u \in \mathbb{R}^m$, state $x \in \mathbb{R}^n$ and output $y \in \mathbb{R}^p$ depend on the discrete time $k \in \mathbb{Z}$. For ease of notation, we use *subscripts* instead of square brackets to indicate the dependence on discrete time k. The saturation function $\boldsymbol{\sigma}(.)$ is defined as in Definition 13 on page 23. We assume that (4.59) is a *neutrally stable* system, which implies that every eigenvalue λ of A is on the unit circle. Stated differently,

$$\lambda\bar{\lambda} = 1. \tag{4.60}$$

This is equivalent to the condition that A is a *unitary* matrix:

$$A^*A = AA^* = I, \quad A \in \mathbb{C}^{n\times n}. \tag{4.61}$$

With some work, a similarity transformation can be found such that

$$A^TA = AA^T = I, \quad A \in \mathbb{R}^{n\times n} \tag{4.62}$$

in some basis. Without loss of generality, we assume that the basis choice is such that (4.62) holds. Hence, A is invertible and $A^{-1} = A^T$. Furthermore, controllability of (A, B) and observability of (A, C) is assumed.

Example 8. *Consider the autonomous discrete-time system*

$$x_{k+1} = Ax_k = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} x_k, \qquad \alpha \in \mathbb{R}. \tag{4.63}$$

If α is an integral multiple of 2π, we have that $A = I$, hence relation (4.62) *holds. On the other hand, if α is not an integral multiple of 2π, the system describes a discrete-time oscillator where the state trajectories have a constant amplitude. Indeed,* (4.63) *is neutrally stable:*

$$A^TA = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.64}$$

4.2.2 Introduction of the Solution Concept

For simplicity, we start with the assumption that the initial condition x_0 is given. Then, given the input on $[0, k^*)$, x_k is known for all $k \in [0, k^*]$. We decide to apply a state feedback of the form

$$u_k = F x_k, \tag{4.65}$$

where $F \in \mathbb{R}^{m \times n}$ is a parameter matrix of the system. The state equation of (4.59) reduces to the autonomous system

$$x_{k+1} = (A + BF) x_k. \tag{4.66}$$

Notice that $x = 0$ is an equilibrium point of (4.67). Since (A, B) is controllable, there exists a matrix F such that $A+BF$ has its eigenvalues in the open unit circle, see Section 2.2.4. Unfortunately, a stabilising feedback cannot be chosen as convenient as for the continuous-time system (4.7) defined on page 39. A scaling factor is introduced, which is used in the definition of the feedback law. After defining a feedback law, Lyapunov theory and LaSalle's invariance principle help us to prove global asymptotic stability of the equilibrium point $x = 0$.

For discrete-time autonomous systems, Lyapunov's stability theorem is as follows.

Theorem 14 (Lyapunov Stability Discrete-Time Systems [10])**.** *Let $x = 0$ be an equilibrium point for the autonomous system*

$$x_{k+1} = f(x_k), \tag{4.67}$$

where f is locally Lipschitz, and let $D \subset \mathbb{R}^n$ be a domain (open set) containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuous such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D \setminus \{0\}, \tag{4.68}$$

$$\Delta V(x) := V(f(x)) - V(x) \leq 0 \text{ in } D. \tag{4.69}$$

Then, $x = 0$ is stable. Moreover, if

$$\Delta V(x) < 0 \text{ in } D \setminus \{0\}, \tag{4.70}$$

then $x = 0$ is asymptotically stable.

Opposed to continuous-time Lyapunov functions, for which the criterion of a negative semidefinite derivative is used, discrete-time Lyapunov functions are characterised by the difference between two consecutive values of V.

For the system (4.67), define the candidate Lyapunov function

$$V_5(x) = x^T x. \tag{4.71}$$

This function is strictly positive for all $x \neq 0$, and $V_5(0) = 0$, so the criteria for a candidate Lyapunov function hold. For the difference $\Delta V_5(x)$, we obtain

$$\begin{aligned} \Delta V_5(x) &= (Ax + Bu)^T (Ax + Bu) - x^T x \\ &= x^T (A^T A - I)x + x^T F^T B^T BFx + x^T A^T BFx + x^T F^T B^T Ax \\ &= x^T F^T B^T BFx + 2x^T A^T BFx. \end{aligned} \tag{4.72}$$

Now we select the feedback gain

$$F = -\gamma B^T A, \qquad \gamma > 0. \tag{4.73}$$

Here γ is a scalar to be determined. As we will see, the scaling factor γ is necessary to attain a negative semidefinite Lyapunov difference. To intuitively explain this, we substitute (4.73) in equation (4.67). This gives the autonomous dynamics

$$x_{k+1} = (I - \gamma BB^T)Ax_k. \tag{4.74}$$

Assume that the elements of B are large. Then the factor γ must be small to prevent the state from blowing up. To derive a criterion for γ, we substitute (4.73) in (4.72). This yields

$$\begin{aligned} \Delta V_5(x) &= \gamma^2 x^T A^T B(B^T B)B^T Ax - 2\gamma x^T A^T BB^T Ax \\ &= \gamma \left\{ x^T A^T B \left(\gamma B^T B - 2I \right) B^T Ax \right\}. \end{aligned} \tag{4.75}$$

The aim is to choose γ such that the matrix $2I - \gamma B^T B$ is positive definite, since $\Delta V_5 \leq 0$ then. Hence,

$$\begin{aligned} & z^T (2I - \gamma B^T B)z > 0, \qquad \forall z \in \mathbb{R}^m \setminus \{0\}, \\ \Leftrightarrow \quad & 2\|z\|_2^2 - \gamma \|Bz\|_2^2 > 0, \qquad \forall z \in \mathbb{R}^m \setminus \{0\}, \\ \Leftrightarrow \quad & \sup_{z \neq 0} \frac{\|Bz\|_2}{\|z\|_2} < \sqrt{\frac{2}{\gamma}}, \\ \Leftrightarrow \quad & \|B\|_2 < \sqrt{\frac{2}{\gamma}}, \\ \Leftrightarrow \quad & \gamma < \frac{2}{\|B\|_2^2}. \end{aligned} \tag{4.76}$$

A useful relation is

$$\|B\|_2 = \sqrt{\lambda_{\max}(B^T B)}, \tag{4.77}$$

see Appendix A. This allows us to easily calculate an upper bound for γ:

$$\gamma < \frac{2}{\lambda_{\max}(B^T B)}. \tag{4.78}$$

Stability of the equilibrium point $x = 0$ can be concluded when (4.78) holds.

Investigation of *(global) asymptotic stability* requires the notion of positive invariant sets, as in continuous-time. For discrete-time, the definition is quite similar. A set M is *positively invariant* with respect to (4.67) if

$$x_0 \in M \Rightarrow x_k \in M, \qquad \forall k \in \mathbb{Z}_+. \tag{4.79}$$

Here $\mathbb{Z}_+$ denotes the set of all positive integers. Now we state a theorem for global asymptotic stability of autonomous systems, which uses LaSalle's Invariance Principle.

Theorem 15 (Global Asymptotic Stability Discrete-Time Systems [8])**.** *If there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying*

1. *$V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $V(0) = 0$,*
2. *$\Delta V(x) = V(f(x)) - V(x) \leq 0$ for all $x \in \mathbb{R}^n$,*
3. *Let M be the largest positively invariant set contained in*

   $$E = \{x \in \mathbb{R}^n \mid V(f(x)) - V(x) = 0\},$$

   and let $0 \in M$. 0 is asymptotically stable for the dynamics restricted to M.
4. *All the solutions of the system $x_{k+1} = f(x_k)$ are bounded,*

then the origin is globally asymptotically stable*.*

Now we prove global asymptotic stability of the equilibrium point $x = 0$ of the autonomous system (4.74), provided that (4.78) is true.

Theorem 16. *Assume that (A, B) is controllable and $A^T A = I$. Furthermore, take $\gamma < 2/\lambda_{\max}(B^T B)$. Then $x = 0$ is a globally asymptotically stable equilibrium point of the autonomous system*

$$x_{k+1} = (I - \gamma B B^T) A x_k. \tag{4.80}$$

Proof. First of all, define $V(x) = x^T x$. Then the first requirement of theorem 15 holds. We already calculated that $\Delta V(x) \leq 0$ if (4.78) holds, so the second requirement is also fulfilled. Now we prove that the third requirement holds. Analogous as in theorem 15, we define

$$E = \{x \in \mathbb{R}^n \mid V(f(x)) - V(x) = 0\} = \{x \in \mathbb{R}^n \mid B^T A x = 0\}. \tag{4.81}$$

The last step follows from (4.75). Looking at (4.74), observe that the system dynamics in M are given by

$$x_{k+1} = Ax_k. \tag{4.82}$$

Take $x_0 \in M$. Since M is positively invariant, we have that

$$x_k = A^k x_0 \in M, \quad \forall k \geq 0. \tag{4.83}$$

Therefore

$$B^T A x_k = 0, \quad \forall k \geq 0. \tag{4.84}$$

In general, using that $A^T A = I$ and (4.84), we have

$$B^T (A^T)^k x_n = B^T (A^T)^k A^k x_{n-k} = B^T x_{n-k} = B^T A x_{n-k+1} = 0 \tag{4.85}$$

for $k = 0, \ldots, n-1$. Thus

$$\begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} x_n = 0, \tag{4.86}$$

or equivalently

$$\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}^T x_n = 0. \tag{4.87}$$

The controllability matrix has full rank, due to controllability of (A, B). Hence $x_n = 0$. This proves that $x = 0$ is an asymptotically stable equilibrium point for the dynamics restricted to M. The fourth requirement is obvious, and global asymptotic stability of the equilibrium point $x = 0$ can be concluded. □

4.2.3 State Observation

Given the initial state, we proved that a stabilising controller exists for discrete-time neutrally stable systems. We seek a controller globally asymptotically stabilising the system (4.59) for unknown initial state. In this section an observer design is discussed which improves the state estimate with time. Thereafter, in the next section, the state observer is combined with the static feedback derived in the previous section to globally asymptotically stabilise the system (4.59). In analogy with the continuous-time case, see 4.1.3, we denote the state estimate at time k by $\hat{x}_k$. Then the interesting quantity to investigate is the estimation error defined as

$$e_k := x_k - \hat{x}_k. \tag{4.88}$$

We desire a decrease in the error as time progresses. To achieve this, we define an observer of the form

$$\hat{x}_{k+1} = \underbrace{A\hat{x}_k + Bu_k}_{\text{copy plant}} + K \underbrace{(\boldsymbol{\sigma}(Cx_k) - \boldsymbol{\sigma}(C\hat{x}_k))}_{\text{measurement error}}, \tag{4.89}$$

including the saturation function in the observer dynamics. $K \in \mathbb{R}^{n\times p}$ is the observer gain to be defined. K must be such that the observer dynamics are at least *stable*, and preferably *asymptotically stable* where possible.

To derive a gain matrix K, we focus on the region where both Cx_k and $C\hat{x}_k$ are not saturated first. Obviously, in this region *asymptotic stability* of the equilibrium point $e = 0$ is desired. For the error dynamics we have

$$e_{k+1} = (A - KC)e_k. \tag{4.90}$$

We propose the observer gain

$$K = \eta AC^T, \qquad \eta > 0. \tag{4.91}$$

Thus the error dynamics become

$$e_{k+1} = A(I - \eta C^T C)e_k. \tag{4.92}$$

Comparing (4.92) with (4.74) shows great resemblance. The dynamics (4.92) are in fact globally asymptotically stable, provided that the system (4.59) is observable, neutrally stable and the bound

$$\eta < \frac{2}{\lambda_{\max}(CC^T)} \tag{4.93}$$

is satisfied. The derivation of (4.93) can be done in a similar manner as the derivation of (4.78), by defining the Lyapunov function $V_6(e) = e^T e$.

Theorem 17. *Assume that (A, C) is observable and $A^T A = I$. Furthermore, take $\eta < 2/\lambda_{max}(CC^T)$. Then $e = 0$ is a globally asymptotically stable equilibrium point of the autonomous system*

$$e_{k+1} = A(I - \eta C^T C)e_k. \tag{4.94}$$

Proof. Take η such that (4.93) holds. Then $\Delta V_6(e) \leq 0$ and stability of (4.94) can be concluded. The duality property gives that

$$(A, C) \text{ observable } \Leftrightarrow (A^T, C^T) \text{ controllable.} \tag{4.95}$$

Due to neutral stability, application of theorem 16 to the controllable pair (A^T, C^T) gives that the system

$$e_{k+1} = (I - \eta C^T C)A^T e_k \tag{4.96}$$

is globally asymptotically stable. This implies that the matrix $(I - \eta C^T C)A^T$ is *Schur*, hence its transpose

$$[(I - \eta C^T C)A^T]^T = A(I - \eta C^T C)^T = A(I - \eta C^T C) \tag{4.97}$$

is also Schur. This proves global aymptotic stability of (4.94). □

We select $K = \eta AC^T$ for the observer (4.89), pursuing global asymptotic stability in the linear region where Cx and $C\hat{x}$ are not saturated. The observer dynamics become

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + \eta AC^T(\boldsymbol{\sigma}(Cx_k) - \boldsymbol{\sigma}(C\hat{x}_k)). \tag{4.98}$$

We have to investigate stability of (4.98) in the saturated regions, and distinguish between two different cases. Those cases are similar as for the continuous-time case. For convenience we restate them here.

- $(Cx)_i$ and $(C\hat{x})_i$ are saturated and have the *same sign*. In this case we have that $(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}))_i = 0$, so the i^{th} output component has no effect on the correction. This is intuitive, because it is unclear how the i^{th} component should correct for the estimate. If all state elements are saturated with the same sign, the observer actually turns into a *predictor*, letting the state estimate undergo the same evolution as the state.
- At least one of the terms $(Cx)_i$ and $(C\hat{x})_i$ is saturated, and $(Cx)_i \neq (C\hat{x})_i$. The beauty of the observer (4.98) is that the i^{th} output component causes a correction now. This is intuitive, since we know in which direction to steer the estimate. We have to check whether a correction is done in the right direction. This is the case if
$$\text{sgn}\{(Ce)_i\} = \text{sgn}\{\sigma((Cx)_i) - \sigma((C\hat{x})_i)\}. \tag{4.99}$$
Without loss of generality we assume that $(Cx)_i \geq 1$ and $(C\hat{x})_i < 1$. Then $(Cx)_i - (C\hat{x})_i > 0$. Furthermore, $\sigma((Cx)_i) - \sigma((C\hat{x})_i) > 0$. Hence, (4.99) is true.

We have shown that the sign of the measurement error term in the observer (4.98) is always correct, i.e. (4.99) holds. However, we still need to prove stability for the general observer (4.98). In full generality, error dynamics for the observer are

$$e_{k+1} = Ae_k - \eta AC^T(\boldsymbol{\sigma}(Cx_k) - \boldsymbol{\sigma}(C\hat{x}_k)). \tag{4.100}$$

To investigate stability, we define the candidate Lyapunov function

$$V_6(e) = e^T e. \tag{4.101}$$

This function is strictly positive for all $e \neq 0$ and $V_6(0) = 0$. For $\Delta V_6(e)$ we obtain

$$\begin{aligned}
\Delta V_6 &= [Ae - \eta AC^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}))]^T[Ae - \eta AC^T(\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}))] - e^T e \\
&= e^T(A^TA - I)e - 2\eta e^T A^T AC^T[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})] \\
&\quad + \eta^2[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})]^T CA^T AC^T[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})] \\
&= -2\eta e^T C^T[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})] \\
&\quad + \eta^2[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})]^T CC^T[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})] \\
&= \eta\Big\{-2e^T C^T[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})] \\
&\quad + \eta[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})]^T CC^T[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})]\Big\} \\
&= \eta\Bigg\{ -2\sum_{i=1}^{p}(Ce)_i[\sigma(Cx) - \sigma(C\hat{x})]_i \\
&\quad + \eta[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})]^T CC^T[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})]\Bigg\} \qquad (4.102)
\end{aligned}$$

In the third step we used the relation $A^TA = I$. To show that ΔV_6 is negative semidefinite, note that the following implications hold:

$$[\sigma(Cx) - \sigma(C\hat{x})]_i > 0 \quad \Rightarrow \quad (Ce)_i > [\sigma(Cx) - \sigma(C\hat{x})]_i\,, \qquad (4.103)$$

$$[\sigma(Cx) - \sigma(C\hat{x})]_i < 0 \quad \Rightarrow \quad (Ce)_i < [\sigma(Cx) - \sigma(C\hat{x})]_i\,. \qquad (4.104)$$

Elaborating on (4.102), we get

$$\begin{aligned}
\Delta V_6 &\leq \eta\Big\{-2[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})]^T[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})] \\
&\quad + \eta[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})]^T CC^T[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})]\Big\} \\
&= \eta[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})]^T\left(-2I + \eta CC^T\right)[\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})]. \qquad (4.105)
\end{aligned}$$

If the bound (4.93) holds, the matrix $2I - \eta CC^T$ is positive definite. Recalling theorem 14, this proves stability of the error dynamics for suitable η. Hence, the error dynamics are *stable*. To explain that the error does not converge to zero asymptotically, note that

$$e_{k+1} = Ae_k \qquad (4.106)$$

if $\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})$ is equal to zero. Obviously, (4.106) is neutrally stable. This shows that the error dynamics are stable. In the next section we tackle this problem, by combining the observer with an *estimated state feedback*. This will let the error, and hence the state, converge to zero.

4.2.4 Global Asymptotic Stability of the Closed-Loop System

In this section we will combine the feedback law and the state observer derived in the previous sections, and prove global asymptotic stability of the resulting closed loop system. For the system (4.59), we restate the observer

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + \eta AC^T(\boldsymbol{\sigma}(Cx_k) - \boldsymbol{\sigma}(C\hat{x}_k)). \tag{4.107}$$

By applying the feedback control

$$u_k = -\gamma B^T A\hat{x}_k, \tag{4.108}$$

the closed-loop dynamics become

$$\begin{aligned} e_{k+1} &= Ae_k - \eta AC^T(\boldsymbol{\sigma}(Cx_k) - \boldsymbol{\sigma}(C\hat{x}_k)), & (4.109)\\ \hat{x}_{k+1} &= (I - \gamma BB^T)A\hat{x}_k + \eta AC^T(\boldsymbol{\sigma}(Cx_k) - \boldsymbol{\sigma}(C\hat{x}_k)), & (4.110)\\ e_k &= x_k - \hat{x}_k. & (4.111) \end{aligned}$$

Furthermore it is assumed that

$$\gamma < \frac{2}{\lambda_{\max}(B^T B)} \quad \text{and} \quad \eta < \frac{2}{\lambda_{\max}(CC^T)}. \tag{4.112}$$

See Figure 4.6 for an intuitive interpretation of the closed-loop system.

Before proving global asymptotic stability, we first intuitively explain the behaviour of the closed-loop dynamics (4.109)-(4.111). Assume that initially $\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}) = 0$ and $e \neq 0$. Looking at (4.109), we see that the error remains constant then. However, looking at (4.110), the cross terms vanish. Hence the state estimate converges to zero, due to Theorem 16. This leads to desaturation of the elements of $C\hat{x}$ at some point in time. Two possibilities arise:

- $\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})$ becomes nonzero. This yields that the norm of the error decreases monotonically. In other words, the quality of the estimate can only increase. Looking at (4.110), the state estimate converges to zero, since $\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x}) \to 0$. (4.111) yields that $x_k \to 0$ then.
- $\boldsymbol{\sigma}(Cx) - \boldsymbol{\sigma}(C\hat{x})$ remains zero. If $C\hat{x}$ desaturates, then Cx desaturate as well. Thus, the observer dynamics become linear, and the error dynamics locally behave according to (4.94). Theorem 17 guarantees that the error decreases monotonically, such that $\boldsymbol{\sigma}(C\hat{x}) \to \boldsymbol{\sigma}(Cx)$. The dynamics (4.110) let the state estimate converge to zero. Consequently, $x_k \to 0$.

We prove the following theorem:

Theorem 18. *Let* (4.59) *be a neutrally stable system, i.e. there is a basis choice such that* $A^T A = I$. *Take* $\gamma < 2/\lambda_{max}(B^T B)$ *and* $\eta < 2/\lambda_{max}(CC^T)$. *Furthermore, assume that* (A, B) *is controllable and* (A, C) *is observable. Then* $(e, \hat{x}) = (0, 0)$ *is a globally asymptotically stable equilibrium point of the closed-loop system* (4.109)-(4.111).

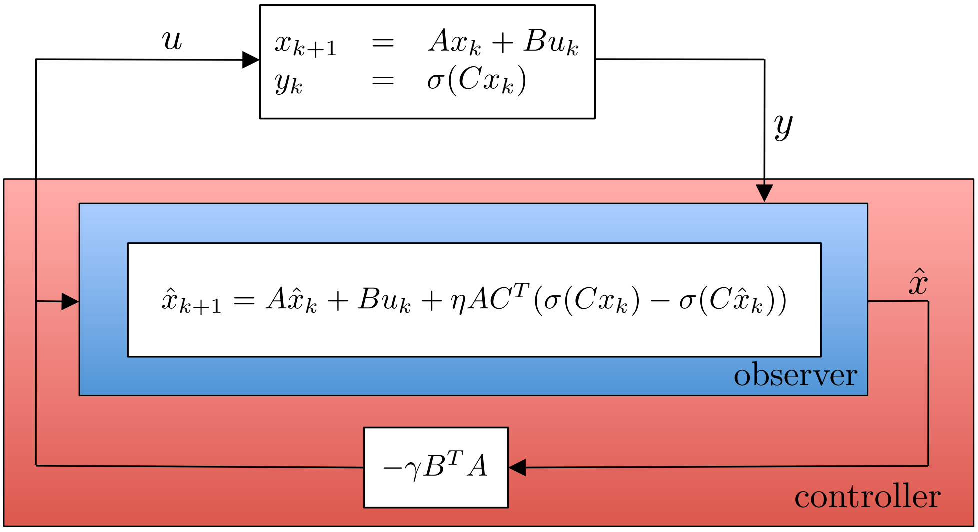


Figure 4.6: The closed-loop system (4.109)-(4.111)

Proof. To prove global asymptotic stability of the equilibrium point $(e, \hat{x}) = (0, 0)$, we define the candidate Lyapunov function

$$V_7(e, \hat{x}) := e^T e. \tag{4.113}$$

When $(e, \hat{x}) = (0, 0)$, then $x = 0$ automatically follows. For V_7 we have

$$V_7(0, 0) = 0 \text{ and } V_7(e, \hat{x}) \geq 0 \text{ for } (e, \hat{x}) \neq (0, 0). \tag{4.114}$$

Notice that $V_7(e, \hat{x})$ is only positive *semidefinite*, so V_7 is not a candidate Lyapunov function. However, the Lyapunov difference is negative semidefinite. Looking at theorem 15, it is seen that a positive semidefinite function is also allowed. Selecting the function $V_7(e, \hat{x}) = e^T e$, the first and second condition of theorem 15 are satisfied. To check the third condition, we define the set E as the set of all points where $\Delta V_7 = 0$:

$$E = \{(e, \hat{x}) \mid \boldsymbol{\sigma}(Cx) = \boldsymbol{\sigma}(C\hat{x}),\ e \in \mathbb{R}^n,\ \hat{x} \in \mathbb{R}^n\}.$$

The last step is derived from (4.102). We denote the largest positive invariant set in E by M. Since $(0, 0)$ is an equilibrium point, $(0, 0) \in M$. The dynamics in M are governed by

$$e_{k+1} = Ae_k \tag{4.115}$$
$$\hat{x}_{k+1} = (I - \gamma BB^T)A\hat{x}_k \tag{4.116}$$
$$e_k = x_k - \hat{x}_k. \tag{4.117}$$

Global asymptotic stability of the equilibrium point $\hat{x} = 0$ follows from Theorem 16. So, at some point in time, all elements of $C\hat{x}$ desaturate. Together with $\boldsymbol{\sigma}(Cx) = \boldsymbol{\sigma}(C\hat{x})$, we obtain that $Cx = C\hat{x}$ then. Consequently, Theorem 17 implies a decrease of the error. For the dynamics restricted to M, both the error and the state estimate converge to zero. Hence, from (4.111) we have that

$$\lim_{k\to\infty} x_k = 0 \tag{4.118}$$

for the dynamics restricted to M. This proves the third condition of theorem 15. Due to stability, the error dynamics (4.109) are bounded. Furthermore, since $\eta AC^T(\boldsymbol{\sigma}(Cx_k) - \boldsymbol{\sigma}(C\hat{x}_k))$ is bounded and $(I - \gamma BB^T)A$ is Schur, the dynamics (4.110) are also bounded, proving the last condition of Theorem 15. This proves global asymptotic stability of the closed-loop system (4.109)-(4.111). □

4.2.5 Extension to Stabilisation of Stable LTI Systems

As for the continuous-time case, a stabilising controller for discrete-time neutrally stable LTI systems is obtained. The desired equilibrium point is globally asymptotically stabilised for any initial state and any initial state estimate. In a similar manner as for continuous-time systems, the class of systems stabilised by the controller proposed in Section 4.2.4 can be extended to the class of *stable* discrete-time systems, i.e. A is a *Schur* matrix. To do this, we prove the existence of a basis choice for A such that $A^TA \leq I$. Taking a close look at the Lyapunov differences (4.75) and (4.102) reveals that stability properties of the closed-loop system (4.109)-(4.111) do not change.

Using the *Jordan normal form*, it is obvious that a Schur matrix A can be divided in two independent parts:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \tag{4.119}$$

where A_1 is *Schur* and A_2 is *neutrally stable*. Thus, all eigenvalues of A_1 have modulus strictly smaller than one, and all eigenvalues of A_2 lie on the unit circle.

To prove that $A^TA - I \leq 0$ in some basis, we use the discrete-time Lyapunov equation

$$A_1^TPA_1 - P < 0. \tag{4.120}$$

If A_1 is Schur, there exists a $P = P^T > 0$ such that the linear matrix inequality (LMI) (4.120) holds. P has a unique positive definite square root [3], which we denote by $P^{1/2}$. Due to the invertibility of a positive definite matrix, $P^{-1/2}$ exists. Exploiting the existence and the symmetry of $P^{-1/2}$, we find

$$\begin{aligned} P^{-1/2}(A_1^T P A_1)P^{-1/2} - P^{-1/2}PP^{-1/2} &= (P^{-1/2}A_1^T P^{1/2})(P^{1/2}A_1P^{-1/2}) - I \\ &= (P^{1/2}A_1P^{-1/2})^T(P^{1/2}A_1P^{-1/2}) - I \\ &= \tilde{A}_1^T\tilde{A}_1 - I \\ &< 0. \end{aligned}$$

In fact, the corresponding similarity transformation is given by

$$\begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} P^{-1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} P^{1/2} & 0 \\ 0 & I \end{bmatrix}. \tag{4.121}$$

This proves the existence of a basis such that $A^TA - I \leq 0$ when A is Schur.

4.3 Simulation

To verify the theory developed in the previous sections, simulation of some examples of stable discrete-time LTI systems subject to output saturation is done in this section. We use the Simulink package of Matlab to implement the closed-loop system and obtain plots. For the Simulink model and its corresponding code, we refer to Appendix D.

4.3.1 SISO Neutrally Stable System

First of all we consider the neutrally stable LTI system

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k \end{cases}, \quad k \in \mathbb{Z}. \tag{4.122}$$

with

$$A = \begin{bmatrix} \cos(\pi/5) & \sin(\pi/5) & 0 \\ -\sin(\pi/5) & \cos(\pi/5) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}. \tag{4.123}$$

The measured output is given by $\sigma(y_k)$, where $\sigma(\cdot)$ denotes the saturation function. It may be checked that $A^TA = I$, and that this system is both controllable and observable. Looking at matrix A, the upper left 2×2 sub-matrix serves as the oscillating part of the system. The lower right identity element is independent of this oscillation, and describes the constant part. Depending on the initial condition, Cx_k has a certain initial phase, amplitude and mean. We show by simulation that the system can be stabilised for some

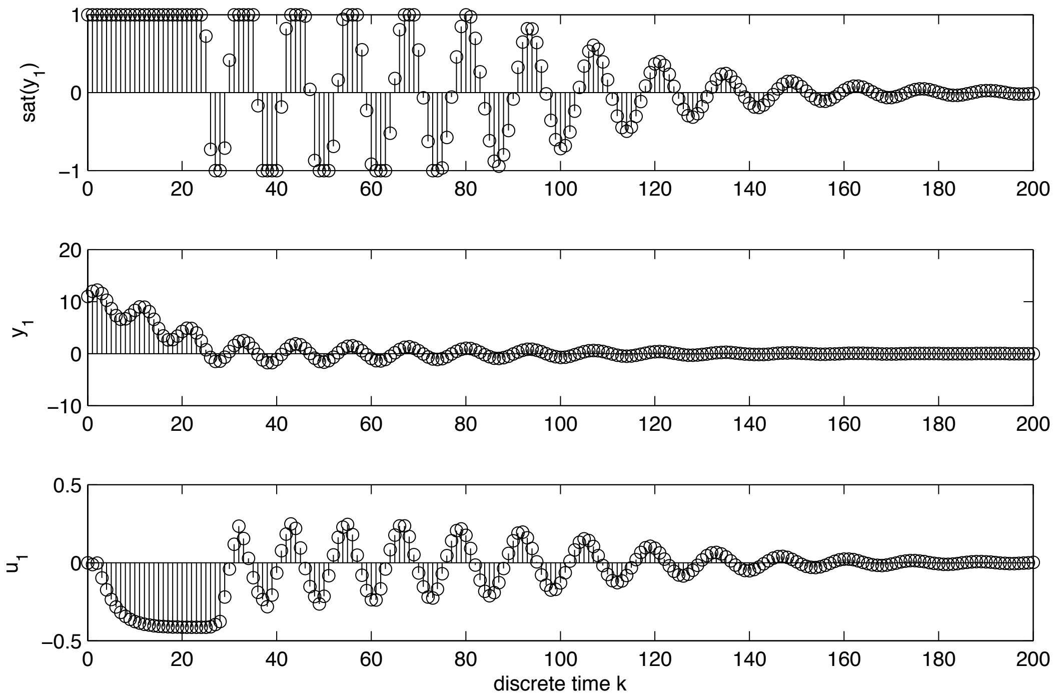


Figure 4.7: Saturated output, actual output and input of the neutrally stable system (4.122) in closed-loop with the stabilising controller.

arbitrary initial condition.

Provided that $\gamma < 2/\lambda_{\max}(B^T B)$ and $\eta < 2/\lambda_{\max}(CC^T)$, $x = 0$ is a globally asymptotically stable equilibrium point of (4.122) according to theorem 18. To this extent we select

$$\gamma = \frac{1}{2}, \quad \eta = \frac{1}{2}. \tag{4.124}$$

Furthermore, we specify the initial conditions

$$x(0) = \begin{bmatrix} 1 & 2 & 10 \end{bmatrix}^T \quad \text{and} \quad \hat{x}(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \tag{4.125}$$

for the plant and the observer respectively. Simulation results of the closed-loop system (4.109)-(4.111) are plotted in Figure 4.7. The first two plots display the saturated output $\sigma(y_k)$ and the actual output y_k respectively. The input u_k is depicted in the third plot. It is seen that the output converges to zero asymptotically. Moreover the input remains bounded and converges to zero as expected.

For completeness we add a simulation with measurement noise. The same parameters as above are used, giving us the opportunity to compare the result to the noise free results.

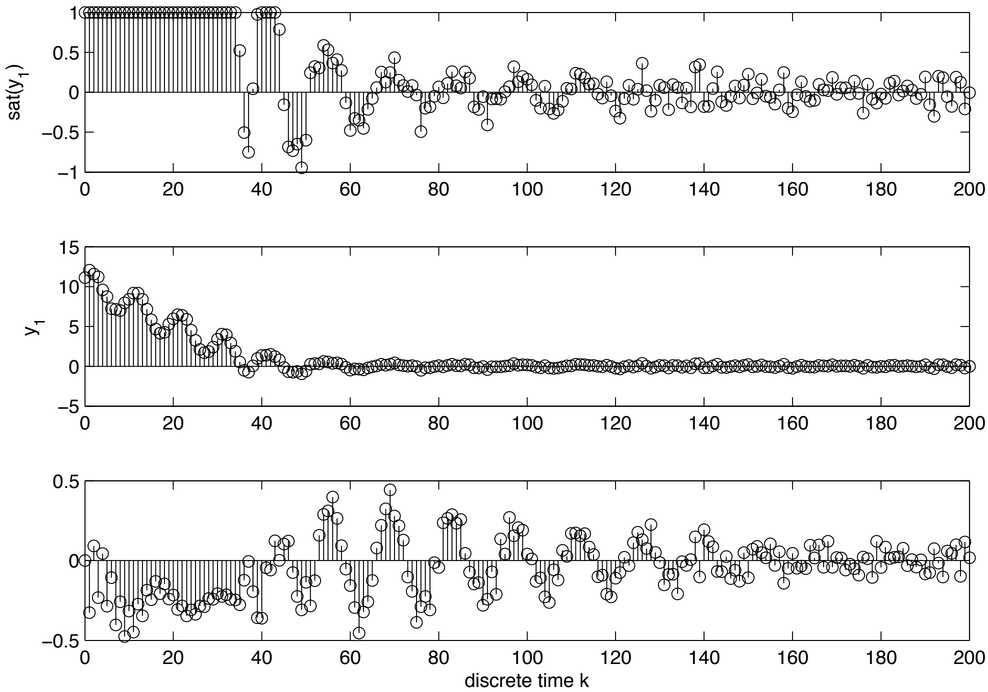


Figure 4.8: Saturated output, actual output and input of the neutrally stable system (4.122) in closed-loop with the stabilising controller and added measurement noise.

A normally distributed random number with mean zero and variance 0.01 is added to each measurement. The results are depicted in Figure 4.8. The simulation shows that the controller indeed has a certain noise rejection.

4.3.2 MIMO Neutrally Stable System

To verify that our controller also works for MIMO LTI systems, we propose to simulate

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k \\ y_k &= \sigma(Cx_k) \end{cases}, \quad k \in \mathbb{Z}. \tag{4.126}$$

with

$$A = \begin{bmatrix} \cos(1/2) & \sin(1/2) & 0 & 0 \\ -\sin(1/2) & \cos(1/2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{4.127}$$

The input and the output of this neutrally stable system both have dimension two. Furthermore (A, B) is controllable and (A, C) is observable. We use theorem 18 to derive appropriate values for γ and η:

$$\gamma = \frac{1}{2}, \quad \eta = \frac{1}{3}. \tag{4.128}$$

Simulating the system (4.126) with initial conditions

$$x(0) = \begin{bmatrix} 4 & 4 & 10 & 12 \end{bmatrix}^T \quad \text{and} \quad \hat{x}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T, \tag{4.129}$$

we obtain the results depicted in Figure 4.9. As for the SISO system, both the input and the output nicely converge to zero. Especially for the second element of the output vector it takes a while before it desaturates. This is due to the large initial conditions. Global asymptotic stability of the closed-loop system is verified. Figure 4.10 depicts the response of the closed-loop system with added measurement noise. To each measurement a random number with mean zero and variance 0.01 is added. This verifies the claimed noise immunity.

In summary, a solution for stabilisation of both continuous- and discrete-time stable LTI systems subject to output saturation is given. This is achieved by finding Lyapunov functions and using Lasalle's Invariance principle. In continuous time the structure is more beautiful, while more technical details are required to prove stability. For discrete-time systems, this is exactly the opposite. The discrete-time solution opens the door to implementable controllers. Implementability is verified by simulation of both SISO and MIMO neutrally stable LTI systems subject to output saturation. The next step is to extend the theory, and find stabilising controllers for (a class of) unstable systems.

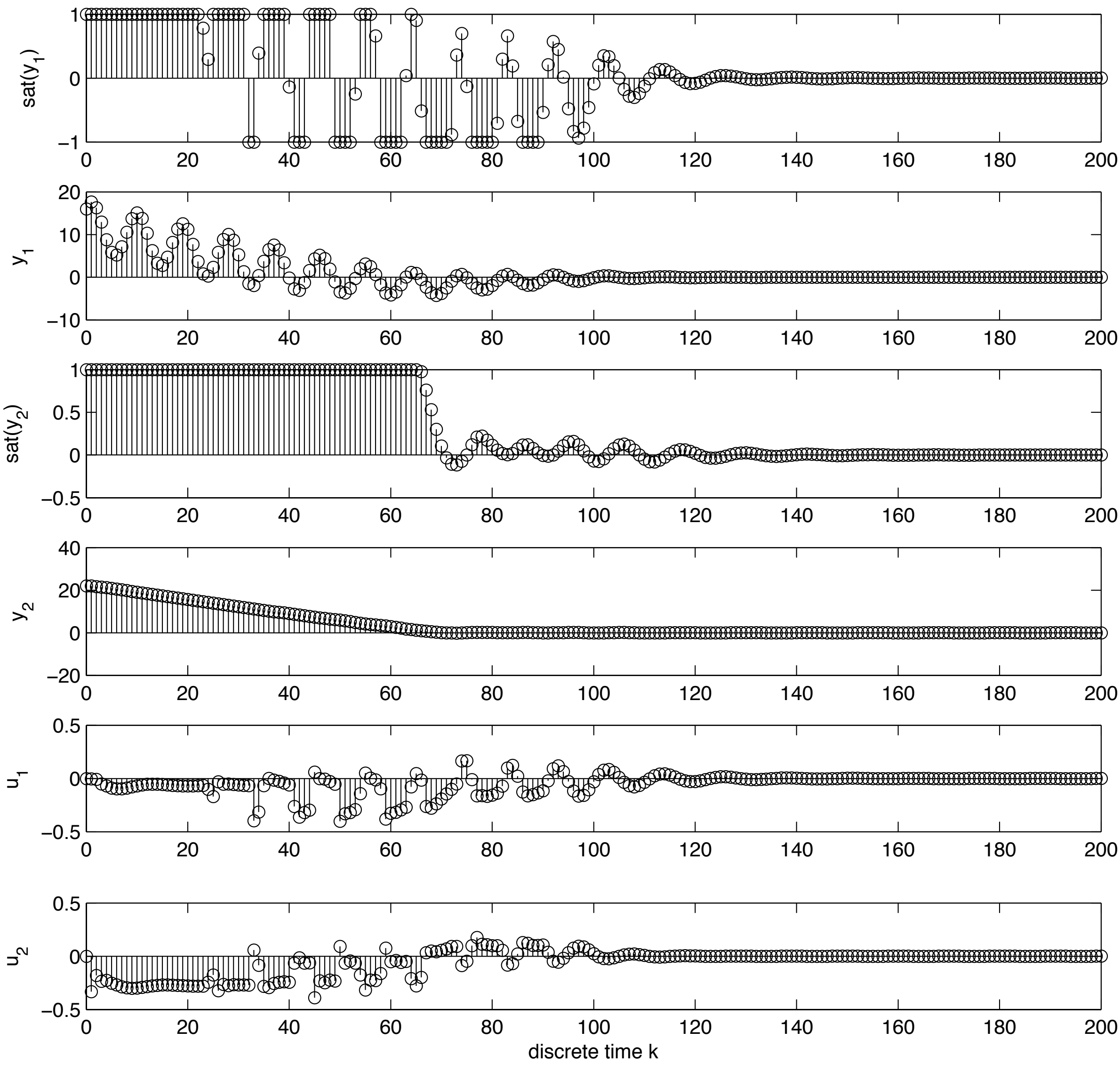


Figure 4.9: Saturated output, actual output and input of the neutrally stable system (4.126) in closed-loop with the stabilising controller.

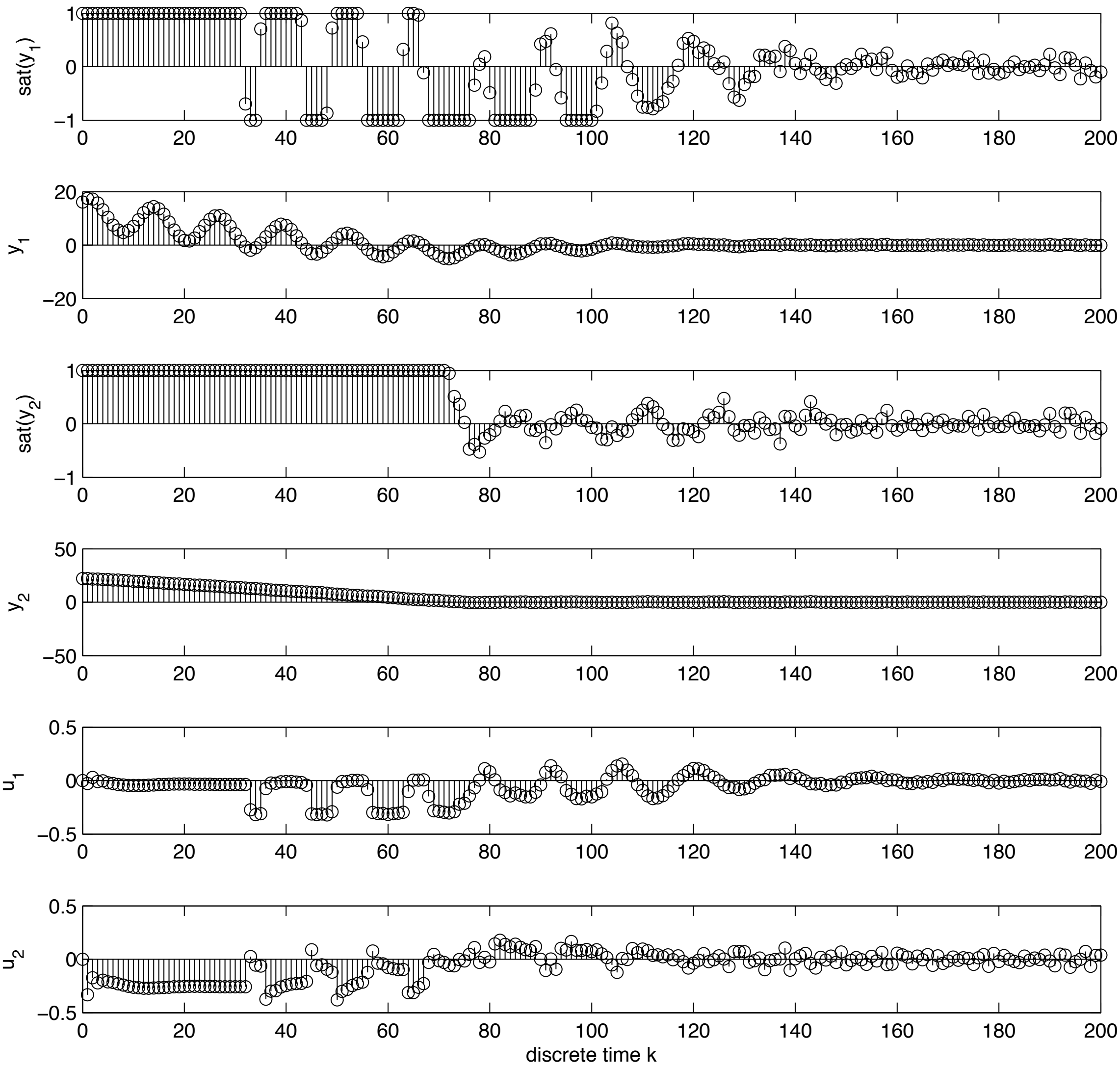


Figure 4.10: Saturated output, actual output and input of the neutrally stable system (4.126) in closed-loop with the stabilising controller and added measurement noise.

Chapter 5

Conclusions and Recommendations

In this chapter the important conclusions regarding this thesis are discussed. Thereafter, recommendations for future research are suggested.

Within this thesis, insight into dynamical systems subject to output saturation is given. The main difficulties are explained, together with the results so far from [4, 12]. This has given a good understanding of the bottlenecks to be tackled. Additionally, elegant stabilising controllers are designed for the class of stable LTI systems with output saturation. The proposed controllers have a certain immunity to measurement noise. Since we found a stabilising controller for the discrete-time system, the implementability issue is also resolved. Stable LTI systems arise in a variety of applications, that can now be controlled in a proper manner.

However, we are not there yet. It has proved to be a major challenge to stabilise discrete-time unstable LTI systems subject to output saturation. Considering discrete-time systems, the most difficult tasks are to guarantee desaturation of the output, and to attain stable error dynamics while the output is saturated. Further research in this direction is suggested.

To start with, one could consider the problem of stabilising a discrete-time double integrator subject to output saturation. Such a system may grow linearly without application of an input. The first task is to find a control law steering the output towards the region where it can be measured correctly. However, if the state is not observed adequately, this results in unstable behaviour. Information about the state can be deduced by looking at the time instants the output changes sign, in case it does not desaturate. We suggest to improve the state estimate, by taking into account this information. Once a stabilising controller is found, the methodology can possibly be expanded to stabilise the class of systems exhibiting polynomial growth. Those systems intrinsically have the same structure as a double integrator. Eventually, a stabilising controller for the entire class of discrete-time LTI systems subject to output saturation is desired.

In summary, we have shown that stabilising controllers exist for the class of stable LTI systems subject to output saturation. This is proved using quadratic Lyapunov functions in combination with LaSalle's invariance principle. The resulting closed-loop system is globally asymptotically stable. The implementability issue is resolved by finding a solution for discrete-time LTI systems. It is discussed how a continuous- time system can be discretised, and under which conditions an implementable controller design can be achieved. Finally simulations of a discrete-time LTI system with output saturation are done, and verify the theoretical results.

Appendix A

Matrix Norm

The definition is taken from [3]. $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is a *matrix norm* if the following three properties hold:

$$\begin{array}{lll} f(A) \geq 0, & A \in \mathbb{R}^{m\times n}, & (f(A) = 0 \text{ if and only if } A = 0) \\ f(A+B) \leq f(A) + f(B), & A, B \in \mathbb{R}^{m\times n}, & \\ f(\alpha A) = |\alpha| f(A), & \alpha \in \mathbb{R}, & A \in \mathbb{R}^{m\times n}. \end{array}$$

We denote the matrix norm by $\|A\| = f(A)$.

A.1 Vector p-Norms

An important subclass of matrix norms are the so-called vector p-norms defined by

$$\|x\|_p := (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}, \qquad p \geq 1. \tag{A.1}$$

The 2-norm is used extensively throughout this thesis. The 2-norm is given by

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{x^T x}. \tag{A.2}$$

A.2 Matrix p-Norms

The matrix p-norms are induced by the vector p-norms:

$$\|A\|_p := \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \|Ax\|_p. \tag{A.3}$$

In fact, the p-norm of some matrix is a measure on how much this matrix can amplify a vector. It may be verified that (A.3) agrees with the definition. The most important p-norm for us is the 2-norm. The matrix 2-norm can conveniently be calculated using the relation (see [13])

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}, \tag{A.4}$$

where $\lambda_{\max}(A^T A)$ denotes the largest eigenvalue of $A^T A$.

Appendix B

Jordan Normal Form

Every square matrix can be brought into *Jordan normal form* by a nonsingular transformation matrix [15]. Hence, there exists a nonsingular matrix S such that

$$A = SJS^{-1}, \tag{B.1}$$

where J is a so-called Jordan matrix

$$J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_r \end{bmatrix}. \tag{B.2}$$

All sub-matrices $J_1, \ldots J_r$ are *Jordan blocks*, and have the form

$$J_k = \begin{bmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{bmatrix}, \tag{B.3}$$

where λ_k is an eigenvalue of matrix A. (If all Jordan blocks are 1×1, then J is a diagonal matrix and A can be diagonalised.)

Appendix C

LaSalle's Theorem

The following result is from [10]:

Theorem 19 (LaSalle's Theorem Continuous-Time Systems [10])**.** *Let $\Omega \subset D$ be a compact set that is* positively invariant *with respect to $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest* invariant *set in E. Then every solution starting in Ω approaches M as $t \to \infty$.*

For consistency, we use a specific version of the above theorem:

Theorem 20 (LaSalle's Theorem Continuous-Time Systems)**.** *If there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying*

1. *$V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $V(0) = 0$,*
2. *$\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$,*
3. *For the dynamics restricted to E^*, $0 \in E^*$ is asymptotically stable, where E^* is the largest* positively invariant *set contained in*

$$E = \{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\},$$

4. *All solutions of the system $\dot{x} = f(x)$ are bounded,*

then the origin is globally asymptotically stable.

To prove that theorem 20 follows from 19, we need a couple of theorems first:

Theorem 21 (The Closure of a positively Invariant Set)**.** *The closure of a positively invariant set is also positively invariant.*

The proof is stated, and is analogous to the proof for *invariant* sets stated in [5].

Proof. Let Ω be a positively invariant set, and let $\bar{\Omega}$ denote its closure. $\bar{\Omega}$ is the union of Ω and the set of all accumulation points of Ω. Let $x(0) \in \bar{\Omega}$. If $x(0) \in \Omega$, then $x(t) \in \Omega$ for all $t \geq 0$ by the definition of a positively invariant set. Hence, $x(t) \in \bar{\Omega}$ for all $t \geq 0$. Now suppose that $x(0) \notin \Omega$, and choose a sequence $x_n(0) \in \Omega$ such that $x_n(0) \to x(0)$. Then $x_n(t) \to x(t)$, implying that $x(t) \in \bar{\Omega}$ for all $t \geq 0$. □

Theorem 22. *For all $x(0) \in \mathbb{R}^n$ there exists a* positively invariant compact set Ω *such that $x(0) \in \Omega$ if and only if for all $x(0) \in \mathbb{R}^n$, $x(t)$ is bounded as a function of t.*

Proof. '⇒': Since $x(0)$ in Ω and Ω is positively invariant, $x(t) \in \Omega, \quad \forall t \geq 0$. The compactness of Ω implies that it is a bounded set. Boundedness of $x(t)$ follows.
'⇐': $x(t)$ is bounded, hence there exists a bounded set Ω_0 such that $x(t) \in \Omega_0, \quad \forall t \geq 0$. By definition Ω_0 is positively invariant. According to theorem 21, the closure of Ω_0, denoted by $\bar{\Omega}_0$, is a positively invariant set. Since $\bar{\Omega}_0$ is closed and bounded, it is compact. Now define $\Omega := \bar{\Omega}_0$. □

Now we are ready to prove that theorem 19 implies theorem 20. To start with, we substitute $D = \mathbb{R}^n$ and $M = 0$ in theorem 19. For this specific choice we obtain:

Theorem 23. *Let $\Omega \subset \mathbb{R}^n$ be a compact set that is* positively invariant *with respect to $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let $\{0\}$ be the largest* invariant *set in E. Then every solution starting in Ω approaches* 0 *as $t \to \infty$.*

We have to prove that the conditions in theorem 23 are equivalent to the conditions in theorem 20. This is proved now.

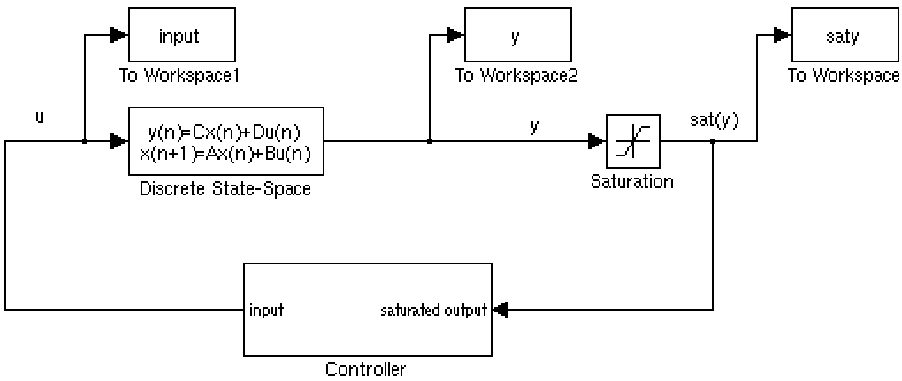
Proof. '⇒' (From theorem 23 to theorem 20): Ω is positively invariant and compact with respect to $\dot{x} = f(x)$. Therefore, we have from theorem 22 that $x(t)$ is bounded as a function of t. So requirement 4 of theorem 20 holds. Since V is continuous on the compact set Ω, V is bounded from below. Without loss of generality, we can assume that V is bounded from below by 0 and $V(0) = 0$. Requirement 1 and 2 are fulfilled.

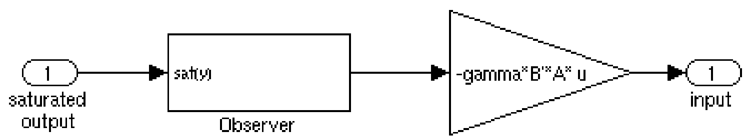
Lastly, 0 being the largest invariant set in $E = \{x \in \Omega \mid \dot{V}(x) = 0\}$ implies that 0 is the only equilibrium point for the dynamics restricted to E. This implies requirement 3.

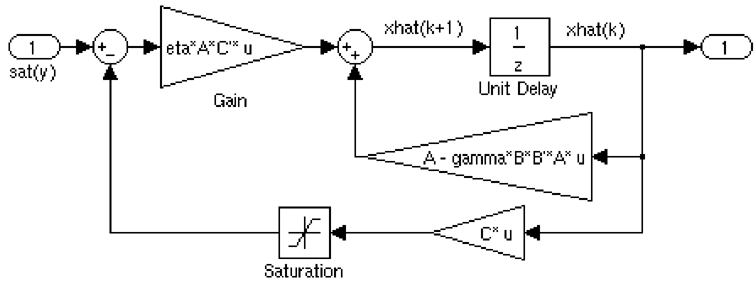
'$\Leftarrow$' (From theorem 20 to theorem 23): All solutions of the system $\dot{x} = f(x)$ are bounded. Using theorem 22 again, the existence of a compact set Ω that is positively invariant with respect to $\dot{x} = f(x)$ is clear. Looking at conditions 1 and 2 of theorem 23, there exists a continuously differentiable $V : \mathbb{R}^n \to \mathbb{R}$ such that $\dot{V}(x) \leq 0$ in Ω. 0 is asymptotically stable for the dynamics restricted to the largest positively invariant set contained in $E = \{x \in \Omega \mid \dot{V}(x) = 0\}$, so the unique equilibruim point. Therefore, 0 is the largest invariant set in E. This finishes the proof. $\square$

Appendix D

Simulink Model for Stable LTI Systems







D.1 Matlab Code

```
% simulation of discrete-time neutrally stable system in
% closed-loop with the stabilising controller.

%% Model Parameters
A = [cos(pi/5) sin(pi/5) 0; -sin(pi/5) cos(pi/5) 0; 0 0 1];
B = [1; 0; 1];
C = [1 0 1];
D = [0];

x0 = [1; 2; 10];
xhat0 = [0; 0; 0];

Contr = [B A*B A^2*B];
Obs = [C; C*A; C*A^2];
rank(Contr);
rank(Obs);

gamma = 1/max(eigs(B'*B));
eta = 1/max(eigs(C*C'));

%% Numerical Simulation
tStart=0;
tFinal=200;

sim('neutrally_stable',[tStart tFinal]);

%% Plots of the results
figure(1)
SUBPLOT(3,1,1)
stem(tout,saty)
ylabel('sat(y_1)')
SUBPLOT(3,1,2)
stem(tout,y,'r')
ylabel('y_1')
SUBPLOT(3,1,3)
stem(tout,input)
xlabel('discrete time k')
```

Appendix E

Discretised Algorithm of Kreisselmeier

```
%This .m-file computes the response of an LTI system as
%described in Kreisselmeier (1996). An initial state X0
%is assumed. The system response using the control law
%as described in section 3 of Kreisselmeier is plotted.

clear all;

%define linear system x' = Ax + Bu, y  = cx
A    = [0 1; 0 0];
b    = [0; 1];
c    = [1 0];
sys = ss(A,b,c,[]);

%time vector
T     = 10e-3;          %interval length
dtau = T*10e-5;         %step size
tau  = 0:dtau:T-dtau;   %vector tau
Ltau = length(tau);
N     = 99;             %#intervals

%control parameters
B      = [0 -T; -T 1];    %B(T)
alpha = exp(2*T*norm(A));
h      = [1;0];           %h s.t. c*h > 0
eAT    = expm(A*T);       %exp(AT)

%initial conditions
y0    = 0;
yd0   = 0.0001;
x0    = [y0 ; yd0];     %initial state
```

```
%input on [0,T) (also when |y0|<1)
U0 = (B\[alpha.*h - eAT*h])*sat(y0);
u(1:Ltau) = -abs(b'*U0)*sign(y0);

%simulate for interval [0,T)
[y, t, x] = lsim(sys,u(1:Ltau),(1:Ltau),x0);

%construct input for interval [kT,(k+1)T)
%using output of interval [(k-1)T,kT)
for k=1:N                %for every interval of length T
    %determine sigma(k)
    sigma(k) = Ltau - sum(abs(sat(y(((k-1)*Ltau+1):(k*Ltau)))));
    %determine new input
    if sigma(k) == 0     %if y saturated in [(k-1)T,kT)
        %determine U_k (vector)
        U(:,k) = (alpha^k).*(B\(alpha.*h-eAT*h)).*sat(y(length(y)));
        %determine u(kT+tau)
        u((k*Ltau+1):((k+1)*Ltau)) ...
            = -abs(b'*-eAT'*U(:,k)).*sign(y(k*Ltau));
    else                 %if y is not completely saturated
        k
        %determine D(k) and xi(k)
        D11(k) = sum(1-abs(sat(y(((k-1)*Ltau+1):(k*Ltau)))));
        D12(k) = 0;
        D21(k) = 0;
        D22(k) = 0;
        xi(:,k) = zeros(2,1);
        for i=1:Ltau
            D12(k) = D12(k) + tau(i)*(1-abs(sat(y((k-1)*Ltau+i))));
            D21(k) = D21(k) + tau(i)*(1-abs(sat(y((k-1)*Ltau+i))));
            D22(k) = D22(k) + tau(i)^2*(1-abs(sat(y((k-1)*Ltau+i))));
            if k == 1   %exception for U0
                xi(:,k) = xi(:,k) + [1 0; tau(i) 1]*c'.*...
                (sat(y((k-1)*Ltau+i)) + c*[0 -tau(i); -tau(i) 1]*...
                (B\[alpha.*h - eAT*h]).*sat(y0)).*...
                (1-abs(sat(y((k-1)*Ltau+i))));
            else
                xi(:,k) = xi(:,k) + [1 0; tau(i) 1]*c'.*...
                (sat(y((k-1)*Ltau+i)) + c*[0 -tau(i); -tau(i) 1]*...
                U(:,k-1)).*(1-abs(sat(y((k-1)*Ltau+i))));
            end
        end
        %determine U_k vector
        if k == 1
            U(:,k) = B\(eAT*(eAT*([D11(k) D12(k);D21(k) D22(k)]\xi(:,k))))...
                - U0;
        else
            U(:,k) = B\(eAT*(eAT*([D11(k) D12(k);D21(k) D22(k)]\xi(:,k))))...
                - U(:,k-1);
        end
        %u(kT+tau)
        u((k*Ltau+1):((k+1)*Ltau)) = -abs(b'*-eAT'*U(:,k))*sign(y(k*Ltau));
    end
```

```
    %simulate
    [y2, t2, x2] = lsim(sys,u((k*Ltau+1):((k+1)*Ltau)),...
        (k*Ltau+1):((k+1)*Ltau),...
        x(length(x),:));
    %augment vectors y,t,x
    y = [y; y2];
    t = [t; t2];
    x = [x; x2];
end
%plot results
subplot(3,1,1);
plot(t/Ltau,y)
subplot(3,1,2);
plot(t/Ltau,sat(y))
subplot(3,1,3);
plot(t/Ltau,u,'r')
```

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