MASTER THESIS

# CONTROLLER DESIGN OF LTI SYSTEMS SUBJECT TO HYSTERESIS

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# Abstract

LTI systems subject to hysteresis are investigated, especially the case where the hysteretic effect lies between controller and plant. Difficulties of stabilizability of this particular class of systems are explored. Then controller design is considered, where two different control strategies are investigated; a fixed sign controller and a bang-bang controller. Theorems are stated to check fixed sign controllability and fixed sign stabilizablility. Stability properties, practical-$\Omega$-stability and quasi-stability of the systems with these controllers are investigated. Furthermore, a bang-bang controller is explored, and finally a switched controller is presented which combines the best of both strategies.

# Chapter 1

# Introduction to Control

The theory of controlling dynamical systems has a long history. Automated control of dynamical systems becomes more and more important, even more with the developments of robotics. Many applications are developed because of the natural laziness of humans, and desire for comfort. One can think of house thermostats, cruise control, automatic gear transmission, segways, etc. Automatic control is also important because a human being can not control manually everywhere, where control is needed, one can think of satellite movement corrections to keep a satellite in its orbit. Also, a human is often not capable to control the system fast and precisely enough, one can think of balancing a multiple inverted pendulum or a robot on stairs, or an automated system is much faster, cheaper, more predictable and hopefully more reliable then a human being. For examples of this last case one can think of stock market, assembly lines, etc.

Automatized control or not, in most of the cases a dynamical system generates output, which could be measured and (possibly) compared with a reference value. This is called feedback control. As it can be seen in Figure 1.1, the difference between reference and measurement is sent to a controller and will be used to design an input for the system.

Many dynamical systems have beautiful behavior and even more beautiful control mechanisms, but only when no disturbances, measuring errors, saturation or hysteresis occurs. It is important to design stablizers which can also handle these distortions, to obtain robustness in the system. Bosgra et al. [3] wrote about robustness of controllers when system suffers from perturbations. A nice report about stabilization of systems, subject to measurement saturation is written by Hilhorst [7].

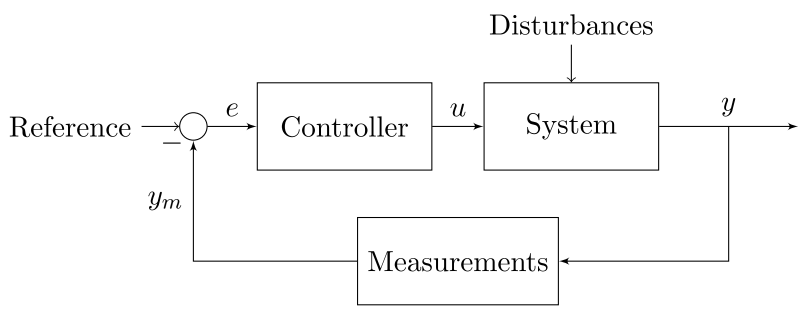



**Figure 1.1:** A dynamical system, which output $y$ is measured as $y_m$, and possibly compared with a reference value, to feed the controller for an appropriate input $u$.

Here, we will handle the control of systems subject to hysteresis. First, a brief introduction to dynamical systems and control is given. Then hysteresis is introduced, with various ways of representing hysteretic behavior. On the basis of an example, the difficulties of hysteresis are sketched. With that example in mind, some suggested solutions are given. Later on, the practical challenges will be generalized.

## 1.1 Basic definitions

It is desirable to analyse objects or situations in their environment to understand phenomena in their behavior. Mathematical modelling of these objects or situations reduces them to variables which describe their state. Dynamic interaction in their states and passing of time have an important role in the evolution of these variables. This arouses our curiosity on how to describe what we see and, even more, how to control the process to obtain desirable behavior. Therefore, we have to define exactly what a dynamical system is.

**Definition 1.1 (Dynamical system [16, 17]).** *A* dynamical system $\Sigma$ *is a triplet* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ *where* $\mathbb{T} \subseteq \mathbb{R}$ *and* $\mathfrak{B} \subseteq \{w : \mathbb{T} \to \mathbb{W}\}$.

Here, $\mathbb{T}$ denotes the *time axis*, which in the continuous case is often equal to $\mathbb{R}^+$. $\mathbb{W}$ describes the *signal space*, i.e. all values which the signals can adopt. $\mathfrak{B}$ is the *behavior* of the system. This is the collection of all trajectories which can be adopt by the system, due to constrains. This is illustrated in the next example.

**Example 1.1 (Train driver):** A train driver wants to describe the behavior of his train which has a mass of $m$ kilogram. The machinist lets the engine provide a force of $F(t)$ Newton at time $t$, and the user manual of the train shows that the air and rolling resistance are related linearly to the velocity of the locomotive with a constant factor $b$. Therefore, he can describe the dynamics of the train with

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + b\frac{\mathrm{d}}{\mathrm{d}t}x(t) = F(t), \tag{1.1}$$

with $x(t)$ the distance which is a function of time. In this case, $\mathbb{T} = \mathbb{R}^+$, and $\mathbb{W} = \mathbb{R}^2$. Its behavior $\mathfrak{B}$ can be represented as

$$\mathfrak{B} :=\{(F, x) : \mathbb{R}^+ \to \mathbb{R}^2 \mid m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + b\frac{\mathrm{d}}{\mathrm{d}t}x(t) = F(t),\ x(0) = 0\}. \tag{1.2}$$

Although the behavior $\mathfrak{B}$ is uniquely defined, it can be represented by easy or difficult equations with several parameters, as long as it 'projects' the *time* to the *force* and *distance*: $\{(F, x) : \mathbb{T} \to \mathbb{W}\}$. In the case of the train driver, an other correct representation of the behavior is

$$\mathfrak{B} := \{(F, x) : \mathbb{R}^+ \to \mathbb{R}^2 \mid x(t) = c_1 + \int_0^t \frac{F(\tau)}{b}\big(1 - \mathrm{e}^{-\frac{b}{m}(t-\tau)}\big)\mathrm{d}\tau,\ \text{for certain } c_1 \in \mathbb{R}\}. \tag{1.3}$$

The reader should verify that this expression represents the same behavior as given in the example. Because the behavior of the system is the important issue, and not the way of representing it, this is called a behavioral approach [17].

Two important properties of dynamical systems are linearity and time-invariance.

**Definition 1.2 (Linearity [17]).** *A system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ *is* linear *if*

$$w \in \mathfrak{B} \quad \text{implies} \quad \lambda w \in \mathfrak{B} \quad \forall \lambda \in \mathbb{R}, \qquad \text{and} \tag{1.4}$$

$$w_1, w_2 \in \mathfrak{B} \quad \text{implies} \quad w_1 + w_2 \in \mathfrak{B}. \tag{1.5}$$

**Definition 1.3 (Time-invariance [7, 17]).** *A system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ *is* time-invariant *if for all* $\tau \in \mathbb{T}$ *holds*

$$w \in \mathfrak{B} \quad \text{implies} \quad \sigma_\tau w \in \mathfrak{B} \tag{1.6}$$

*where* $\sigma_\tau$ *denotes the* shift operator $\sigma_\tau w(t) = w(t - \tau)$.

Looking to our example of the train driver, we see that this system is linear and time-invariant.
A common way to describe a linear, time-invariant (LTI) dynamical system is the state-space representation:

$$\begin{cases} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{cases} \tag{1.7}$$

where $u(t) \in \mathbb{R}^m$ denotes the input at time $t$, $y(t) \in \mathbb{R}^p$ the output and where $x(t) \in \mathbb{R}^n$ describes the state at time $t$. The matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ are all given. Further, $\dot{x}$ denotes the derivative of $x$ with respect to time. The straightforward solution of this system is

$$x(t) = \mathrm{e}^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} Bu(\tau)\, \mathrm{d}\tau \tag{1.8}$$

$$y(t) = Cx(t) = C\mathrm{e}^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} C\mathrm{e}^{A(t-\tau)} Bu(\tau)\, \mathrm{d}\tau. \tag{1.9}$$

Mechanical systems like mass-damper-spring systems and electrical circuits with inductors, resistors and capacitors can be modelled as LTI systems; although their physical appearances are different, their mathematical representation is similar.

**Example 1.2 (Train representation):** Our train driver decides that the applied force provided by the engine is the input and the travelled distance is the output of his LTI system. He wants to represents the trains behavior as a state-space model and he wisely defines the state vector $\boldsymbol{x}$ as $[x(t), \dot{x}(t)]^T$. After a crash course in linear algebra, he found that the following state-space equation represents his LTI system:

$$\begin{cases} \dot{\boldsymbol{x}}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t) \end{cases}, \tag{1.10}$$

which he based on the differential equation (1.1).

To make a start with the analysis of stability, we need a definition of an equilibrium point.

**Definition 1.4 (Equilibrium point [9]).** *Consider the system $\dot{x} = f(x)$. A point $\bar{x}$ is an* equilibrium point *if $f(\bar{x}) = 0$.*

This definition tells us that if $x = \bar{x}$, then $\dot{x} = 0$, which implies that $x$ remains in this equilibrium for further time. So in the case that the system reaches an equilibrium, the system remains in this equilibrium forever, when it is not exposed to any kind of disturbance.

As stated by Khalil [9], the coordinates of the equilibrium point can be shifted towards arbitrary coordinates by changing the system variables, without losing its characteristic behavior. Therefore, we can assume without loss of generality that the equilibrium $\bar{x}$ lies in the origin $x = 0$.

The behavior around an equilibrium plays a crucial role. We want to investigate the characteristics of equilibrium points, which is essential in analysis of dynamical systems. Consider a system in its equilibrium and suppose a small distortion is given. A major question in the analysis of systems is: 'Does the system tend away due to the distortion, or does it nicely return to its equilibrium, or will it keep moving around the equilibrium, without returning or leaving?' To make this more formal, we need a definition of stability. The following definitions are used.

**Definition 1.5 (Stability [9]).** *The equilibrium point $\bar{x}$ of a system $\dot{x} = f(x)$ is*

*(i)* stable*, if for each $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\|x(0)\| < \delta \quad \text{implies} \quad \|x(t)\| < \varepsilon, \quad \text{for all } t \geq 0 \tag{1.11}$$

*(ii)* unstable*, if not stable.*

*(iii)* asymptotically stable*, if $\bar{x}$ is stable and $\delta$ can be chosen such that*

$$\|x(0)\| < \delta \quad \text{implies} \quad \lim_{t\to\infty} x(t) = 0 \tag{1.12}$$

*(iv)* globally asymptotically stable*, if it is stable and*

$$\lim_{t\to\infty} x(t) = 0 \tag{1.13}$$

*for all initial conditions.*

**Definition 1.6 (Attractivity [17]).** *The equilibrium point $x = 0$ of a system $\dot{x} = f(x)$ is an* attractor *if there exist an $\varepsilon > 0$ such that*

$$\|x(0)\| < \varepsilon \quad \text{implies} \quad \lim_{t\to\infty} x(t) = 0 \tag{1.14}$$

Remark that a stable attractor is an asymptotical stable equilibrium point. To illustrate the difference between attractors and stability, we give the the following example.

**Example 1.3 (Unstable attractor):** Consider the following non-linear dynamical system, written in polar coordinates:

$$\begin{cases} \dot{r} &= 1 - r \\ \dot{\theta} &= \sin^2(\theta/2) \end{cases} \tag{1.15}$$

Clearly, the only equilibrium point is $\bar{x} = (1, 0)$. A sketch of this situation is given in Figure 1.2. By first observation, we see that the first expression of this system ensures that a distortion of modulus is compensated. The system will be sent back to the unit circle. However, the second expression results in the fact that a small distortion directs the system to an angle of the next multiple of $2\pi$, no matter how small this distortion is chosen. In all cases where the disturbance is above the $x$-axis, the system goes around, to approach the equilibrium point from below. Therefore, the equilibrium point is an *unstable attractor*.

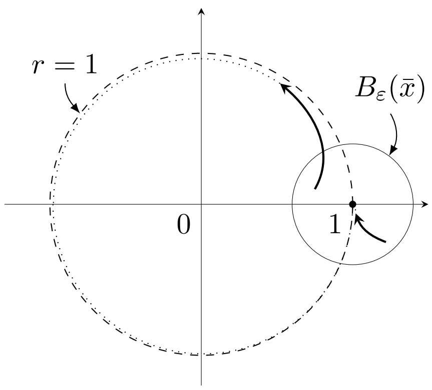



**Figure 1.2:** A plane which shows the behavior of the system (1.15). Two initial conditions are chosen, unequal to the equilibrium point. The trajectory does not always remain inside the $\varepsilon$-neighbourhood around the equilibrium, therefore it is not *stable*. It is, however, an *attractor*, since all trajectories approach the equilibrium when time goes to infinity.

Sometimes, as in the previous example, the trajectory is converging to a certain region. To make distinction between this type of systems and unstable ones, which does not have that kind of behavior, we need an extension of the definition of stability. After this extension we are able to describe stability of certain regions.

**Definition 1.7 (Invariant set).** *A set of states $M$ of a system $\dot{x} = f(x)$ is called an* invariant set *of the system if for all $x_0 \in M$ and for all $t \geq 0, x(t) \in M$.*

Based on ideas of Khalil [9], we define an $\varepsilon$-neighbourhood of the area $M$ by

$$M_\varepsilon = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, M) < \varepsilon\} \tag{1.16}$$

where $\operatorname{dist}(x, M)$ is a function to describe the minimal distance from $x$ to a point in $M$:

$$\operatorname{dist}(x, M) = \inf_{y \in M} \|x - y\|. \tag{1.17}$$

According to the ideas behind stability for equilibrium points, one can state similar definitions of stability of sets:

**Definition 1.8 (Stability of an invariant set [9]).** *An invariant set $M$ of a system $\dot{x} = f(x)$ is*

*(i)* stable *if for each $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$x(0) \in M_\delta \quad \text{implies} \quad x(t) \in M_\varepsilon, \quad \text{for all } t \geq 0 \tag{1.18}$$

*(ii)* asymptotically stable *if it is stable and $\delta > 0$ can be chosen such that*

$$x(0) \in M_\delta \quad \text{implies} \quad \lim_{t \to \infty} \operatorname{dist}(x(t), M) = 0 \tag{1.19}$$

Remark that if we reduce this set $M$ to an equilibrium point $\bar{x}$, this definition is equal to Definition 1.5.

**Example 1.4 (Asymptotically stable invariant set):** Consider the dynamical system (1.15) of Example 1.3, defined in the domain $\mathbb{R}^2 \setminus 0$. We define $M$ as the annular region $\{x \in \mathbb{R}^2 \mid r = 1\}$. This is an invariant set, since $\dot{r} = 0$ for all $x \in M$. Given $\varepsilon > 0$, we choose $\delta = \varepsilon$ and see that all solutions in $M_\delta$ will remain in $M_\varepsilon$. Therefore $M$ is stable. Moreover, we have

$$\lim_{t \to \infty} \operatorname{dist}(x(t), M) = 0, \tag{1.20}$$

no matter how large $\delta$ is chosen. Therefore, this invariant set $M$ is asymptotically stable.

Many tools for analysis of systems are based on LTI systems, for example the analysis of stability of equilibria. The first step of analysis of non-linear systems is therefore linearization. This linearization is a good approximation of the system in the neighbourhood of its equilibrium, because the higher order terms are very small in comparison to the first order term. Furthermore, we have the advantage that the typical stability characteristics are not lost when linearization is applied. The proof is not stated here, but for a formalization we refer to Khalil [9]. We have to keep in mind that linearization has some major limitations. Since a linearized system approximates only the neighbourhood of an equilibrium point, only local behavior can be estimated. Unfortunately nothing can be said about the system when it behaves far away from the chosen equilibrium point, let alone about the global behavior.
To test the stability of equilibria, Lyapunovs first method is a usable method for linear or linearized systems, written as $\dot{x} = Ax$. For this method, we need the definition of semisimple eigenvalues.

**Definition 1.9 (Semisimple eigenvalue [7, 17]).** *Let $A \in \mathbb{R}^{n\times n}$ and let $\lambda$ be an eigenvalue of $A$. An eigenvalue is* semisimple *if the dimension of the null-space*

$$\mathrm{Null}(\lambda I - A) := \{v \in \mathbb{R}^n \mid (\lambda I - A)v = 0\} \tag{1.21}$$

*is equal to the multiplicity of $\lambda$ as a root of the characteristic polynomial of $A$.*

**Theorem 1.10 (Lyapunovs first method for stability [9, 17]).** *Consider an autonomous LTI system $\dot{x} = Ax$. This system is*

*(i)* asymptotically stable *if and only if all eigenvalues of $A$ have negative real part.*

*(ii)* stable *if and only if for all eigenvalues of $A$ holds either $\mathrm{Re}(\lambda) < 0$, or $\mathrm{Re}(\lambda) = 0$ and $\lambda$ is semisimple.*

*(iii)* unstable *if $A$ has an eigenvalue with positive real part and/or a non semisimple eigenvalue with zero real part.*

For an intuitive idea behind this theorem, suppose $t \in \mathbb{T} = \mathbb{R}^+$. We see with equation (1.9), (remark that $u(t) = 0$), that $\mathrm{e}^{At}$ must be bounded to obtain a stable system. By construction of the matrix exponential $\mathrm{e}^{At}$, this is only the case when eigenvalues of $A$ are negative. This is stated intuitively, and compared with the scalar case of $\mathrm{e}^{at}$, which is only bounded for all $t > 0$ when $a < 0$. For a formal proof we refer to Khalil [9, p. 130], and Polderman and Willems [17, p. 243].
The behavior of the non-linear system is nearly equal to the behavior of its linearization, when the system is in the neighbourhood of its equilibrium point. This fact is the idea behind the first method of Lyapunov: When the function is sufficient smooth, the behavior of the linearization is a proper approximation of the non-linear system in the neighbourhood of the equilibrium point.
The key idea of the second method of Lyapunov is based on an energy function: if there is some dissipation of energy (e.g. due to friction), then the signals of a system will converge to zero. For example, if one passively releases a yo-yo, then after a while there is no movement nor height (mechanical energy) any more, since the energy dissipates due to friction. The only way to keep a yo-yo in motion is actively playing with it, which will add energy to the system of the yo-yoing yo-yo. However, this second method of Lyapunov is not used in the investigation of hysteresis in this thesis, and therefore it is omitted here.
It is desirable to have a system where the controller is able to steer the behavior in a desired position. Otherwise, one could just look, do nothing and see that the system behaves as it is determined to do, without the possibility to intervene. This steering is called controllability, and the following definition is used.

**Definition 1.11 (Controllability [17]).** *Consider a time invariant system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$. Then $\Sigma$ is* controllable *if for any two trajectories $w_1, w_2 \in \mathfrak{B}$ there exist a $t_1 \geq 0$ and a third trajectory $w \in \mathfrak{B}$ such that*

$$w(t) = \begin{cases} w_1(t) & t \leq 0 \\ w_2(t - t_1) & t \geq t_1 \end{cases} . \tag{1.22}$$

Close to this definition are the terms of null controllability and reachability. Null controllability is the weaker situation, where the trajectory $w_2$ is replaced by the equilibrium signal. A system is *null controllable* if starting from an arbitrary trajectory $w_1$ the system can be steered in the equilibrium (in finite time). Reachability is almost analogue, but then $w_1$ is replaced by the equilibrium. Thus a system is *reachable* if each trajectory $w_2$ can be reached from its equilibrium position. Precise definitions are omitted, since they are analogue to the definition of controllability.

Remark that if and only if a system is both reachable and null controllable, then it is fully controllable. This follows directly from the definitions of these properties.

In the above definition, trajectories $w$ of the system are mentioned. An equivalent property of systems is *state controllability.* This describes the possibility to steer any state $x_1$ to another arbitrary state $x_2$ within finite time. The equivalence is proven in [16, 17]. In this report the term controllability is used for both properties.

**Theorem 1.12 (Controllability of LTI systems [17]).** *A system defined by equation* (1.7) *is controllable if and only if the controllability matrix*

$$\mathfrak{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{1.23}$$

*has full row rank.*

**Example 1.5 (Controllable train):** The train driver read an article about train accidents.[a] It makes him worried and he wants to know if his train-system is stable and controllable. Therefore, he calculates the eigenvalues of the $A$ matrix in his self-made equation (1.10) in Example 1.2:

$$\mathrm{Eig}\left(\begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix}\right) \quad \rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = -\frac{b}{m}. \tag{1.24}$$

Concerned because of the non-negative value of $\lambda_1$ which has a multiplicity of one, he starts to calculate the null-space:

$$\mathrm{Null}\left(\begin{bmatrix} 0 & -1 \\ 0 & \frac{b}{m} \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{1.25}$$

So the dimension of the nullspace is one. Now he knows that this non-negative eigenvalue is semisimple, and although his system is not asymptotically stable, it is still stable according to Theorem 1.10. Also, he calculates the controllability matrix

$$\mathfrak{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & \frac{-b}{m^2} \end{bmatrix}, \tag{1.26}$$

with matrices $A$ and $B$ taken from equation (1.10). The train driver sees that the rank of this matrix equals two. Reassured, he concludes that his train is fully controllable.

[a] U.F. Malt et al., The effect of major railway accidents on the psychological health of train drivers, *Journal of Psychosomatic Research*, 37(8):793 - 805, 1993

Following the definition of controllability, Definition 1.11, remark that all trajectories must stay in the defined behavior. In our example, the behavior of the train is prescribed to remain on the railway. So if the train driver complains about lack of controllability because he is not able to jump off of the rails with his locomotive, it has nothing to do with the mathematical controllability according to our definition.

If a system is not controllable, it could still be possible to steer to a constant trajectory, e.g. an equilibrium point. If this is possible, the system is called *stabilizable*. This is formalized by the following definition.

**Definition 1.13 (Stabilizability [17]).** *Consider a time invariant system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$. *Then* $\Sigma$ *is* stabilizable *if for every trajectory* $w \in \mathfrak{B}$, *there exist a trajectory* $w_1 \in \mathfrak{B}$ *with the property*

$$w_1(t) = w(t) \text{ for } t \leq 0 \qquad \text{and} \qquad \lim_{t\to\infty} w_1(t) = 0. \tag{1.27}$$

A test to check whether a system is stabilizable will be stated in the next section, in Theorem 1.14 where feedback control is mentioned.

## 1.2 Basic controller design

To get feeling for some elementary control theory, we consider an extensive example. This example will be extended in the next chapters. This section will show how a design of a controller is made.

**Example 1.6 (Inverted pendulum):** Consider an inverted pendulum with point mass $m$, and rod length $l$. Assume that this pendulum is positioned in a fixed pivot position subject to some damping, proportional to the angular velocity with factor $b$. According to rotational dynamics, the moment of inertia of this pendulum will be $I = ml^2$. Let there be a controller, which can apply a torque $\tau$ on this pivot position. An illustration is given in Figure 1.3. Some basic mechanics (see Resnick et al. [20]) learns that the following differential equation can be derived:

$$I\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = mgl\sin(\theta(t)) - b\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} + \tau(t) \tag{1.28}$$

which can be rewritten as

$$\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = \frac{g}{l}\sin(\theta(t)) - \frac{b}{I}\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} + \frac{\tau(t)}{I} \tag{1.29}$$

with $g$ the gravitational constant. From now on, for the sake of brevity, $\dot{\theta}$ and $\ddot{\theta}$ will denote the first and second derivative of $\theta$ with respect to time.
Linearizing this system in it equilibrium point $\bar{\theta} = 0$ (standing position), and writing it in the state-space notation (as in equation (1.7)), gives

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{I} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{array}\right. \tag{1.30}$$

where $x(t) \in \mathbb{R}^2$ contains the state variables *angular displacement* $\theta(t)$ and *angular velocity* $\dot{\theta}(t)$ and $u(t) \in \mathbb{R}$ contain the input variable *torque* $\tau(t)$. Assume that in this model the

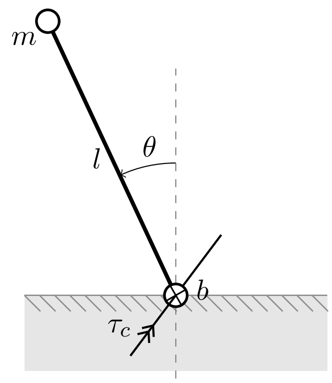



**Figure 1.3:** An inverted pendulum with point mass $m$ and rod length $l$ in a fixed pivot position, with friction coefficient $b$. To stabilize, a torque $\tau$ is applied on the pivot.

position of the pendulum is measured. Therefore, $y(t) \in \mathbb{R}$ equals the first element of $x(t)$, the angular displacement. This justifies the output equation.

Remark that if this system is uncontrolled, i.e. if $u(t) = 0$, it will be unstable in the equilibrium point $\bar{\theta} = 0$. This is quite intuitive with use of Definition 1.5: if there is a distortion $\delta > 0$ from this equilibrium, the orbit of $\theta(t)$ will be unbounded. So, for any specific boundary $\varepsilon > 0$, there is no initial condition $\delta > 0$ for which the pendulum will not eventually pass that boundary $\varepsilon$. With use of the eigenvalues $\lambda_i$, found by the solution of the characteristic polynomial

$$\det\left(\begin{bmatrix} \lambda & -1 \\ -\frac{g}{l} & \lambda + \frac{b}{I} \end{bmatrix}\right) = \lambda\left(\lambda + \frac{b}{I}\right) - \frac{g}{l} = \lambda^2 + \frac{b\lambda}{I} - \frac{g}{l} = 0, \tag{1.31}$$

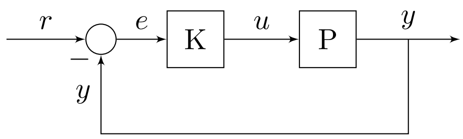
we see that they do not both have negative real part. This can be concluded according to the Routh test [17] and because $g > 0$, $l > 0$ and $I > 0$. Therefore, according to Theorem 1.10, our intuition is confirmed that this system is unstable.

### 1.2.1 Feedback control

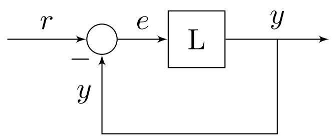
Working with a system as given in Figure 1.1, we see that the output of the system is led back. This output contains crucial information and can be used when a desired output must be reached by controlling. This way of control is called feedback control. Looking to the feedback loop, as depicted in Figure 1.4a and the output equation of (1.7), we see that $u = Ke = K(r - y)$. When $r = 0$, the input can be rewritten as $u = -Ky = -KCx$. Because then the input completely depends on the output (and thus on the state of the system), the state equation of (1.7) can be reduced to

$$\dot{x}(t) = Ax(t) - BKCx(t) = (A - BKC)x(t). \tag{1.32}$$

Since $\dot{x}(t)$ now only depends on $x(t)$, the rewritten equation (1.32) becomes an autonomous system for which stability can be checked by the eigenvalues of $(A - BKC)$ and Theorem 1.10. This system can be drawn as Figure 1.4b. In literature ([3, 17, 16]), general LTI systems with feedback are written as $\dot{x} = Ax + Bu$ with $u = Nx$, where the eigenvalues of $(A + BN)$ play a crucial role in stabilizability.
The following theorem can be used for design of this state feedback controllers, this is discussed in more detail in Chapter 3.




**(a)** plant ($P$) with separate controller ($K$)




**(b)** controller and plant combined in one system ($L$)

**Figure 1.4:** Schematic view of controller design with a feedback loop.

**Theorem 1.14 (Stabilizability [17, 16]).** *Consider a system $\dot{x} = Ax + Bu$, with feedback control $u = Nx$. This system is* stabilizable *if $N$ can be chosen such that all the eigenvalues $\lambda_k$ of $(A + BN)$ lie in the open left half complex plane, i.e. for all such $\lambda_k$ holds*

$$\mathrm{Re}(\lambda_k) < 0. \tag{1.33}$$

**Example 1.6 (continued):** We want to design a stabilizing controller $K$ for our inverted pendulum system, as described before. The pendulum must be standing up, so the reference value will be $r = 0$. Suppose the controller depends linearly on the position of the pendulum, and the chosen gain will be $k$, then

$$A - kBC = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{I} \end{bmatrix} - k \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} - \frac{k}{I} & -\frac{b}{I} \end{bmatrix}. \tag{1.34}$$

According to Theorem 1.10(i), we know that if we want to obtain a system which is asymptotically stable, the real values of the eigenvalues of equation (1.34) must be strictly negative. To achieve this with a controller gain of $k_s$, the characteristic polynomial

$$\det\left(\begin{bmatrix} \lambda & -1 \\ -\frac{g}{l} + \frac{k_s}{I} & \lambda + \frac{b}{I} \end{bmatrix}\right) = \lambda\left(\lambda + \frac{b}{I}\right) - \frac{g}{l} - \frac{k_s}{I} = \lambda^2 + \frac{b}{I}\lambda - \frac{g}{l} - \frac{k_s}{I} \tag{1.35}$$

must have strictly negative solutions, and similar to equation (1.31) and the Routh test we see

$$\frac{k_s}{I} - \frac{g}{l} > 0, \qquad k_s > \frac{gI}{l}, \qquad k_s > mgl, \tag{1.36}$$

which is in words quite intuitive; the control torque must be necessarily higher than the torque due to gravity to obtain a stable system.

### 1.2.2 Damping

Consider the following system, written as ordinary differential equation and also as a state-space respresentation:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\zeta\omega_0 \frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0, \qquad \dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \boldsymbol{x}(t). \tag{1.37}$$

**Definition 1.15 (Damping [16]).** *The system* (1.37) *is*

*(i)* Overdamped, ($\zeta > 1$)*: The system returns to its equilibrium without oscillating. Larger values of the damping ratio return to the equilibrium slower.*

*(ii)* Critically damped, ($\zeta = 1$)*: The system returns with the minimum amount of damping to its equilibrium point, without oscillating.*

*(iii)* Underdamped, ($0 < \zeta < 1$)*: The system oscillates (with $\omega < \omega_0$) with the amplitude gradually decreasing to zero.*

*(iv)* Undamped, ($\zeta = 0$)*: The system oscillates at its natural frequency ($\omega_0$).*

**Example 1.6 (continued):** We want to design a controller which makes the system (1.30) critically damped. Therefore, we consider equation (1.34) and search for a $k$, such that $\zeta = 1$ in equation (1.37). Looking at these equations, we see that

$$-\frac{b}{I} = -2\omega_0 \qquad \text{and} \qquad \left(\frac{g}{l} - \frac{k_c}{I}\right) = \omega_0^2. \tag{1.38}$$

Solving these equations gives

$$\left(-\frac{b}{I}\right)^2 = -4\left(\frac{g}{l} - \frac{k_c}{I}\right) \quad \rightarrow \quad k_c = mgl + \frac{b^2}{4I}. \tag{1.39}$$

When $k_c$ becomes even more larger, the system will be underdamped, according to Definition 1.15. While the amplitude is still decreasing to zero, it will not decrease monotonically; the system oscillates. In our case, the applied torque is linearly related to the pendulum-position. Oscillation of the system means that the controller also oscillates; the direction of the application of a torque will change. If the pendulum is securely fixed into its pivot, then this switching is no problem, but when there is a kind of slackness in the pivot position, then this switching behavior will play a more important role. Controlling such a dynamic system then becomes a more complex problem. To handle this, we take a closer look on hysteresis, which will be done in Chapter 2.

# Chapter 2

# Introduction to Hysteresis

The phenomenon hysteresis is a broad subject which has branches in physics, chemistry, mechanics and economics. Mathematical generalizations were made in the 70's by Krasnoselskii et al. [11]. To introduce the problems which arise due to hysteresis, the phenomenon is described and an example will be explored. This simplified example will be used to handle problems, which could be a solution of more complicated situations.

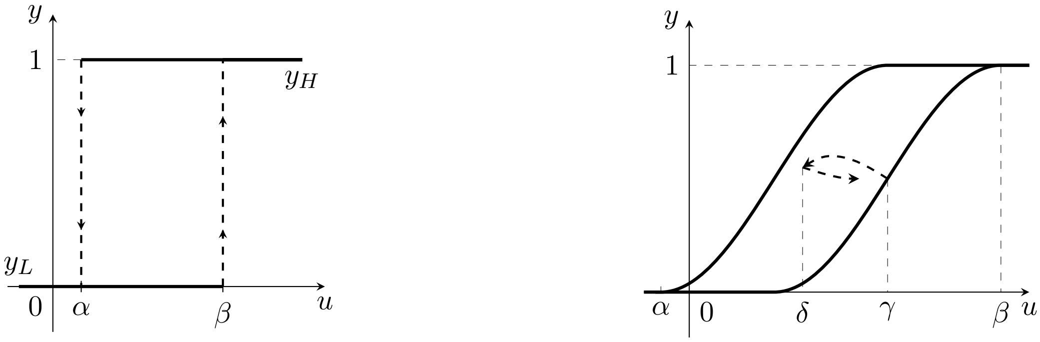
## 2.1 The phenomenon of hysteresis

Hysteresis is the effect that a system not only depends on its current state, but also on its past. The word is derived from ὑστέρησις, an ancient Greek word meaning "deficiency" or "lagging behind". That is because of the character of the relation between input and output; with a regular system, an input has direct influence on the state, while a hysteretic system can have some delay between input and state, but even more: it depends on the past of the input how the output behaves. Two general types of hysteresis can be described [13]: *relay hysteresis* and *active hysteresis.*

### 2.1.1 Relay hysteresis

Relay hysteresis can be described by an input $u(t)$ which gives an output $y_L$(t) when $u(t)$ is below a certain threshold $\alpha$, and $y_H(t)$ when it is above another threshold $\beta$, with $\alpha < \beta$. Between those thresholds, the system will maintain the value of the last threshold that is attained, so it is indeed "lagging behind". Remark that there is only switching at the thresholds and nowhere else. Therefore this type of hysteresis is also called *passive* hysteresis. Based on Mayergoyz [15] and Rasskazov et al. [19], we can represent this behavior formally with the following equations, restricted to $t \geq t_0$:

$$y(t) = \mathcal{H}_{\alpha,\beta}[t_0, y_0, u(t)] = \begin{cases} y_0, & \text{if } \alpha < u(\tau) < \beta, \text{ for all } \tau \in [t_0, t] \\[2ex] y_H(t), & \text{if there exists } t_1 \in [t_0, t] \text{ such that} \\ & u(t_1) \geq \beta \text{ and } u(\tau) > \alpha \text{ for all } \tau \in [t_1, t] \\[2ex] y_L(t), & \text{if there exists } t_1 \in [t_0, t] \text{ such that} \\ & u(t_1) \leq \alpha \text{ and } u(\tau) < \beta \text{ for all } \tau \in [t_1, t] \end{cases} \tag{2.1}$$




**(a)** A non-ideal elementary relay $\mathcal{R}_{\alpha,\beta}$, or *hysteron*, with only two places where switching is possible.

**(b)** A hysteresis loop of active hysteresis. It allows behavior inside the hysteretic region.

**Figure 2.1:** Two types of hysteresis: *relay* or *passive* (Fig. 2.1a) and *active* (Fig. 2.1b) hysteresis.

where $y(t)$ is the output function with $y_0 \in \{y_L(t_0), y_H(t_0)\}$ as the initial output, $u(t)$ is the input function and $\mathcal{H}_{\alpha,\beta}[t_0, y_0, \cdot]$ describes the relay. In most cases and also in this report, $y_L(t)$ and $y_H(t)$ do not depend on time, therefore they can be denoted as $y_L$ and $y_H$. However, the output depends on time, because the function will attain other values at certain switch times. Between these switch-times, the output is constant. The values $\alpha$ and $\beta$ are the points where the relay will switch: at $\alpha$ the relay will switch from 'high' to 'low', at $\beta$ the relay will switch from 'low' to 'high'. Investigation shows that if $u(t) \leq \alpha$, the output is always $y(t) = y_L$ and $u(t) \geq \beta$ implies $y(t) = y_H$. The natural assumption is made that $\alpha < \beta$. Figure 2.1a shows an example of an elementary (non-ideal) relay function $\mathcal{R}[u(t)]$, with $y_L = 0$ and $y_H = 1$. Krasnoselskii and Pokrovskii called this basic hysteretic operator a *hysteron* [11]. This function can easily be shifted, scaled, stretched and rotated, to achieve every desired *relay* hysteretic function.

This type of hysteresis is used in favor of controlling systems which have switching behavior. It is often undesirable to have chattering in the control, think for example of a central heating, which must be switched on and off. It is not desirable to switch extremely fast between these states. Automatic gear transmission uses also relay hysteresis. For example, take an automatic gear box that shift gear up at 50 km/h, and shift gear down at 40 km/h. Imagine what happens if this gear box will shift both up and down at 45 km/h, and that speed is exactly the desired speed of the car. Then it will shift gear frequently, due to irregularities. However, in this thesis we only discuss continuous states, where relay hysteresis is often an undesirable phenomenon.

### 2.1.2 Active hysteresis

In contrast to relay hysteresis, *active* hysteresis allows behavior inside the hysteretic region. Figure 2.1b is an illustration of active hysteresis, in the situation that input $u$ increases to $\gamma$, then decreases to $\delta$ and increases afterwards. Therefore, this behavior is essentially different to relay hysteresis, and has a richer transition state. There are several mathematical models to describe this kind of behavior. Two of them will be pointed out here, based on Macki et al. [13], in which a more extended overview can be found. We encourage the reader to look at that whole paper.

**Example 2.1 (Controlling temperature):** A train driver wants to model the control of the temperature $T$ in the boiler of his steam locomotive. When the temperature is below $\alpha = 90$, he starts to insert extra coals. The temperature evolves according $\dot{T} = -\frac{T}{20} + 8$. When the temperature reaches $\beta = 130$, the train driver stops with inserting coal. The temperature decreases then with $\dot{T} = -\frac{T}{20}$. The initial temperature is $y_0 = 50$, so he can describe this temperature behavior with

$$\dot{T} = -\frac{T}{20} + \mathcal{H}_{[90,130]}(0, y_0, T), \tag{2.2}$$

$$\text{with} \quad y_L = 8,\ y_H = 0. \tag{2.3}$$

When he only wants to use elementary relay hysterons $\mathcal{R}$, he can write this as

$$\dot{T} = -\frac{T}{20} + 8\big(1 - \mathcal{R}_{[90,130]}(T)\big). \tag{2.4}$$

Remark that this system is non-linear, due to the hysteron.

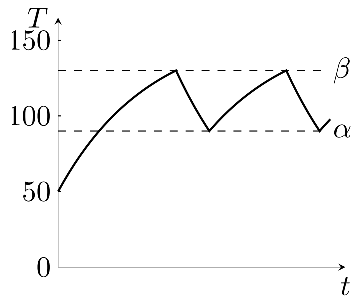



**Figure 2.2:** Behavior of the temperature of the boiler, as explained in Example 2.1. When the thresholds $\alpha$ and $\beta$ are reached from above and below respectively, the hysteron $\mathcal{H}$ switches. When the hysteron is switched off, the temperature increases, while the temperature decreases when it is on.

**Duhem model** The Duhem model of hysteresis is based on the fact that the behavior of the system only changes its character when the input changes direction. This relation is given by the following differential equation:

$$\dot{y}(t) = f_I(y,u)\dot{u}^+(t) + f_D(y,u)\dot{u}^-(t) \tag{2.5}$$

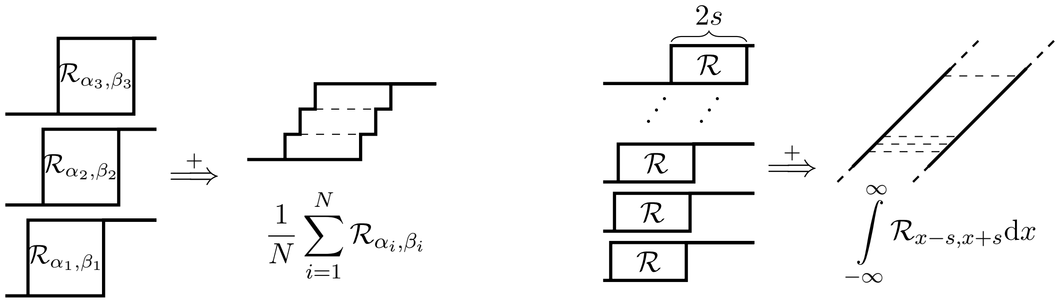
with $\dot{u}^+(t) = \max[0, \dot{u}(t)]$, $\dot{u}^-(t) = \min[0, \dot{u}(t)]$ and where $f_I$ and $f_D$ are the functions for the increasing and the decreasing input respectively. A slightly different, often used representation of the same behavior could be

$$\dot{y}(t) = \begin{cases} f_D(y,u)\dot{u}(t) & \text{for } \dot{u}(t) \leq 0 \\ f_I(y,u)\dot{u}(t) & \text{for } \dot{u}(t) \geq 0 \end{cases}. \tag{2.6}$$

**Preisach model** Probably the most used model of active hysteresis is the Preisach model. This model is developed by Preisach, to describe the hysteretic behavior of magnetization of a coil core of an FeNi-alloy [18]. In fact, this model is an infinite sum of relay functions according to

$$y(t) = \iint \mu(\alpha,\beta)\mathcal{R}_{\alpha,\beta}[u(t)]\,\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{2.7}$$

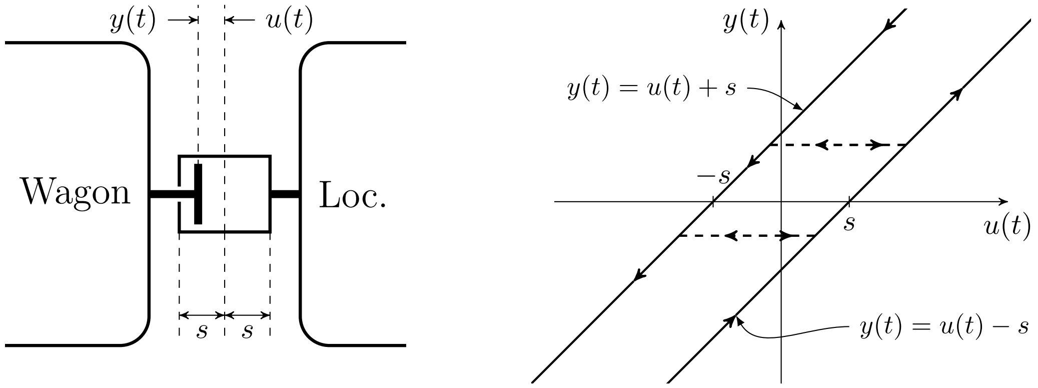
where $\mu(\alpha,\beta)$ is a non-negative weight function, also called *Preisach measure*. This weight function usually has compact support in the $(\alpha,\beta)$-plane [13]. The $\mathcal{R}_{\alpha,\beta}[u(t)]$ is an elementary hysteron with switches at $\alpha$ and $\beta$. The idea of the summation of elementary hysterons with given weight and given relay intervals is given in Figure 2.3. This summation gives the flexibility to model a lot types of hysteresis. To show how this model describes a system, we give a modified example of a mechanical play, based on [5, 11, 13].




**(a)** Three hysterons summated, $N = 3$.

**(b)** Integration of (relay) hysterons creates active hysteretic behavior.

**Figure 2.3:** Summation of elementary relays is the idea behind active hysteresis in a Preisach model. In this sketch, all hysterons have equal weight.




**(a)** Slackness in coupling between locomotive and wagon, as described in the example.

**(b)** Relation between the position of the locomotive $u(t)$ as input, and the position of the rail wagon $y(t)$ as output, with given slackness $2s$.

**Figure 2.4:** Schematic view of behavior of slackness in a coupling, according to Example 2.2.

**Example 2.2 (Train shunting):** The train driver has to shunt a rail wagon. On the shunting school he has learned that there is some slackness in the coupling of the locomotive and the wagon, see Figure 2.4a. Suppose that this slackness is $2s$, and the position of the locomotive is the input $u(t)$. Furthermore, suppose that the wagon has low mass, and a lot of friction with the tracks (the unexperienced train driver forgot to take off the handbrake). The driver knows that if he simply moves the locomotive forward, the position of the wagon $y(t)$ is eventually given by $y(t) = u(t) - s$, until he stops. But when he reverses the locomotive, it does not affect the wagon directly. After a locomotive displacement of $-2s$ from the reversing point, the wagon starts to move, according to $\dot{y} = \dot{u}$ and its position is given by $y(t) = u(t) + s$. He draws this relation in a figure and calls it Figure 2.4b.
The relation between input and output is given in terms of elementary relay hysterons, as defined in Figure 2.1a, using the superposition principle of equation (2.7). We know that all our relays have equal weights ($\mathrm{d}x$) and constant width ($2s$), and therefore if we take the elementary hysterons

$$\mathcal{R}_{\alpha,\beta}[u(t)] = \begin{cases} -1 & \text{if } u < \alpha, \\ 1 & \text{if } u > \beta, \\ \text{unchanged} & \text{if } \beta < u < \alpha, \end{cases} \tag{2.8}$$

then the operator is given by

$$y(t) = \mathcal{P}_s[u](t) = \frac{1}{2} \int_{-\infty}^{\infty} \Big( \mathcal{R}_{x-s,x+s}[u(t)] \Big) \mathrm{d}x, \tag{2.9}$$

with the initial values of the ambiguous hysterons (thus, where $-s < x < s$) attain either $+1$ or $-1$, such that they describe properly the initial output.

In the previous example, we see that the weight and intervals of the hysterons remains constant, but in general, this is not the the case. Then it is necessary to integrate over two variables, which also shown up in expression (2.7).
Looking at this previous example, this hysteretic operator $\mathcal{P}$ can also be given by

$$y(t) = \mathcal{P}_s[u, y_0](t) = \min\big(u(t) + s, \max(u(t) - s, y(t_i))\big), \tag{2.10}$$

where the $t_i$ are times, such that $0 < t_1 < t_2 < \ldots < t_n$ and $u(t)$ is monotonically increasing or decreasing on $[t_i, t]$. In fact, the $t_i$'s are switching times; it are the moments at which the input changes direction.
The two Examples 2.1 and 2.2 show that relay hysteresis and active hysteresis are essentially different. The behavior of a system with relay hysteresis is restricted to the two possible outcomes, $\mathfrak{B} = \{\, \{u, y\} \mid y \in \{y_L, y_H\} \,\}$. There the behavior of the system switches instantanously from one to another state. A system with active hysteresis can freely exist on the whole diagonal band, $\mathfrak{B} = \{\, \{u, y\} \mid y \in [u{-}s, u{+}s]\}$.
The last example can also be represented with a Duhem model, because the switching behavior depends on the input direction. If we define the functions $f_D$ and $f_I$, used in equation (2.6), as $f_D(u, y) = 1 + \mathrm{sgn}(y - (u{+}s))$ and $f_I(u, y) = 1 - \mathrm{sgn}(y - (u{-}s))$, we get

$$\dot{y}(t) = \begin{cases} [1 + \mathrm{sgn}(y - (u{+}s))]\dot{u}(t) & \text{for } \dot{u}(t) \leq 0 \\ [1 - \mathrm{sgn}(y - (u{-}s))]\dot{u}(t) & \text{for } \dot{u}(t) \geq 0 \end{cases}, \tag{2.11}$$

where $\mathrm{sgn}(\cdot)$ is defined as

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}. \tag{2.12}$$

## 2.2 Hybrid dynamical systems

Based on Heemels and De Schutter [6], we can model a system subject to hysteresis in a third way, namely as a hybrid dynamical system. The idea behind hybrid dynamical systems is that a system will remain in a node, and behave according to certain dynamics as long as the invariants of that particular node are hold. If this invariant becomes false, the system will switch towards a connected node. Before it switches to an other node, it has to fulfil the requirements of a guard to this node. Before the system enters the new node, an optional reset is applied to the state variables.

Both relay and active hysteresis can easily be modelled by automatons, with invariants, guards and resets. A sketch is given in Figure 2.5. This specific scheme models active hysteresis. However, relay hysteresis can be modelled with the same philosophy. Then the *transition state* is removed, and the left and right bound are directly linked to each other.

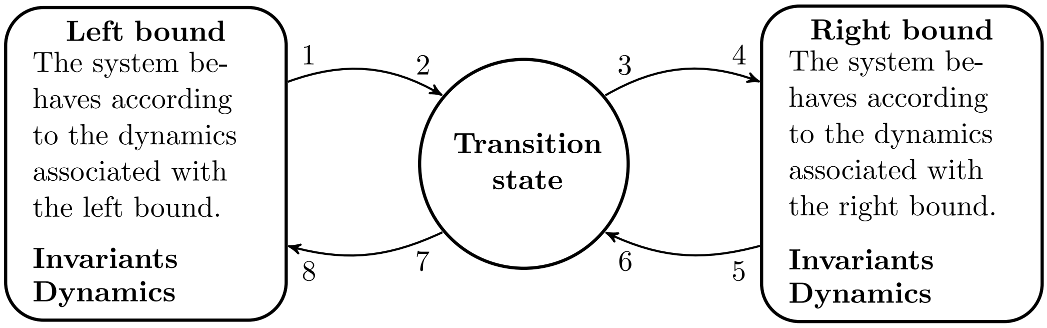



**Figure 2.5:** Several states of hysteresis, used in hybrid system modelling. The odd numbers refer to guards, the even numbers refer to resets.

> **Example 2.2 (continued):** Recall the example of the train shunting problem. The dynamics of the states which represent the bounds are given by: $\dot{y} = \dot{u}$. The dynamics of the transition state are $\dot{y} = 0$. Furthermore: Invariants of the left and right bound will be respectively $\dot{u} \leq 0$ and $\dot{u} \geq 0$. The guards (1) and (5) are given by $\dot{u} > 0$ and $\dot{u} < 0$. The invariant of the transition state will be $u - s < y < u + s$, with the corresponding guards (3) and (7): $y = u - s$ and $y = u + s$ respectively. All resets are optional in this case. They will only be used when this system is implemented and some numerical problems occur.
> Remark that switching only occurs when an invariant of a node is not fulfilled any more. If the system is in the left node and $\dot{u} > 0$, it switches towards the transition state. When short thereafter $\dot{u} \geq 0$, the system remains in the transition state. It only switches back when $y = u + s$.

## 2.3 Hysteretic dynamical system

In this section, we model a hysteretic system, and point out what the difficulties are in controlling such a system. Thereafter we modify our pendulum, described in Example 1.6, such that it illustrates the problems rising due to hysteresis.
A general hysteretic dynamical system can be modelled by the scheme given in Fig-

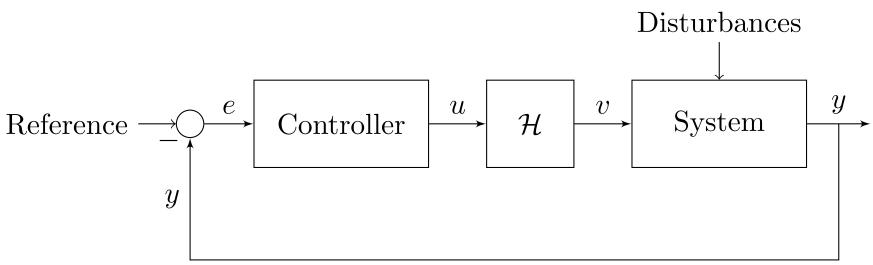



**Figure 2.6:** A dynamical system, c.f. Figure 1.1 but with a hysteretic element $\mathcal{H}$. The output of this element feeds the system block, the element on which we have set the goal to control it properly.

ure 2.6. Variables $e$, $u$, $v$ and $y$ are used as sketched in this figure. In this report, measurement errors are neglected, while disturbances in the system are taken into account. Of course, the hysteretic element can also be situated inside the feedback loop, or (similar) directly after the system. This implies that the measurements are lacking behind the true values of the output of the system. We will only discuss the case where the hysteretic element is between controller and system, as it is sketched in the figure.
In all cases, we assume that the system itself contains regular linear dynamics. A non-linear situation can be described, but it will always be linearized. Thus the system block can be described by

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bv(t) \\ y(t) &= Cx(t) \end{cases} \tag{2.13}$$

with $v$ as input variable and $y$ as output variable, as explained in Chapter 1.
In the simple case, the controller is a static function of the error $e$, the difference between reference value and output of the system. However, the model can be extended such that the controller contains dynamics. This will be explored in the next chapter.

Due to the character of hysteresis, the hysteretic function $v = \mathcal{H}(u)$ does contain dynamics, since the temporary past of the input $u$ plays a role in the output $v$. This element can be described in a Duhem or Preisach representation. However, the whole system can also be modelled according to a hybrid dynamical system. This is all illustrated with an example.

**Example 2.3 (Slackness in inverted pendulum):** We continue with Example 1.6, but modify it such that it illustrates the problems of controlling hysteretic systems. Suppose that the pendulum is mounted in a disk with a (small) slackness of $2\varphi$, as depicted in Figure 2.7. The angle of the pendulum becomes $\theta_p$ and suppose its inertia is $I_p$. Investigation of the physics of this model shows that the equilibria of the pendulum will still be $\bar{\theta}_p = 0$ and $\bar{\theta}_p = \pi$. Further, we neglect (1) the gravitational influence on the disk $\tau_{g,d}$. This is admissible, because of the assumed (2) small size of $\varphi$. A small size of $\varphi$ implies $I_d \approx \frac{1}{2}MR^2$ and a small shift of center of mass, $l_{\text{cm}} \approx 0$. Moreover, this value can be neglected in comparison to the radius of the disk, $R$, and therefore

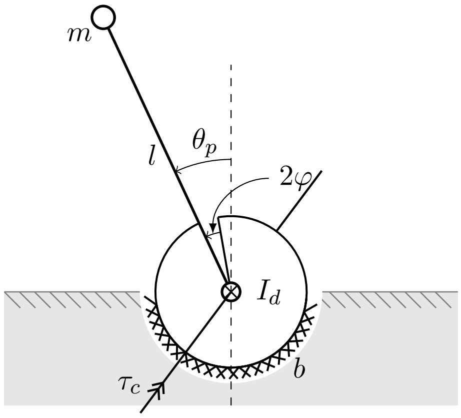
$$\tau_{g,d} = \frac{Mg_z l_{\text{cm}}}{I_d} \overset{(2)}{\approx} \frac{g_z l_{\text{cm}}}{\frac{1}{2}R^2} \overset{(1)}{\approx} 0. \tag{2.14}$$

Neglecting this gravitational influence causes that all positions of the disk are equilibria. However, since the disk and pendulum are connected to each other, upwards equilibria are all positions where $\theta_p = 0$ and $-\varphi < \theta_d < \varphi$. From now on, we denote this continuum of equilibrium points as $\Omega$, which is in words nothing else then the system in rest, with the pendulum standing up and the disk within the range of the slackness.
According to classical mechanics, the following linearized equations hold:

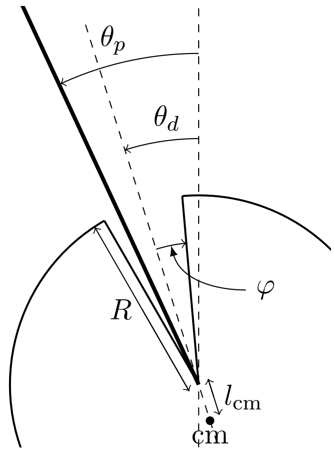
$$I_p\ddot{\theta}_p = mg_z l\theta_p - \tau_n \tag{2.15}$$

$$I_d\ddot{\theta}_d = -b\dot{\theta}_d + \tau_n + \tau_c, \tag{2.16}$$

where $\tau_n$ is the normal torque created by the force of the disk acting on the pendulum. The variable $\tau_c$ is the torque which is performed by an external input, to make it possible to control the system. Notice that the system block of this model remains a dynamical system, c.f. Example 1.6. We assume that (3) the system will be coupled most of the time. Therefore,



**(a)** Modification of Figure 1.3: The pendulum is mounted on a disk with inertia $I_d$



**(b)** Detailed view of the position of angles and the center of mass ($\mathrm{cm}$)

**Figure 2.7:** The pendulum and the pivot, with a slackness $2\varphi$.

although the pendulum is frictionless mounted in the disk, it can be still modelled with the damping constant, as in equation 1.30. The state-space equation of the pendulum becomes

$$\dot{\theta}_p(t) = \begin{bmatrix} 0 & 1 \\ \frac{g_z}{l} & \frac{-b}{I_p} \end{bmatrix} \theta_p(t) + \begin{bmatrix} 0 \\ \frac{1}{I_p} \end{bmatrix} v(t). \tag{2.17}$$

The disk has also dynamics. Suppose it has an inertia of $I_d$ and a friction linearly depending on the velocity with coefficient of $b$, then the state-space equation becomes

$$\dot{\theta}_d(t) = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-b}{I_d} \end{bmatrix} \theta_d(t) + \begin{bmatrix} 0 \\ \frac{1}{I_d} \end{bmatrix} u(t). \tag{2.18}$$

The normal torque acts on the pendulum if and only if the angular difference of the position of the disk and the pendulum is equal to the slackness, $|\theta_p - \theta_d| = \varphi$. When the pendulum behaves inside the slackness, $|\theta_p - \theta_d| < \varphi$, no external forces but the gravitational force will act on the pendulum. It will behave autonomous, until the angular difference will be $\varphi$ again. We model this gap between disk and pendulum as $g(t) = \theta_d(t) - \theta_p(t)$. This gap behaves dynamical, and it must certainly be taken into account. Observations reveal that if $g = -\varphi$, $g$ will be non-decreasing and moreover only increasing when the input is smaller than zero. For $g = \varphi$ the opposite holds, because of symmetry. When the state is in between these values, the evolution of $g$ depends on the input. Further, we assume $(4)$ that $I_d$ is small compared to $I_p$. This causes that the disk moves quicker than the pendulum, when certain control is applied. The evolution of the gap depends on the velocities of disk and pendulum. Since the disk moves quicker than the pendulum, we can assume $(5)$ that the velocity of the pendulum has negligible influences on the rate of change of the gap.
With these assumptions we state

$$\dot{g} = \dot{\theta}_d - \dot{\theta}_p \overset{(5)}{\approx} \dot{\theta}_d, \qquad I_d\ddot{g} \overset{(5)}{\approx} I_d\ddot{\theta}_d = -b\dot{\theta}_d - u \overset{(4)}{\approx} 0. \tag{2.19}$$

When the gap is already $\pm\varphi$, the gap remains constant, unless the input is in the direction of the gap, so we describe this hysteretic element as follows

$$v = \mathcal{H}(u) = \begin{cases} u & \text{if } g = -\varphi \\ 0 & \text{if } -\varphi < g < \varphi \\ u & \text{if } g = \varphi \end{cases} \qquad \text{with} \qquad \dot{g} = \begin{cases} \max(0, \frac{u}{b}) & \text{if } g = -\varphi \\ \frac{u}{b} & \text{if } -\varphi < g < \varphi \\ \min(0, \frac{u}{b}) & \text{if } g = \varphi \end{cases} \tag{2.20}$$

where $g$ is the state variable of the hysteresis, which serves as memory.
Remark that

$$\dot{g} = \begin{cases} \max(0, u) & \text{if } g = -\varphi b \\ u & \text{if } -\varphi b < g < \varphi b \\ \min(0, u) & \text{if } g = \varphi b \end{cases} \tag{2.21}$$

is another representation of the same input-output behavior. Although the internal state behaves different, the switches are still similar.
This example is simulated with MATLAB, with a normal, underdamped controller, as described in the previous chapter. The values for the parameters are given in Table 2.9. A plot of the state of the pendulum is given in Figure 2.8. We see two types of behavior. (I) The approach of the equilibrium as a curved line, due to the controller which depend on the position of the pendulum. In this situation the pendulum is coupled to the disk, and $\mathcal{H}(u) = u$. (II) The uncoupled pendulum, as a straight line. In this situation, the pendulum is inside the hysteresis, and $\mathcal{H}(u) = 0$. Therefore, the pendulum behaves as a free fall.

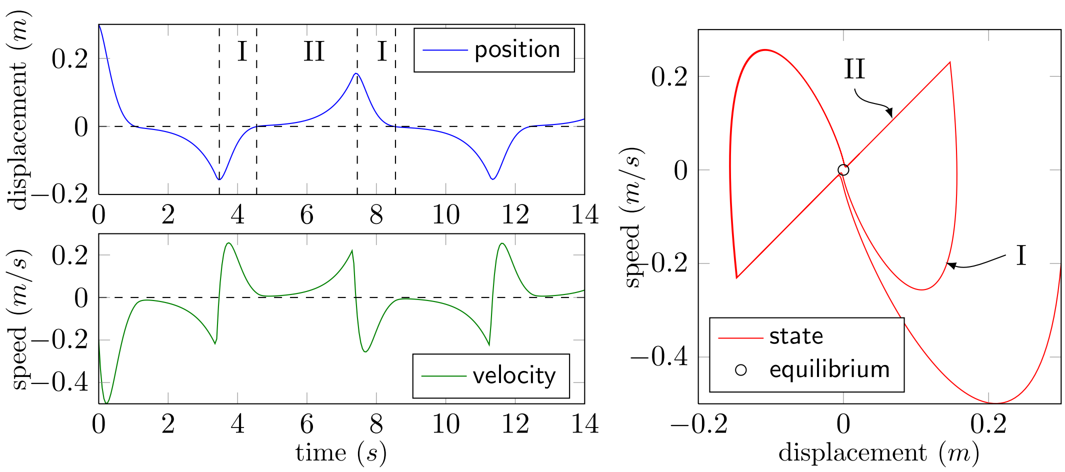



**Figure 2.8:** Simulation of an inverted pendulum with slackness in the pivot position. An underdamped feedback controller is used. Left: the position (blue) and velocity (green) of the pendulum, as function of the time. Right: The phase plot (red) of the pendulum. The equilibrium (black) is shown at $(0, 0)$.

**Duhem model** Looking atthe normal torque $\tau_n$ as input of the pendulum, the behavior changes its character when this input changes sign. Therefore, $\tau_n$ should be the output $v$ of the hysteresis. The input of this element is given by the controller, as sketched in Figure 2.6.
Duhem modelling uses the change of direction, by using the derivatives of input and

**Table 2.9:** Physical constants of the example.

| | | | |
|---|---|---|---|
| $m = 0.5$ | Mass of pendulum | $I_p = ml^2$ | Inertia of pendulum |
| $l = 0.8$ | Length of pendulum | $\varphi = \pi/16$ | Slackness of disk |
| $g_z = 9.81$ | Gravitational constant | $[\theta_0, \dot{\theta}_0] = [0.3, 0.2]$ | Initial value of the state |
| $b = 2$ | Damping constant | $g_0 = -\varphi$ | Initial value of the gap |

output, respectively $\dot{u}$ and $\dot{v}$. If we take the regular input $u$ instead of its derivative, the behavior of the system can be described, while this small modification preserves the Duhem concept. The mapping of the hysteresis from input to output can be given by

$$v = \begin{cases} \left[1 - \text{sgn}(\int u/b + g_i + \varphi)\right]u, & \text{if } u \leq 0 \\ \left[1 - \text{sgn}(\int u/b - g_i - \varphi)\right]u, & \text{if } u \geq 0 \end{cases} \tag{2.22}$$

The variable $g_i$ is the state of the hysteresis, at the $i$-th switch point. This is not known but regularly $g_i = \varphi$ if the above described function switch towards the $u \leq 0$-part. Vice versa, $g_i = -\varphi$ if the function switch towards the $u \geq 0$-part. Further, $\int u/b$ is the integral of the input $u$ from $t_i$ to $t$. Here, $t_i$ is the time of the $i$-th switch point. This description is not directly corresponding to the template which is given previously. However, instead of variable and its derivative, this function uses a variable and its integral. The philosophy of Duhem modelling is kept.

Because it is assured that $g \in [-\varphi, \varphi]$, the output $v$ must be an element of $\{0, u\}$. The fact that there are two possible outcomes triggers to model this phenomenon also with relay hysterons.

**Hybrid model** The way of modelling the problem in the previous example is more or less the hybrid dynamical system philosophy, classified as a piecewise affine system. However, it should be noticed that $\dot{g}$ is orthogonal to $v$. They can not both be non-zero. This notion is used within the class of linear complementarity systems, and the system can be modelled as

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bv(t) \\ y(t) &= Cx(t) \\ \dot{g}(t) &= u(t) - v(t) \end{cases} \tag{2.23}$$

$$\dot{g}(t) \perp v(t) \tag{2.24}$$

**Relay model** Because the system has three states, the system can be modelled with multiple relay hysterons. Again, $\tau_n$ will be the output of the model. This model can be made by combining two relay hysterons. We describe the two hysterons following the precise definition. First we define the outcomes: it must be zero when the pendulum and the disk are uncoupled, and otherwise it should be equal to the input. Also the switch points are given: when the gap $g$ reaches $\varphi$ or $-\varphi$, the model must switch from 0 to $u$. When the supplied torque $u(t)$ changes sign, the model switches from $u$ to 0. This leads to the following description, where the hysterons are already combined.

$$v(t) = \mathcal{H}[t_0, v_0, g(t), u] = \begin{cases} v_0, & \text{if } -\varphi < g(\tau) < \varphi, \text{ for all } \tau \in [t_0, t] \\ u, & \text{if there exists } t_1 \in [t_0, t] \text{ such that} \\ & g(t_1) \geq \varphi \text{ and } u(\tau) > 0 \text{ for all } \tau \in [t_1, t] \\ 0, & \text{if there exists } t_1 \in [t_0, t] \text{ such that} \\ & u(t_1) = 0 \text{ and } -\varphi < g(\tau) < \varphi \text{ for all } \tau \in [t_1, t] \\ u, & \text{if there exists } t_1 \in [t_0, t] \text{ such that} \\ & g(t_1) \leq -\varphi \text{ and } u(\tau) < 0 \text{ for all } \tau \in [t_1, t] \end{cases} \tag{2.25}$$

Remark that the range of this relay function is given by $\{0, u\}$ instead of the regular $\{0, 1\}$. However, it can be written as a regular relay, which can be eventually multiplied by $u$. Only the initial condition $v_0$ must be redefined when the function is rewritten.

Figure 2.10 gives the relation between input $u(t)$ and output $v(t)$ of the hysteretic element. In the example, the input $u(t)$ is the torque which is supplied by the controller. The output $v(t)$ will be the normal torque, the torque of the disk which act on the pendulum.

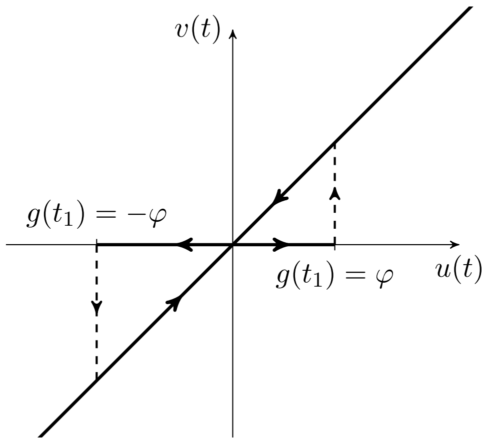



**Figure 2.10:** Sketch of the relation between input and output of the hysteretic element.

Again we see that the state $g$ of the hysteresis is a variable, and necessary to model this phenomenon. Since the derivative $\dot{g}$ in equation (2.21) is defined by terms of $u$, it is possible to describe the switching time $t_1$ as an integral in terms of $u$.

$$g(t_1) = \frac{1}{b} \int_{t_0}^{t_1} u(\tau) \mathrm{d}\tau = \pm\varphi, \quad \text{with} \quad g(t_0) = \mp\varphi \tag{2.26}$$

However, in all cases the recent past of the input must be taken into account when the hysteron is modelled. This 'lagging' has strong influences on the behavior of the system when feedback control is applied. This is what we will investigate in the next chapter.

## 2.4 Controllability of a hysteretic system

In the first chapter, a definition of controllability is given. If we consider the system

$$\begin{cases} \dot{x}(t) &= Ax(t) + B\mathcal{H}[u(t)] \\ y(t) &= Cx(t) \end{cases} \tag{2.27}$$

we can apply the given test in Theorem 1.12. If the controllability matrix

$$\mathfrak{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{2.28}$$

has no full row rank, certainly the system is not controllable, so this condition is necessary. However, since the input of the system is not always directly the input which is supplied by the controller, the sufficiency should be questioned.
Therefore, a closer look at the hysteron must be taken. Questions about this controllability are passed on to the next chapter. First the assumptions which are made are noticed.

## 2.5 General assumptions

In this section, the assumptions which are made in this assignment are summed up. It was already mentioned in the previous section, that the hysteretic element occurs between controller and the system. Also, disturbances occur only in the state of the system, not in the hysteron states or in the output of the controller. In line with the previous assumption, also no measurements errors are taken into account. Further, the system itself is linear, such that its dynamics can be written as an LTI system.
Many types of hysteresis can occur. Each hysteretic function has a certain number of possible states. In addition, each state has a specific mapping from input to output. Finally, for each state, transitions to other states are defined. If we should describe all the different types of hysteresis, and thereby design a controller for the general case, this would take ages. Therefore we restrict ourselves to a specific category of hysteresis that meets the following requirements:

**States** The hysteretic function $\mathcal{H}$ in this report, has always three states, $S_1$, $S_2$ and $S_3$. From states $S_1$ and $S_3$, the hysteresis could possible switch to state $S_2$, as drawn in the sketch of Figure 2.5. From $S_2$ both other states are reachable. The state $S_2$ is thereby, in all cases a transition state.

**Mapping** The states $S_1$ and $S_3$ both have a mapping in such way that the output is linearly proportional to the input. This results in the fact that if the input is unbounded, the output also is. This excludes saturation. The other restriction to our hysteretic function is that the output of state $S_2$ is always zero.

**Transitions** The transition state is relatively small, although (trivially) not empty. When $S_2$ is empty, hysteresis is possible, but it acts as a piecewise function where the input always propagates to the output. Furthermore, we assume that the moment that a transition takes place, depends on the input of the hysteretic function. This can be a highly non-linear relation, but may not be restricted or limited by a certain time. It also may not depend on time.

With these restrictions, the hysteretic function can in general be described as

$$\mathcal{H}(u) = \begin{cases} \alpha u & \text{if } g \in S_1 \\ 0 & \text{if } g \in S_2 \\ \beta u & \text{if } g \in S_3 \end{cases} \tag{2.29}$$

where $g$ behaves as a dynamic variable according to

$$\dot{g} = f_i(u) \quad \text{if} \quad g \in S_i \quad \text{for} \quad i = 1, 2, 3. \tag{2.30}$$

These assumptions are made to make sure that the hysteretic region of a system exists, but only has a local influence on the behavior. The example of the inverted pendulum is a realization of this kind of hysteresis, but in general, mechanical systems with a certain slackness in a joint can be described by this type of hysteresis. That slackness is represented by the transition state, while the normal behavior of the system is described by the input propagating states.
Other systems, like the example of temperature controlling, have only two states. Although this kind of hysteresis is interesting for control design, that type will not be discussed in this assignment.

At last, an assumption is made with respect to the input of the system. In all cases, this will be a scalar. In the real world, this input could be a force, a torque, a voltage, etc. All cases where the input is multi dimensional are not taken into account. The definition of *preserving sign* is then easy to explain: If the input is once positive, it remain positive and if it is negative, it stays negative. With this assumption, the dimensions of the input matrices are also fixed: $B \in \mathbb{R}^{n\times 1}, N \in \mathbb{R}^{1\times n}$.

## 2.6 Explicit trajectory

In this last section of this chapter, a brief analytical description of a general trajectory is given. After this generalization, the trajectory is investigated of Example 2.3. To ease the calculations, the time $t$ is reset to zero after a switch-action.
Since the system has two different kinds of behavior, coupled ($g \in \{S_1, S_3\}$) and uncoupled ($g \in S_2$), we can determine how the behavior will evolve piecewise. By assumption, we see that in $S_2$ the system can be rewritten as

$$\dot{x}(t) = Ax(t), \tag{2.31}$$

with initial conditions $\theta_0$, and thus a straightforward solution of

$$x(t) = \mathrm{e}^{At}x_0. \tag{2.32}$$

Assume that at time $t_1$, the hysteron will switch, and the final values are $x_1 = \mathrm{e}^{At_1}x_0$. Remark that this are also the initial values for the system when governed by the coupled equation. We can work this out by assuming a feedback control of $u = Nx$. The coupled system ($g \in \{S_1, S_3\}$) can then be written as

$$\dot{x}(t) = Ax(t) + Bu(t) = (A + BN)x(t), \tag{2.33}$$

with initial conditions $x_1$, and thus a straightforward solution of

$$x(t) = \mathrm{e}^{(A+BN)t-t_1}x_1. \tag{2.34}$$

In general, when $t_i$ is the $i$-th switch time of the hysteron, and $n$ is a counter of switches which already occurs at time $t$, and $M_i$ is the matrix which defines the behavior of the system between the switch times $t_{i+1}$ and $t_i$, we can describe the trajectory

$$x(t) = \mathrm{e}^{M_n(t-t_n)}\mathrm{e}^{M_{n-1}(t_n-t_{n-1})}\cdots\mathrm{e}^{M_1(t_2-t_1)}\mathrm{e}^{M_0(t_1-t_0)}x_0 \tag{2.35}$$

$$= \mathrm{e}^{M_n(t-t_n)}\Big(\prod_{i=0}^{n-1}\mathrm{e}^{M_i(t_{i+1}-t_i)}\Big)x_0 \tag{2.36}$$

This fact will be used when analysing some controllers in the next chapter. Before we switch to this chapter, we illustrate how this explicit trajectory can be worked out.

**Example 2.4 (Trajectory of inverted pendulum):** In this example we illustrate how the trajectory can be described. In the end, this is used to determine the switch times of the system. First, we write the system, starting in $S_2$ in a state space representation:

$$\dot{\theta}(t) = \begin{bmatrix} 0 & 1 \\ \frac{g_z}{l} & \frac{-b}{I_p} \end{bmatrix}\theta(t). \tag{2.37}$$

To solve this ordinary differential equation, we use the eigenvalues of the matrix, calculated in Example 1.6, and see that $\lambda_1 \neq \lambda_2$. This gives:

$$\theta(t) = c_1\mathrm{e}^{\lambda_1 t} + c_2\mathrm{e}^{\lambda_2 t} \tag{2.38}$$

where $c_1$ and $c_2$ can be solved using the initial values $\theta_0$ and $\dot{\theta}_0$:

$$\theta_0 = c_1 + c_2, \qquad \dot{\theta}_0 = \lambda_1 c_1 + \lambda_2 c_2 \qquad \Longrightarrow \tag{2.39}$$

$$c_1 = \frac{-\lambda_2\theta_0 + \dot{\theta}_0}{\lambda_1 - \lambda_2}, \qquad c_2 = \frac{\lambda_1\theta_0 - \dot{\theta}_0}{\lambda_1 - \lambda_2}. \tag{2.40}$$

Due to symmetry, we can assume that $g_0 = g(0) = -\varphi$, and then the time $t_1$, at which the hysteron switches, can be written as the solution of

$$g(t_1) = \varphi. \tag{2.41}$$

This implies

$$2\varphi = g(t_1) - g(0) = \int_0^{t_1}\dot{g}(\tau)\mathrm{d}\tau = \int_0^{t_1}\frac{1}{b}u(\tau)\mathrm{d}\tau \tag{2.42}$$

$$= \int_0^{t_1}\frac{k_c}{b}\Big(c_1\mathrm{e}^{\lambda_1\tau} + c_2\mathrm{e}^{\lambda_2\tau}\Big)\mathrm{d}\tau = \frac{k_c c_1}{b\lambda_1}\mathrm{e}^{\lambda_1\tau} + \frac{k_c c_2}{b\lambda_2}\mathrm{e}^{\lambda_2\tau}\Big|_{\tau=0}^{\tau=t_1} \tag{2.43}$$

Due to unstability of the system under $S_2$, we know that at least one eigenvalue has $\mathrm{Re}(\lambda_i) > 0$. If we take the largest eigenvalue, and neglect the other one, we are still able to find an upper bound $T$ for the switch time $t_1$, since if we choose $T$ such that

$$2\varphi = \frac{k_c c_i}{b\lambda_i}\Big(\mathrm{e}^{\lambda_i T} - 1\Big) \tag{2.44}$$

$$1 + \frac{2\varphi b\lambda_i}{k_c c_i} = \mathrm{e}^{\lambda_i T} \tag{2.45}$$

$$\log\Big(1 + \frac{2\varphi b\lambda_i}{k_c c_i}\Big)/\lambda_i = T, \tag{2.46}$$

we certainly know that $t_1 < T$. In the worst case, it costs $T$ time to switch to $S_1$ or $S_3$. This fact is used, and we assume that the system behaves like equation (2.37) until time $T$. We denote $\theta_1$ and $\dot{\theta}_1$ as final conditions of the environment where $S_2$ is governing, and take this as initial conditions of the system which rules when $g \in \{S_1, S_3\}$. This second behavior can be represented as:

$$\dot{\theta}(t) = \begin{bmatrix} 0 & 1 \\ \frac{g_z}{l} - \frac{k_c}{I_p} & \frac{-b}{I_p} \end{bmatrix} \theta(t). \tag{2.47}$$

This system can be solved, analogue to the previous one, resulting in

$$\theta(t) = c_3 e^{\kappa t} + c_4 t e^{\kappa t} \tag{2.48}$$

with $\kappa$ the eigenvalue of $(A + BN)$ with multiplicity of 2, because of the critically damped controller. Initial values are given by $\theta_1$ and $\dot{\theta}_1$. So analogue to equations (2.39)-(2.40), we see

$$\begin{aligned} \theta_1 &= c_3, \qquad \dot{\theta}_1 = \kappa c_3 + c_4 \qquad \Longrightarrow & (2.49) \\ c_3 &= \theta_1, \qquad c_4 = \dot{\theta}_1 - \kappa\theta_1. & (2.50) \end{aligned}$$

Remark that $\theta(t)$ and $\dot{\theta}(t)$ have same sign and that $\theta(t)$ will reach its maximum at $t_2$ when $\dot{\theta}(t_2) = 0$. Setting $\dot{\theta}(t_2) = 0$, gives

$$\begin{aligned} c_3 \kappa e^{\kappa t_2} + c_4 e^{\kappa t_2} + c_4 \kappa t_2 e^{\kappa t_2} &= 0 & (2.51) \\ \left(c_3 \kappa + c_4 + c_4 \kappa t_2\right) e^{\kappa t_2} &= 0 & (2.52) \\ c_3 \kappa + c_4 + c_4 \kappa t_2 &= 0 & (2.53) \\ \frac{c_1 \kappa + c_2}{-c_2 \kappa} &= t_2, & (2.54) \end{aligned}$$

since $e^{\kappa t_2} \neq 0$ for all $t_2 > 0$ and $\kappa$.
These both time instances, $T$ and $t_2$ can be used to find a boundary in the behavior of the system. This boundary will be used in the next chapter.

# Chapter 3

# Controller design

In the previous chapter we investigated some hysteretic concepts, and introduced an extensive example. In Chapter 1 we stabilized a regular system with a linear feedback controller, which depended on the output of the system. However, this is not sufficient when a system suffers from hysteresis, so other controllers must be designed. The aim of this chapter is to design a controller which stabilizes a hysteretic system. Further, proper design of a controller depends on the goals that must be met by the system. It is often a trade-off between speed and precision.

In this chapter, both types of controllers will be designed: first a constant low gain controller, which acts rather slow but has high accuracy. Also a high gain controller will be made. Simulations with a mixture of both controllers, a switched controller, are also done. This switching is quite normal in daily life. For example, think of opening a door with a key: the first movement of the arm, to move the key from the pocket to the neighbourhood of the key hole is quite fast and inprecise, but when the key comes near the keyhole, the speed of the arm decreases and the movement becomes more accurate. This idea will be worked out at the end of this chapter.

## 3.1 Fixed sign controllability

In this section a controller is designed, according to the strategy of ignoring the hysteresis. Actually, the results are subject to hysteresis, nothing can be done about it. But with a proper controller design we can circumvent the troubles. We assumed already that each hysteresis has three states, and our plan is to design a controller such that the hysteresis variable remains in the state where it starts. Further more, we saw in the Duhem model that the dynamics of the system switches at certain points. This switch in the piecewise function of the generalized model occurs when the sign of the input changes. Therefore we define a specific kind of controllability, based on Definition 1.11, and the notion that controllability is equivalent to state-controllability.

**Definition 3.1 (Fixed sign controllability).** *Consider a time invariant system* $\Sigma$*, governed by* $\dot{x} = f(x, u)$*. Then* $\Sigma$ *is called* fixed sign controllable *if for any two states* $x_1(t), x_2(t) \in \mathbb{R}^n$ *there exist a* $t_1 > 0$ *and a third state* $x \in \mathbb{R}^n$ *where*

$$x(t) = \begin{cases} x_1(t) & t \leq 0 \\ x_2(t - t_1) & t \geq t_1 \end{cases} \tag{3.1}$$

*and where the input* $u(t)$ *preserves sign for all* $0 \leq t \leq t_1$*.*

Remark that the sign of the input should be chosen after the trajectories are given. So, every time that the states $x_1$ and $x_2$ are redefined, the sign may be chosen again.

To get a feeling with fixed sign controllability, we first give some examples. Thereafter, we dive into discrete LTI systems and look at the system while time steps are taken.

**Example 3.1 (Train driver):** As a last example, consider the train drivers system of Example 1.1. This system is not fixed sign controllable. To illustrate this, take two states $x_1(t)$ and $x_2(t)$, where the states are given by the positions $x_1 < x_2$ and velocities $\dot{x}_1 = \dot{x}_2 = 0$. Both states are in the behavior of the system. When the objective is to steer from state $x_1$ to $x_2$, one should choose a controller which increase the velocity. However, it is not possible to decrease the velocity any more, and although the train has some friction, the train can not stop at $x_2$. So this system is not fixed sign controllable.

**Example 3.2 (Controlling temperature with gas incineration):** Reconsider the controlling of the temperature of a boiler in Example 2.1, but now with a gas burner. There it is possible to control the temperature by burning gas, which drives up the temperature. Theoretically, 'unburning gas' will lower the temperature. Hence, if we take gas incineration as the input of our system, we can steer from any state $x_1$ to arbitrary state $x_2$, with the following simple strategy: burn (extra) gas if $x_1 < x_2$, and 'unburn gas' if $x_1 > x_2$. Doing this until the state $x_2$ is reached. Everything that happens after $x_2$ is reached, does not matter any more. Therefore, this system is fixed sign controllable.

Note that unburning gas is rather unrealistic. In real life, only non-negative input is possible. This small modification is defined as *positive controllability.* For literature about positive controllability we refer to Klamka [10].

**Example 3.3 (Position of a swing):** Consider a swing, which can be controlled by pushing ($u \geq 0$) and pulling ($u \leq 0$). This system is still controllable when only input of one particular sign is allowed. Suppose the case that only pushing is allowed, the swing is approaching $\dot{x} < 0$, and the goal is to let it approach faster. Since pulling the swing is not allowed, one should wait until the swing moves away ($\dot{x} > 0$) and push the swing to increase the speed. Then it swings back, and approaches with a higher speed, which was desired. So, although it takes some extra time to swing the swing in the desired state, each other trajectory is still reachable in a certain time $t_1 > 0$. Remark that the behavior space $\mathfrak{B}$ is reduced, since all trajectories where the swing is in a stationary position with $x < 0$ need a constant input with $u < 0$, which therefore leads to non-allowed behavior.

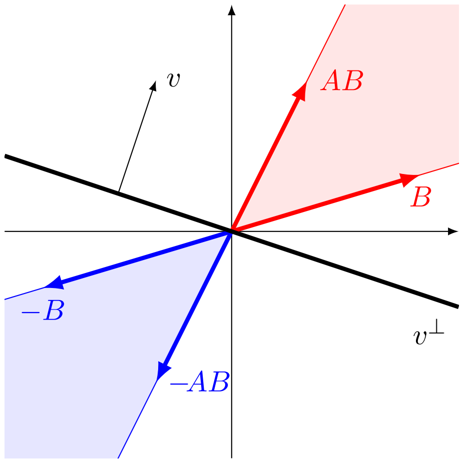



**Figure 3.1:** Two vectors, $B$ and $AB$, covering convex cones. The red area is reachable with a positive combination of these vectors, the blue area is reachable with a negative combination. A separating line is given by the set $v^{\perp}$.

We want to investigate this fixed sign controllability of LTI systems. Therefore, we look at the following discrete LTI system

$$x(k) = Ax(k-1) + Bu(k-1) = A^2x(k-2) + ABu(k-2) + Bu(k-1) \tag{3.2}$$

$$= A^n x(k-n) + \sum_{i=1}^{n} A^{n-i} Bu(k-i) \tag{3.3}$$

Consider two arbitrary states $x_0, x_1 \in \mathbb{R}^n$, which must be connected to each other by an input $u(i)$ with $i = 0, 1, 2, \ldots$. If this is possible for all $x_0$ and $x_1$, the system is state controllable. Take for example $x_0 = A^k x(0)$ and $x_1 = x(k)$. We see that full control is possible in $k$ steps, if and only if the vectors $A^i B$ (for $i = 0, 1, \ldots, k-1$) span the whole $\mathbb{R}^n$, since it is then possible to choose a linear combination of $u(i)$, such that $\sum_{i=1}^{k} A^{k-i} Bu(k-i)$ can be chosen freely. This influence of $u(i)$ on the system will bring the system from $x_0$ to $x_1$, since $\sum_{i=1}^{k} A^{k-i} Bu(k-i) = x(k) - A^k x(0)$, according to equation (3.3). The lowest $k$, where $A^i B, i = 0 \ldots k$ already spans the whole $\mathbb{R}^n$, says that there are at least $k$ steps needed to assure controllability.

This result follows the same philosophy as in the continuous case, described in Theorem 1.12 on page 8, where the controllability matrix $\mathfrak{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^nB \end{bmatrix}$ has to have full row rank.

However, Definition 3.1 concerns fixed sign controllability, and not regular controllability, which is mentioned above. With fixed sign controllability, $u(i)$ must preserve sign, and therefore only positive or only negative combinations of the vectors $A^i B$ may be chosen, which make it less trivial to cover the whole space $\mathbb{R}^n$. In Figure 3.1, it is easy to see that the vectors $B$ and $AB$ span the whole $\mathbb{R}^2$. However, the positive combinations together with the negative combinations do clearly not cover the whole space. The positive cone, which is formed by a convex combination of the vectors $A^i B$ must at least cover a half-plane, to assure that the whole space is covered by both negative and positive cone. This idea will be extrapolated to the $n$-dimensional space and formally proved in the following theorem. First a lemma is stated, from S. Boyd and L. Vandenberghe in *Convex Optimization*, section 2.5:

**Lemma 3.2 (Separating hyperplane [4]).** *Suppose $C$ and $D$ are two convex sets that do not intersect, i.e., $C \cap D = \emptyset$. Then there exist $a \neq 0$ and $b$ such that*

$$a^T x \leq b \quad \text{for all } x \in C, \qquad \text{and} \qquad a^T x \geq b \quad \text{for all } x \in D. \tag{3.4}$$

*The hyperplane $\{x \mid a^T x = b\}$ is called a* separating hyperplane *for the sets $C$ and $D$.*

Remark that if $C$ and $D$ are convex cones with vertex at zero, then $C \cap D = \{0\}$. The lemma then still holds, since 0 is the only element of the intersection [4, p. 50]. The separating hyperplane then separates the sets through the origin, $\{x \mid a^T x = 0\}$.

**Theorem 3.3 (Discrete fixed sign controllability).** *Consider a discrete LTI system in the state space representation*

$$x(k+1) = Ax(k) + Bu(k). \tag{3.5}$$

*This system is* fixed sign controllable in $n$ steps *if and only if there exists no $v$ such that*

$$v^T B > 0 \tag{3.6}$$
$$v^T AB > 0 \tag{3.7}$$
$$v^T A^2 B > 0 \tag{3.8}$$
$$\vdots$$
$$v^T A^n B > 0 \tag{3.9}$$

*holds.*

**Proof:** *First part: FSC $\Rightarrow \nexists v$.*
*By contradiction; suppose such $v$ exists, while the system is fixed sign controllable. Because of the linearity of the system, we choose $x_0$ as the origin, without loss of generality. If we then choose $x_1$ lying on the separating hyperplane, defined by that particular $v^\perp$ which is assumed to be existing, it is clearly not possible to write $x_1$ as a convex combination of $[B, AB, A^2B, \ldots, A^nB]$. This implies that it is not possible to steer from state $x_0$ to an arbitrary $x_1$, which should be, since the system is fixed sign controllable. Therefore, a contradiction appears, and such $v$ could not exist.*

*Second part: $\nexists v \Rightarrow$ FSC. State that such $v$ does not exist, and assume that the system is not fixed sign controllable. No fixed sign controllability means that the $\mathbb{R}^n$ is not covered by the cones. So there is an $x_1$, such that $x_1$ is neither a convex combination of $[B, AB, A^2B, \ldots, A^nB]$, nor a convex combination of $[-B, -AB, -A^2B, \ldots, -A^nB]$. From now on, we name the positive convex cone $C$ and the negative convex cone $D$. Since $C$ and $D$ are closed, there exists an open ball $B_\varepsilon(x_1)$ such that $B_\varepsilon(x_1) \notin \{C, D\}$. We define the convex cone $X$, based on the set $B_\varepsilon(x_1)$. Then $C \cap X = \{0\}$ and $D \cap X = \{0\}$. Geometrically speaking, $x_1$ is located outside the convex cones. By Lemma 3.2, there exists a separating hyperplane, which separates $X$ and $C$, passing through the origin. Since $D$ is the opposite of $C$, $D$ lies in the same half space of $X$. But then each vector $v^\perp \in X$ is a separating hyperplane of $C$ and $D$. However,* *we assumed that no $v$ was allowed to exist, so a contradiction with the preposition is made. Therefore, whenever a $v$ does not exist, the system should be fixed sign controllable.* ∎

In the above theorem, $n$ is finite. This is necessary, since controllability demands that any state could be steered to an other arbitrary state in finite time. We use this theorem to extract some properties of the $A$-matrix, to assure fixed sign controllability. Therefore, first a lemma is stated, and thereafter a theorem about the eigenvalues of $A$.

**Lemma 3.4 (Fixed sign controllability implies controllability).** *Consider a system $\Sigma$. If $\Sigma$ is* fixed sign controllable*, then it is also* controllable.
**Proof:** *This follows from the definitions.* ∎

**Theorem 3.5 (Eigenvalues of discrete fixed sign controllable system).** *The system* (3.5) *is fixed sign controllable if and only if it is controllable and $A$ does not have eigenvalues on the positive real axis.*
**Proof:** *First part: FSC $\Rightarrow \nexists\lambda$ on positive real axis & controllable.*
*Lemma 3.4 shows the controllability. With use of Theorem 3.3, we already knew that there could not exist $v$ such that all equations* (3.6)-(3.9) *hold, with $n$ finite.*
*To prove the part of the eigenvalues, we want to construct a contradiction by supposing the opposite: We can not find $v$ with an eigenvalue $\lambda$ on the positive real axis.*
*Let us take $x$ as a corresponding left eigenvector of the eigenvalue $\lambda$, so $x^TA = \lambda x^T$.*
*Proof by induction: We assume that $x^TA^iB > 0$. The inductive step is to prove that $x^TA^{i+1}B > 0$. With use of the left eigenvectors, we see that*

$$x^TA = \lambda x^T \tag{3.10}$$
$$x^TA^{i+1}B = x^TAA^iB = \lambda x^TA^iB \tag{3.11}$$

*and therefore we see that*

$$\underbrace{x^TA^iB}_{>0} \Rightarrow \lambda \underbrace{x^TA^iB}_{>0} \Rightarrow \underbrace{x^TA^{i+1}B}_{>0} \tag{3.12}$$

*since $\lambda > 0$. The basis step consists of proving that $xA^0B = xB > 0$. We can take this for granted in general. In case this is not true, we could take $-xB > 0$. Then the induction step also holds, so $-xA^iB > 0$ for all $i$, with a given positive eigenvalue, $\lambda > 0$. An equality sign ($xB = 0$) is not allowed, since the system is controllable, which implies full row rank of $\mathfrak{C}$.*
*If we choose this particular eigenvector $x = v$, we found $v$, which was forbidden.*
*So, with fixed sign controllability it is not allowed to have a positive eigenvalue, therefore, all eigenvalues must be negative.*

*Second part: $\nexists\lambda$ on positive real axis & controllable $\Rightarrow$ FSC.*
*To prove this part, we want to construct again a contradiction by supposing the opposite: We state that no eigenvalues lie on the positive real axis and assume that the system is not fixed sign controllable. Then there exist $v$, such that all equations* (3.6)-(3.9) *hold, with $n$ bounded.*

*Let us take that particular $v$, which implicates that $v^T B > 0$. We know that all left eigenvectors of $A$, with inclusion of the generalized eigenvectors, span the $\mathbb{R}^n$, so $v$ can be written as a linear combination of these eigenvectors:*

$$v^T = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n. \tag{3.13}$$

*Each regular eigenvector $y_0 \in \{x_1, x_2, \ldots, x_n\}$ has a corresponding eigenvalue $\lambda$, such that $y_0 A^k = \lambda^k y_0$. If $\lambda$ has a multiplicity, a generalized eigenvector $y_1$ also shows up, such that $y_1 A^k = k\lambda^k y_1$. Higher multiplicities gives more generalized eigenvalues, and with a multiplicity of $m$ holds*

$$y_{m-1} A^k = k^{m-1} \lambda^k y_{m-1}. \tag{3.14}$$

*This fact is used to show that there is at least one dominant eigenvalue. Since the corresponding eigenvalue is not on the positive real axis,* $\text{angle}(\lambda) \neq 0$ *holds. Written in polar coordinates, we can state that* $\text{angle}(\lambda^k) = k \cdot \text{angle}(\lambda)$. *For sure, there is an $i$ such that* $\text{Re}(\lambda^k) < 0$. *Then we choose this $\lambda$, and, with this particular $k$:*

$$v^T A^k \approx y_{m-1} A^k = k^{m-1} \lambda^k y_{m-1} \approx k^{m-1} \lambda^i v^T. \tag{3.15}$$

*If we right multiply with $B$ and take $i$ sufficiently large and such that $\lambda_k^i < 0$, we see that*

$$\underbrace{v^T A^i B}_{<0} = k^{m-1} \underbrace{\lambda^k}_{<0} \underbrace{v^T B}_{>0} \tag{3.16}$$

*which is in contradiction with the fact that all equations* (3.6)-(3.9) *must hold.*
*Suppose there are $n$ different dominant eigenvalues $\{x_{k_1}, \ldots, x_{k_n}\}$, with trivially same multiplicity $m$. Then, with sufficient high $k$:*

$$\begin{aligned} v^T A^k &\approx (x_{k_1} + x_{k_2} + \ldots + x_{k_n}) A^k = k^m \left(\lambda_{k_1}^i x_{k_1} + \lambda_{k_1}^i x_{k_1} + \ldots \lambda_{k_n}^i x_{k_n}\right) & (3.17) \\ &\approx k^m (\lambda_{k_1}^i + \lambda_{k_2}^i + \ldots + \lambda_{k_n}^i) v^T. & (3.18) \end{aligned}$$

*Since no $\lambda_k$ lies on the positive real axis, there is an $i_k$, such that $\lambda_k^{i_k}$ has negative real part. Furthermore, there is a least common multiple $i$ of all $i_k$'s, such that $\lambda_k^i$ has negative real part for all $\lambda_k$'s. Then if we take this $i$, we see that*

$$\underbrace{v^T A^i B}_{<0} = \underbrace{(\lambda_{k_1}^i + \lambda_{k_2}^i + \ldots + \lambda_{k_n}^i)}_{<0} \underbrace{v^T B}_{>0} \tag{3.19}$$

*which contradicts the preposition.*
*We conclude the proof by stating that $v^T A^i B$ can never be zero, because of the full row rank of $\mathfrak{C}$. So if all $\lambda < 0$, there can not exist a separating plane defined by $v$, and thus by Theorem 3.3, the system must be fixed sign controllable.* ■

In the continuous case, we can use the same philosophy. Returning to the straightforward solution of a system, equation (1.9), we remark that at time $t_1$ the effect of the input at time $\tau$, $u(\tau)$, is directly proportional, to a factor $\mathrm{e}^{A(t_1-\tau)} B$. Now we are able to define a convex cone, similar to the discrete case. Each input affects the output with the given factor, and therefore linear combinations of these factors may

be freely chosen, to steer from $x_0$ to $x_1$. Therefore, the convex cone is defined as the continuum of $\mathrm{e}^{A\tau}B$ with $\tau \in [0, t_1]$. Analogue to the discrete case, we state that there is a separating plane defined by $v$ if and only if the system is not fixed sign controllable. This is used to formulate a test to check whether the system $\dot{x} = Ax + Bu$ is fixed sign controllable or not.
To pave the way for the proof of the theorem, the following lemmas are formulated.

**Lemma 3.6 (Eigenvalues of exponential matrices).** *Suppose the eigenvalues of the matrix $A$ are $\lambda_k$, $k = \{1, \ldots, m\}$. Then the eigenvectors of the exponential matrix $\mathrm{e}^{At}$ are $\mathrm{e}^{\lambda_k t}$, $k = \{1, \ldots, m\}$.*
**Proof:** *Suppose $v$ is the corresponding eigenvector of the eigenvalue $\lambda_k$, then $Av = \lambda_k v$. By definition,*

$$\mathrm{e}^{At}v = \left(\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}\right) v = \sum_{n=0}^{\infty} \frac{t^n A^n v}{n!} = \sum_{n=0}^{\infty} \frac{t^n \lambda_k^n v}{n!} = \mathrm{e}^{\lambda_k t} v \tag{3.20}$$

*which implies that $\mathrm{e}^{\lambda_k t}$ is an eigenvalue of $\mathrm{e}^{At}$.* ∎

**Lemma 3.7 (Matrix decomposition [17]).** *Each matrix $A \in \mathbb{R}^{n \times n}$ may be transformed according*

$$S^{-1}AS = J \tag{3.21}$$

*with $S$ non-singular, and where $J$ a Jordan matrix, consisting of Jordan blocks $J_k$,*

$$J = \mathrm{diag}(J_1, J_2, \ldots, J_N), \qquad J_k = \begin{bmatrix} \lambda_k & 1 & & \\ 0 & \ddots & \ddots & \\ & & \lambda_k & 1 \\ & & & \lambda_k \end{bmatrix}. \tag{3.22}$$

*We see that*

$$\mathrm{e}^{At} = S\mathrm{e}^{Jt}S^{-1}, \qquad \mathrm{e}^{Jt} = \mathrm{diag}(\mathrm{e}^{J_1 t}, \ldots, \mathrm{e}^{J_N t}). \tag{3.23}$$

**Theorem 3.8 (Eigenvalues of continuous fixed sign controllable systems).** *Consider a continuous time system*

$$\dot{x} = Ax + Bu. \tag{3.24}$$

*This system is* fixed sign controllable *if and only if it is* controllable *and all eigenvalues $\lambda_k$, $k = \{1, \ldots, m\}$ have non-zero imaginary part.*
**Proof:** *First part: FSC $\Rightarrow$ All $\lambda$ have non zero imaginary part & controllable.*
*We state that a system is fixed sign controllable. By Lemma 3.4 this system is also regular controllable. We also know that the whole $\mathbb{R}^n$ is spanned by $[\mathrm{e}^{At}B]$, because of regular controllability. Furthermore, since $u(t)$ preserves sign, at least a half plane is spanned by $[\mathrm{e}^{At}B]$, so there can not be a separating plane $v$ such that $v^T\mathrm{e}^{At}B > 0$, for all $t$.*

*To construct a contradiction, suppose there is at least one strict real eigenvalue $\lambda$. With use of Lemma 3.6, we see that* $\mathrm{eig}(\mathrm{e}^{At}) = \mathrm{e}^{\lambda_k t}$. *Further, we take the corresponding left eigenvector $x^T$, which is, such that*

$$x^T \mathrm{e}^{At} = \mathrm{e}^{\lambda t} x^T, \qquad \textit{for all } t. \tag{3.25}$$

*Right multiplication with $B$ gives*

$$x^T \mathrm{e}^{At} B = \mathrm{e}^{\lambda t} x^T B, \qquad \textit{for all } t. \tag{3.26}$$

*By the assumption that $\lambda$ is strictly real, we see that* $\mathrm{e}^{\lambda t} > 0$ *for all $t$. Analogue to the discrete case, equation* (3.12), *we see that if* $x^T B > 0$, *then*

$$x^T \mathrm{e}^{At} B > 0, \qquad \text{for all } t. \tag{3.27}$$

*The other case,* $x^T B < 0$ *will keep equation* (3.27) *negative, for all $t$. The case* $x^T B = 0$ *will keep the equation zero, but this is not valid, because of regular controllability. So by choosing this $x$ as $v$, a separating plane is created. This is in contradiction with our starting position. Hence, fixed sign controllability implies imaginary eigenvalues.*

*Second part: All $\lambda$ have non zero imaginary part & controllable $\Rightarrow$ FSC.*
*We state that all $\lambda$ have non-zero imaginary part. We want to construct a contradiction, so we state that there is a separating plane $v$ such that* $v^T \mathrm{e}^{At} B > 0$. *Further, analogue to the discrete case, $v^T$ can be approximated by a linear combination of dominant (generalized) left eigenvectors of* $\mathrm{e}^{At}$, *and (together with Lemma 3.6)*

$$v^T \mathrm{e}^{At} \approx t^m \big(c_1 x_1 \mathrm{e}^{At} + c_2 x_2 \mathrm{e}^{At} + \ldots + c_n x_n \mathrm{e}^{At}\big) \tag{3.28}$$

$$= t^m \big(c_1 \mathrm{e}^{\lambda_1 t} x_1 + c_2 \mathrm{e}^{\lambda_2 t} x_2 + \ldots + c_n \mathrm{e}^{\lambda_n t} x_n\big). \tag{3.29}$$

*Since the separating plane must hold for all $t \in \mathbb{R}^+$, and for all eigenvalues we assumed $\lambda = a + i\omega$ with $\omega \neq 0$, we zoom in to the eigenvalues with largest real value $a$. Suppose there is just one unique eigenvalue with the largest value of $a$, which has an imaginary part of $i\omega$. After a long time, this eigenvalue will dominate, since*

$$\mathrm{e}^{\lambda t} = \mathrm{e}^{(a+i\omega)t} = \mathrm{e}^{at}\big(\cos(\omega t) + i\sin(\omega t)\big), \tag{3.30}$$

*and if another eigenvalue has lower real value, suppose $b < a$, then*

$$\lim_{t\to\infty} \frac{\mathrm{e}^{bt}}{\mathrm{e}^{at}} = 0. \tag{3.31}$$

*Since there is a* $v^T \mathrm{e}^{At} B > 0$, *we know that* $v^T \mathrm{e}^{A(t+\pi/\omega)} < 0$, *which contradicts the preposition.*
*Suppose there are $n$ different eigenvalues with the same largest real value $a$. Then the imaginary parts must differ; $i\omega_1 \neq i\omega_2 \neq i\omega_n$.*

$$\mathrm{e}^{at}\big(\cos(\omega_1 t) + \cos(\omega_2 t) + \ldots + \cos(\omega_n t)\big). \tag{3.32}$$

*Now it is necessary (and sufficient) to show that* $\cos(\omega_1 t) + \cos(\omega_2 t) + \ldots + \cos(\omega_n t)$ *is positive for all $t$, to assure the assumed separating plane $v$, since* $\mathrm{e}^{at} > 0$ *for all $t$.*

*By Bohr [2], we know that the sum of a finite number of continuous periodic functions with arbitrary periods is almost periodic. Also, every almost periodic function is bounded and continuous. Furthermore, for every almost periodic function there exist a mean value. For each trigonometric function, this mean value is zero. Therefore, the mean of the sum is also zero.*
*Further, by reasoning that the mean is zero and the function is bounded, one can state that for each* $k = 1, \ldots, n$ *the integral is bounded, i.e. there exists* $M_k$ *such that for all* $T$

$$\int_0^T \cos(\omega_k t)\mathrm{d}t < M_k \tag{3.33}$$

*holds, and therefore there exists alsoan* $M$ *such that*

$$\int_0^T \sum_{k=1}^{n} \cos(\omega_k t)\mathrm{d}t < M. \tag{3.34}$$

*Hence, if this function is indeed positive for all* $t$*, expression* (3.34) *can not hold any more, since* $\int_0^T f(t)\mathrm{d}t \geq T\varepsilon$ *is unbounded if* $T \to \infty$*. If this function is non-negative, it must always be zero, which is not allowed. At this point a contradiction appears. Therefore the function can not be non-negative, which results in the fact that no separating hyperplane can exist.* ■

We saw already that controllability is equivalent to null-controllability and reachability. This is, however, not the case for fixed sign controllability.

**Theorem 3.9 (Fixed sign controllability implies fixed sign reachability, null controllability).** *Consider a system* $\Sigma$*. If* $\Sigma$ *is* fixed sign controllable*, it is also* fixed sign reachable *and* fixed sign null controllable*.*
**Proof:** *Because* $\Sigma$ *is fixed sign controllable, we can steer from state* $x_1$ *to* $x_2$ *using an input* $u$ *with constant sign. Therefore, we can choose* $x_1$ *as the equilibrium, and steer to* $x_2$ *using an input* $u$ *with constant sign. By definition, this is fixed sign reachable. We can also choose* $x_2$ *as the equilibrium, and steer from any initial condition towards it, using an input* $u$ *with constant sign. Therefore the system is also fixed sign null-controllable.* ■

Vice versa, this statement will does hold always. Suppose a system is fixed sign null controllable and fixed sign reachable. Because of fixed sign null controllability, the system can be steered from $x_1$ to $x_0$, with an input which does not change sign. Because of fixed sign reachability, the system can be steered from $x_0$ to $x_2$, with an input which does also not change sign. These two inputs do not change sign, but this sign can differ from the sign which is used to steer to the origin $x_0$.

## 3.2 Fixed sign stabilizability

In the previous section, we discussed the aspects of fixed sign controllability. This section will handle the notion of stabilizability. This is a slightly weaker property then controllability First the definition will be given, after that some theorems are given and proved, to check wether a system is stabilizable or not. Again, the assumption that the input $u$ is a scalar is also in this section applied.

**Definition 3.10 (Fixed sign stabilizable).** *Consider a time invariant system $\Sigma$ governed by $\dot{x} = f(x, u)$. Then $\Sigma$ is called* fixed sign stabilizable *if for every state $x_1 \in \mathbb{R}^n$, there exist a state $x_0 \in \mathbb{R}^n$ with the property*

$$x_0(t) = x_1(t) \text{ for } t \leq 0 \qquad \text{and} \qquad \lim_{t\to\infty} x_1(t) = 0. \tag{3.35}$$

*where $u(t)$ preserves sign for all $t > 0$.*

To check whether a system is fixed sign stabilizable, the following test can be done. Remark that the trajectories $w$ in Definition 1.13 on page 9 are interchanged with the states $x$ in previous definition. This will also be used in the following theorem. This is allowed due to the fact that stabilizability equals state stabilizability, similar to the fact that controllability equals state controllability.

**Theorem 3.11 (Fixed sign stabilizable).** *Consider a system $\dot{x} = Ax + Bu$, with feedback control $u = Nx$. This system is* fixed sign stabilizable *if $N$ can be chosen such that all the eigenvalues $\lambda_k = \text{eig}(A + BN)$ lies on the negative real axis. That is that for all $\lambda_k$ the following holds:*

$$\text{Im}(\lambda_k) = 0, \qquad \text{Re}(\lambda_k) < 0. \tag{3.36}$$

**Proof:** *It was already clear that there must be an $N$, such that $\text{Re}(\lambda_k) < 0$ holds, otherwise, regular stability is already not guaranteed. This fact is given in Theorem 1.14 on page 11.*
*We know due to Lemma 3.7 that $x(t) = \mathrm{e}^{(A+BN)t}x_0 = S\mathrm{e}^{Jt}S^{-1}x_0$. First we assume that all eigenvalues have multiplicity of one, so $x(t) = S\mathrm{e}^{\Lambda t}S^{-1}x_0$. Hence all state variables will behave in the long run as a linear combination of $\mathrm{e}^{\lambda_k t}$.*
*Remark that $c_i\mathrm{e}^{\lambda_i t}$ will preserve sign, when $\text{Im}(\lambda_k) = 0$. Furthermore, looking to two terms, we see that if $\lambda_i > \lambda_j$, then $\mathrm{e}^{\lambda_i t} > \mathrm{e}^{\lambda_j t}$ for all $t$. Even more, eventually there is a $t_1$, such that*

$$c_i\mathrm{e}^{\lambda_i t} > c_j\mathrm{e}^{\lambda_j t} \qquad \text{for all } t > t_1 \text{ and } c_i, c_j \in \mathbb{R}. \tag{3.37}$$

*So $c_i\mathrm{e}^{\lambda_i t} + c_j\mathrm{e}^{\lambda_j t}$ will switch sign at most 1 time (at time $t_1$). Furthermore, since $u = Nx$, with $N \in \mathbb{R}^{1\times n}, x \in \mathbb{R}^{n\times 1}$, the input will be defined as:*

$$u(t) = \sum_{i=1}^{n} m_i \left( \sum_{k=1}^{n} c_{i,k}\mathrm{e}^{\lambda_k t} \right) \tag{3.38}$$

$$= \sum_{k=1}^{n} \gamma_k \mathrm{e}^{\lambda_k t} \tag{3.39}$$

*with $\gamma_k = \sum_{i=1}^{n} m_i c_{i,k}$.*

*Similar to the previous reasoning, we can state in general that the feedback $Nx(t)$ switches at most $n-1$ times, when $\Lambda = \mathrm{diag}(\lambda_1 \dots \lambda_n)$. The number of switches depends on the initial conditions.*

*It is for sure that the largest eigenvalue ($\lambda_i$), with the largest multiplicity ($m_i$) will dominate the input, since the corresponding term*

$$\gamma_i t^{m_i-1} \mathrm{e}^{\lambda_i t} \tag{3.40}$$

*is eventually larger then all other terms. This term preserve sign by itself, since $\lambda$ is real. Only when $\lambda_i$ has a non zero imaginary part, then sign switching is still possible when $t \gg 0$.*

Although the feedback is switching sign, the control can be fixed sign by $u(t) = \max(Nx(t), 0)$ or $u(t) = \min(Nx(t), 0)$.
Suppose the system will not switch at all, then it is clear. Suppose it will switch once, it will switch, also with control $u = 0$. After this switch, the input will be fixed sign. Further reasoning: when the system will switch twice, then the $u = Nx$, and it will switch sign definetely. After this switch, $u = 0$, so it will switch even faster than with $u = Nx$, and after this switch it will behave as normal.

**Example 3.4 (Simple inverted pendulum):** We look again to the example of the simple inverted pendulum, Example 1.6 on page 9. Eigenvalues of the matrix $A - BkC$ are already discussed, and we saw that if $k$ is large enough to stabilize, but not too large to assure that the system will not oscillate, so $\mathrm{Im(eig)} = 0$, then the system is critically or overdamped damped. Precisely, if we choose $k$ such that

$$mgl < k < mgl + b^2/4I \qquad \text{and} \qquad N = \begin{bmatrix} -k & 0 \end{bmatrix}, \tag{3.41}$$

then we see that $A + BN$ has all eigenvalues on the negative real axis. Therefore the system is fixed sign stabilizable, according to Theorem 3.11.

We saw already in Chapter 2 that a system, subject to hysteresis is governed by another expression when the input changes sign, see equation (**??**). With a fixed sign controller, no input sign switching can occur at all, therefore only one expression rules the system. This fact will make the controlling far easier.
A critically or overdamped linear feedback controller is often a specific realization of a fixed sign stabilizing system, or can be adapted to be such a fixed sign controller, since the switching is finite. The desired trajectory must be reached without oscillations, since oscillations often causes switching of input. In case that the system must be steered towards its equilibrium, then fixed sign stability is enough to prevent oscillations caused by the hysteresis. It should be mentioned that the certain stabilizing strategy with parameter matrix $N \in \mathbb{R}^{1\times n}$ which fulfil the requirement for a system of being fixed sign stabilizable, is not always direct the strategy to obtain the particular fixed sign stabilizability. This has to do with the initial values. This will be shown in the next example with simulation.

**Example 3.5 (Critically damped):** A MATLAB-simulation of the inverted pendulum of Example 2.3 is made with a critical damped linear feedback controller. The autonomous system behaves according to

$$\dot{x}(t) = Ax(t) + B\mathcal{H}(Nx(t)), \qquad \text{with} \qquad N = \begin{bmatrix} -k_c & 0 \end{bmatrix}. \tag{3.42}$$

Due to symmetry of the hysteron, i.e. the outcomes under $S_1$ is equal to the outcomes under $S_3$ and the transitions are symmetric, the system can be rewritten as

$$\dot{x}(t) = Ax(t) + BN\hat{\mathcal{H}}(x(t)) \tag{3.43}$$

The modified hysteron, based on equation (2.20) can be described as

$$v = \hat{\mathcal{H}}(x) = \begin{cases} x & \text{if } g = -\frac{\varphi b}{k_c} \\ 0 & \text{if } -\frac{\varphi b}{k_c} < g < \frac{\varphi b}{k_c} \\ x & \text{if } g = \frac{\varphi b}{k_c} \end{cases} \qquad \text{with} \qquad \dot{g} = \begin{cases} \max(0, x) & \text{if } g = -\frac{\varphi b}{k_c} \\ x & \text{if } -\frac{\varphi b}{k_c} < g < \frac{\varphi b}{k_c} \\ \min(0, x) & \text{if } g = \frac{\varphi b}{k_c} \end{cases} \tag{3.44}$$

The same physical constants are used, given in Table 2.9. The initial conditions are $[\theta_0, \dot{\theta}_0] = [0.01, -0.05]$ and $g_0 = -\varphi$. In Figure 3.2 the results are given. Originally one change of sign of the input occurs, in the beginning of the simulation, due to the initial conditions. This can be evaded, by setting the input on zero for the first while. After a certain $t_1$, the input can be switch on, and then is would not switch sign anymore. In the figure, one swing occurs at $t \approx 5$. After that, the system nicely converges to the equilibrium.
We denote the two sets which maps $v \to u$ as set $S_1$ and $S_3$, and the set which maps $v \to 0$ as set $S_2$.
When the system is governed by the expressions of set $S_2$, we know that the equilibrium of the system of Example 2.3 is unstable. As a consequence, the state variable of the hysteron $g$ grows without bounds, since $\dot{g} = u = N\theta_p$. Furthermore, we have seen in the example that $S_2$ is bounded, since this particular unstable dynamics only hold when $|g| < \varphi$. Therefore, the state of the hysteron will leave $S_2$ in finite time, when a distortion of $\delta > 0$ is applied on the system in equilibrium.
The dynamics of the sets $S_1$ and $S_3$ are also well known. Inside $S_1$, the system behaves simply as a LTI system. Because we know that this inverted pendulum is fixed sign stabilizable (as explained in Example 3.4), we can choose the controller either $u = \max(0, Nx)$ or $u = \min(0, Nx)$, depending on the initial conditions. Trivially these controllers do not change sign. Then the hysteron state variable $g$ will remain constant, since $\dot{g} = 0$. So by definition, the hysteron state is in its equilibrium.

In the previous example, we know that $\mathcal{H}$ has two possible outcomes, $\{u, 0\}$. In general, we can say more about this way of controlling. We assumed that $S_2$ is bounded, and relatively small, but moreover that then the outcomes of the hysteretic function equals zero. Hence, assumed that the autonomous system is linear and unstable, the state of the hysteron will leave $S_2$ in finite time, when a distortion of $\delta > 0$ is applied on the system in equilibrium.

About the dynamics of the sets $S_1$ and $S_3$, with a positive (or negative) controller in general can be said more. Inside these sets the system behaves as a LTI system, as described in Chapter 1, and we have already assumed that the system has a stabilizable equilibrium. If $u$ does not have to change sign to steer towards its equilibrium, which is the fact when the system is fixed sign stabilizable, the hysteron will not switch in behavior. With a critical damped system, this is exactly what happens: Once the state of the hysteron $g$ reaches $S_1$ or $S_3$, it will stay inside this set, since no switching sign does occur. Therefore, the controlled system will converge to its

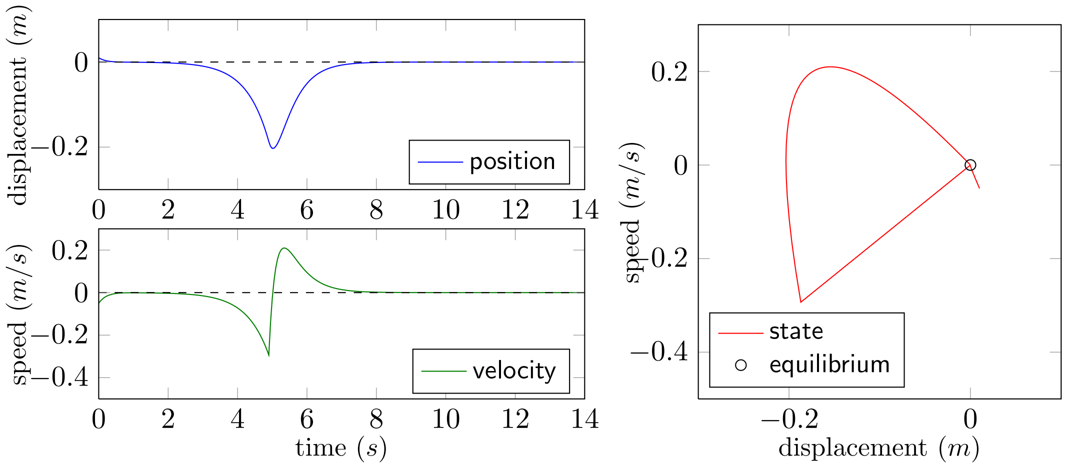



**Figure 3.2:** Simulation of an inverted pendulum with slackness in the pivot position. A critical damped feedback controller is used. Left: the position (blue) and velocity (green) of the pendulum, as function of the time. Right: The phase plot (red) of the pendulum. The equilibrium (black) is shown at $(0, 0)$.

equilibrium. Recall that it is assumed that there exist one stabilizable equilibrium. Together with the existence of set $S_2$, this is exactly the property of an *attractor*, as stated in Definition 1.6.
However, a small distortion ($\delta > 0$) of the equilibrium can cause a change of sign of $u$, which result in a shift from $S_1$ to $S_2$. Recall that $S_2$ is non-empty, and $u(t)$ is bounded. Therefore, there is an $\varepsilon > 0$ and a $t > 0$ such that $\|x(t)\| \geq \varepsilon$, which violates the necessary conditions to be stable (Definition 1.5(i)). This makes the equilibrium *unstable*. Therefore, this equilibrium point of the hysteretic dynamical system, described in Example 3.5 is an *unstable attractor*.

Unstability holds for all hysteretic dynamical systems under the following assumptions: (1) it has an stabilizable equilibrium when $g \in \{S_1, S_3\}$, (2) it has an unstable equilibrium when $g \in S_2$ with $S_2$ bounded and (3) it has an unstable equilibrium of the hysteron state $g$.
When the initial system state is far away from its equilibrium, then this system will work properly, because there is clearly no reason to switch sign. The hysteron will not play a role in it, or at most a relatively small one in the begin of the controlling. Clearly, when the critically damped system is in it equilibrium and is only exposed to perturbations in one direction, such that $u$ does not change sign, the system does also not bother the hysteresis, because the critical damped controller will never let the pendulum further oscillate, and cross its equilibrium position, which causes a switch of sign. However, this is not a realistic scenario, and therefore, other solutions must be found, in case that the system is to close the equilibrium. To get a measure for what is close, we first define other types of stability.

## 3.3 Practical stability

In this section, some comments about stability of the system itself are made. Although only unstability can be concluded with the regular stability definitions, it must be observed in Figure 2.8 that the controlled system remains within certain bounds. Also in general, this is often the case with hysteretic systems. Therefore, based on the ideas of Lakshmikantham et al. [12], we define another type of stability: *practical stability.*

**Definition 3.12 (Practical-$\Omega$-stability).** *An equilibrium point $x = 0$ of a system $\dot{x} = f(x)$ is called* practical-$\Omega$-stable *if there is a certain bounded region $\Omega$ and a $\delta > 0$ such that*

$$\|x(0)\| < \delta \quad \text{implies} \quad x(t) \in \Omega \quad \text{for all} \quad t \geq 0 \tag{3.45}$$

The idea behind this definition is that if there is a distortion on the system in its origin, which is sufficiently small (thus within the range of $\delta$), then the system is stable 'enough' to remain inside a certain region, given by $\Omega$. The essential difference between this practical definition and the definition about stability of invariant sets, Definition 1.8, is that there $\varepsilon$ could be chosen arbitrary small, while this is not necessarily the case for $\Omega$. However, stability always implies practical stability.

**Theorem 3.13 (Stability implies practical-$\Omega$-stability).** *A* stable *equilibrium point $\bar{x}$ of a system $\dot{x} = f(x)$ is always* practical-$\Omega$-stable.
**Proof:** *Choose $\Omega = B_\varepsilon(\bar{x})$. Then because stability, for each $\varepsilon > 0$ holds,*

$$\|x(t)\| < \varepsilon \qquad \Rightarrow \qquad x(t) \in B_\varepsilon(\bar{x}) = \Omega \tag{3.46}$$

*which fulfils the requirements to be practical-$\Omega$-stable.* ■

**Example 3.6 (Practically stable inverted pendulum):** If we look again to our simulation of Example 3.5, and to the behavior as described in Example 2.4, we can choose $\Omega = \mathrm{e}^{(A+BN)t_2}\mathrm{e}^{At_1}x_0$. We know from values of $t_1$ and $t_2$, given in Example 2.4, that the practical-$\Omega$-stability requirement holds, since all eigenvalues of $(A + BN)$ are strict negative. This requirement hold for every initial value $x_0$. If a distortion of $\delta$ is added to our system in its equilibrium position, we can take $x_0$ as the set of $\{x \mid \|x\|_\infty < \delta\}$.
In Figure 3.3, the simulation results are drawn, and the idea of the region $\Omega$ is sketched. In all cases where the initial values are inside $B_\delta(\bar{x})$, the system will always remain inside $\Omega$. Therefore, our critically damped system is *practical-$\Omega$-stable*.
In the simulation, noise is added on our critically damped system, but only added when the system is close enough to the equilibrium. The size of the disturbances is chosen as $\delta = 0.05$. The physical constants, including $\varphi$ remains the same, given in Table 2.9.

In this given example, practical-$\Omega$-stability is a nice feature. In the general, discussing hysteretic systems, this particular stability could be desirable in applications. For example, the $\Omega$-region can be given by a manufacturer of machines, who needs some precision-measure of his devices. This $\Omega$-region requires then a certain controller. In general, the smaller $\Omega$ is, the higher the input of the controller should be.

To assure practical-$\Omega$-stability in general, the system must be investigated, and the following theorem is used to check whether a system subject to hysteresis is practical-$\Omega$-stable. Remark that $\Omega$

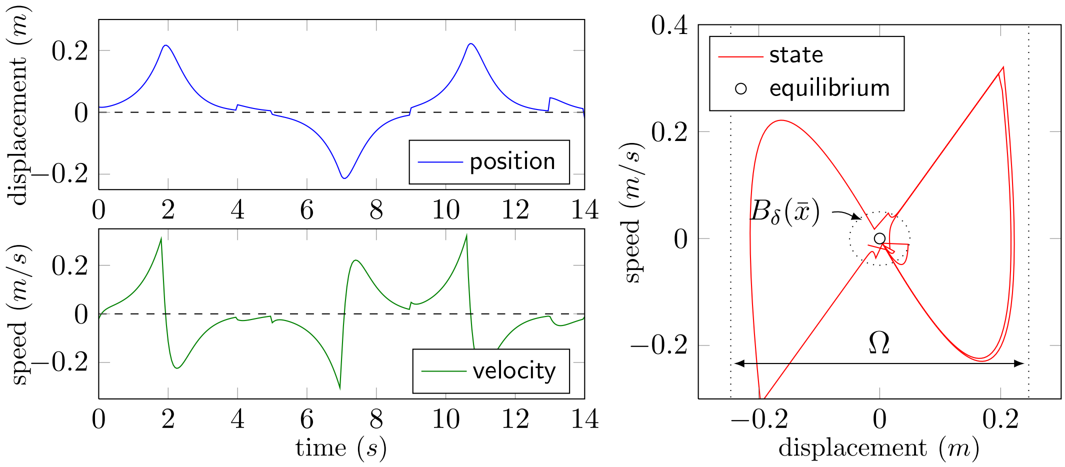



**Figure 3.3:** Simulation of an inverted pendulum with slackness in the pivot position, and noise added. The same critical damped feedback controller as in Example 3.5 is used. Left: the position (blue) and velocity (green) of the pendulum, as function of the time. Right: The phase plot (red) of the pendulum. The equilibrium (black) is shown at $(0,0)$. The boundaries of its $\delta$-neighbourhood and the $\Omega$-region are dotted.

**Theorem 3.14 (Practical-$\Omega$-stability of LTI-systems subject to hysteresis).** *Consider the following LTI-system, subject to hysteresis*

$$\dot{x} = Ax + B\mathcal{H}(u), \quad \text{with} \quad \mathcal{H}(u) = \begin{cases} \alpha u & \text{if } g \in S_1 \\ 0 & \text{if } g \in S_2 \\ \beta u & \text{if } g \in S_3 \end{cases} \tag{3.47}$$

*where $g$ behaves as a dynamic variable according to*

$$\dot{g} = f_i(u) \quad \text{if} \quad g \in S_i \quad \text{for} \quad i = 1, 2, 3. \tag{3.48}$$

*This system is practical-$\Omega$-stable if $S_2$ is bounded, and there exist some $u$, such that $f_2(u) \neq 0$, and the pair $(A, B)$ is controllable.*

**Proof:** *With the fact that $S_2$ is bounded, and there exist some $u$, such that $f_2(u) \neq 0$, we can state immediately that there is a possible input $u$ such that the time that the hysteron is in state $S_2$ is finite.*

*After this time, the hysteron system is certainly $S_{1,3}$, a region where the input of the hysteron directly propagates to the output. The state space equation becomes*

$$\dot{x} = Ax + B\alpha u \quad \text{or} \quad \dot{x} = Ax + B\beta u. \tag{3.49}$$

*which has controllable dynamics, since $(A, B)$ is a controllable pair.* ■

The difficulty of the overshoot can be countered by stretching the allowed region for stability. But to have a useful definition of practical stability, the temptation must be withstood to choose $\Omega$ extremely large. When $\Omega$ equals the whole behavior space, this definition is trivially true for all systems. Although the definition is still valid, the application has become useless. Therefore, the restriction that $\Omega$ must be bounded, is added.

## 3.4 Quasi stability

Although practical-$\Omega$-stability is a nice feature, more things can be said about the system of the given example. It must be observed that after all disturbances, the system converges to its equilibrium. Even more: after a finite time, the system comes arbitrary close to its origin. To catch this kind of behavior, we define an other kind of stability, given in the following definition, also inspired by Lakshmikantham et al. [12].

**Definition 3.15 (Quasi stability).** *A system* $\dot{x} = f(x)$ *is called* quasi stable *if for each* $\varepsilon > 0$ *there exist a* $\delta > 0$ *and a time* $T(\delta, \varepsilon) \geq 0$ *such that*

$$\|x(0)\| < \delta \quad \text{implies} \quad \|x(t)\| < \varepsilon \quad \text{for all} \quad t \geq T. \tag{3.50}$$

Remark that the time $T$ depends on the initial disturbance $\delta > 0$ and on the given boundary $\varepsilon > 0$ which the system must cross after this given time.

**Example 3.6 (continued):** Let us consider the same situation, again with the same critical damped controller, with the purpose to illustrate the quasi stability of the system. Therefore, with Example 2.4 in mind, we reason how long it takes to return to a specific $\varepsilon$-neighbourhood. In the worst case, a distortion of $\delta$ will let the pendulum fall into a free fall. We saw already in the previous simulation that the system reached its maximum at time $t_1 + t_2$. Since we know that the system is fixed sign stabilizable, so it converges monotonically to its equilibrium from this moment. Furthermore, after time $t_1$, the system is governed by

$$x(t) = \mathrm{e}^{(A+BN)t}\mathrm{e}^{At_1}x_0. \tag{3.51}$$

Since $\mathrm{e}^{At_1}x_0$ is given, and $(A + BN)$ has all eigenvalues in the open left half plane, this converges to the origin. So for each $\varepsilon > 0$, there is a $\delta > 0$ and a $T$, such that

$$\|x(0)\| < \delta \qquad \text{implies} \qquad \|x(t)\| < \varepsilon \qquad \text{for all} \qquad t \geq T. \tag{3.52}$$

It should be taken into account, that this particular $T$ not only depends on $\delta$, but also on the initial conditions $x_0$. A system where $\|x_0\| = \delta/2$ will take longer to fall than where $\|x_0\| = \delta$. To assure quasi stability, $\|x(T)\| < \varepsilon$ must hold, but also a fixed sign controller is needed to assure that $\|x(t)\| < \varepsilon$ holds for all $t \leq T$.
To illustrate this theory, we simulate this example with $\delta = 1/20$ and $\varepsilon = 1/100$, and initial conditions $x_0 = \begin{bmatrix}\delta & 0\end{bmatrix}$. The first time that the norm is smaller than $\varepsilon$ is numerically given by $T \approx 3.5$. Thereafter, this system will never leave this $\varepsilon$-neighbourhood, since it is critically damped (see Example 3.5). Therefore, this system is *quasi stable*, with given variables $\delta$, $\varepsilon$, $x_0$ and $T$.

This strategy can be used in general for systems which are time invariant. However, after a distortion, it could cost again $T$ seconds to be again in the $\varepsilon$-neighbourhood. The aim for this stability is useful for systems with not many distortions and/or with a small T, such that the system is returned fast, close to its origin. If there are a lot of distortions, and T is large, this definition will become useless. Remark that in the example all cases above, the controller is chosen, as a fixed sign controller, which makes the system critically damped. With an underdamped system, where the fixed sign notion does not hold any more, this notion of quasi stability will not hold any more.

In practise, this kind of stability can be used if it may take some time to get a desired, arbitrary small precision. For example, we could take the fact of parking a car in a parking lot. This could and may take some time, but when the driver has enough time to reverse and turn, he can park his car exactly in the right position.
Given the two new definitions of stability, our quest for finding a suitable controller can be split up in two pieces. With practical stability we search for *(1) minimizing* $\Omega$, and with quasi stability *(2) minimizing* $T$.

## 3.5 Local stability

Fixed sign controllers have the nice feature to make system stabilizable, in certain cases. However, even small distortions on the system can cause switching to other states, and then suddenly the fixed sign controlling will not have any purpose any more. Therefore, some robustness is desired.

**Theorem 3.16 (Stable hysteretic dynamical system).** *Suppose an autonomous hysteretic dynamical system* $\dot{x} = Ax + B\mathcal{H}(x)$*, where* $\mathcal{H}(x) = u_i$ *when the hysteron state* $g \in S_i$*. If the system* $\dot{x} = Ax + Bu_i$ *has a (asymptotically) stable equilibrium and there exist an* $\eta > 0$ *such that the hysteron state* $g$ *remains in* $S_i$ *for each* $x(t)$ *where* $\|x(t)\| < \eta$*, then* $\dot{x} = Ax + \mathcal{H}(x)$ *has a locally (asymptotically) stable equilibrium.*
**Proof:** *In words, the state will remain in* $S_i$ *when the distortions are not too extreme, i.e. smaller then* $\eta$*. When the state remains in* $S_i$*, then no switching occurs, so the hysteretic dynamical system behaves according to* $\dot{x} = Ax + B\mathcal{H}(x) = Ax + Bu_i$ *for all time* $t > 0$*. It was already assumed that this system has a stable equilibrium.* ■

Inspired by the theorem above, we can modify our example, to show that with a small modification, a stable system can be obtained. Although we realize that modifying the model of physical behavior is completely different of designing controllers, it is done here to illustrate how the theorem can be used.

**Example 3.7 (Sticky pendulum):** Suppose that the pendulum and the disk are sticked together, and at at least a torque of $\alpha$ is needed to decouple them. Then the origin will be locally stable, a small distortion in the system can keep the system inside the given neighbourhood. This will be elaborated.
The implementation of this stickiness can be done by changing the hysteron. We modify the hysteron, described in equation (2.20), into

$$v = \mathcal{H}(u) = \begin{cases} u & \text{if } g = -\varphi b \\ 0 & \text{if } -\varphi b < g < \varphi b \\ u & \text{if } g = \varphi b \end{cases} \quad \text{with} \quad \dot{g} = \begin{cases} \max(0, u - \alpha) & \text{if } g = -\varphi b \\ u & \text{if } -\varphi b < g < \varphi b \\ \min(0, u + \alpha) & \text{if } g = \varphi b \end{cases} \tag{3.53}$$

In words, this says that the slackness remain unchanged in all cases when $-\alpha < u < \alpha$. That means that the pendulum then will be attached to the disk. We observe (again) that the origin of the dynamical system $\dot{x} = Ax + Bu$ is (asymptotically) stable if control can be applied. By definition, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|x(0)\| < \delta \quad \text{implies} \quad \|x(t)\| < \varepsilon, \quad \text{for all } t \geq 0. \tag{3.54}$$

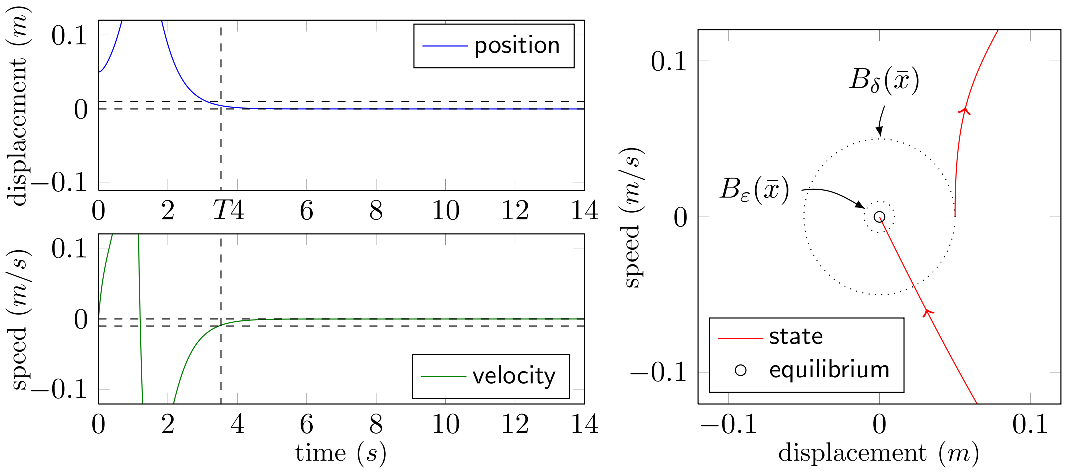



**Figure 3.4:** Simulation to illustrate quasi stability. A critical damped feedback controller is used. Left: the position (blue) and velocity (green) of the pendulum, as function of the time. At time $T$, the norm of the system is decreased to $\varepsilon$. Right: The phase plot (red) of the pendulum. The system starts with $\|x(0)\| = \delta$, and returns after some time in the $\varepsilon$-neighbourhood.

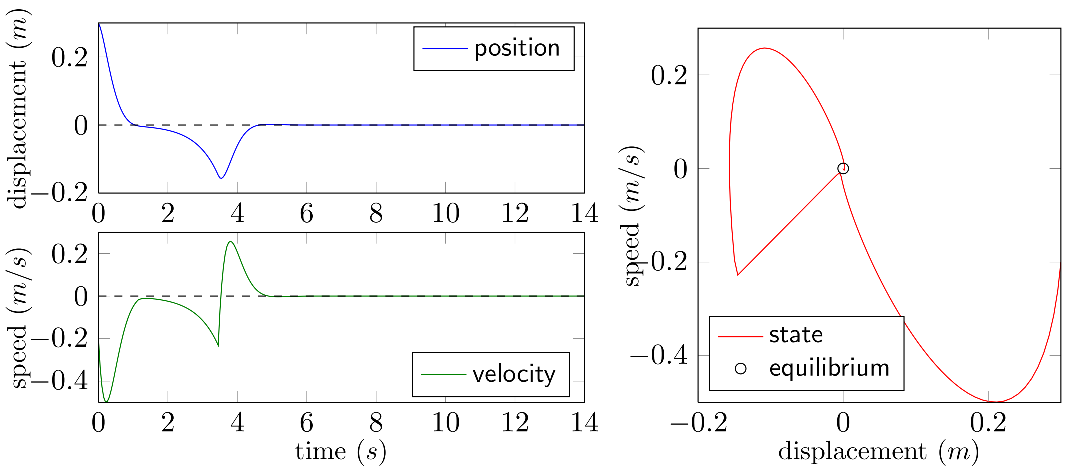



**Figure 3.5:** Simulation of an inverted pendulum with slackness in the pivot position, and some stickiness between pendulum and disk. An underdamped controller is used, the results can be compared with Figure 2.8. Left: the position (blue) and velocity (green) of the pendulum, as function of time. Right: The phase plot (red) of the pendulum. The equilibrium (black) is shown at $(0, 0)$. The 'glue'-value is $\alpha = 0.02$.

Therefore, with regular control of $u = -kx$, we know that $|u(t)| < k\varepsilon$, so when the pendulum remains inside $|x| < \frac{\alpha}{k}$, $\dot{x} = 0$, the hysteron state remains in $\{S_1, S_3\}$, and therefore the hysteretic dynamical system converge to its equilibrium. To achieve this, $\delta < \frac{k}{a}$ must be chosen.
To simulate this phenomenon, we take $\alpha = 0.02$. Further we take the same physical constants, initial values and controller as Example 2.3. Of course, the initial values are too large, $|\theta_0| > \frac{\alpha}{k}$, but when the system comes once close enough to the equilibrium (say, at time $T$), then it stays in that neighbourhood for all $t \geq T$. The results are given in Figure 3.5.
We observe an underdamped controller. The first overshoot is explained by the fact that the pendulum goes too far beyond the equilibrium position. This causes the hysteron state switch to $S_2$, and a free fall occures. After this swing more overshoots occurs since the underdamped controller. However the overshoot is not large enough to causes the hysteron state to switch towards $S_2$. Hence, a free fall does not occur any more.

## 3.6 Target area

In this section we discuss another stabilizing strategy. Inspired by the idea of local stability, and the properties of practical-$\Omega$-stability, we investigate the behaviour when the system is brought to a certain region, the neighbourhood of a desired state. As described earlier, this could be a neighbourhood of an equilibrium, but not necessarily. Then hopefully there can be some stability conclusions taken, when the controller is proper chosen.

A property of our pendulum is that its equilibrium lies on the edge of the sets where the hysteron state $g$ can be in. This is in general exactly the property which makes an equilibrium unstable, instead of locally stable. Therfore, if we define a neighbourhood such that the interior of all sets are represented, then controlling towards this neighbourhood is easier. A sketch of this idea is given in Figure 3.6. In this section, a neighbourhood is defined for our example, to show that if this neighbourhood is not too small, no switching behavior occurs.

**Example 3.8:** First we define the modes where the system can exist. We define $\Psi_{1,2,3}$ as

$$\Psi_1 = \{[\theta_p, g] \mid (g \in S_1) \cap (\theta_p \geq 0)\} \tag{3.55}$$
$$\Psi_2 = \{[\theta_p, g] \mid (g \in S_2) \cup (\theta_p < 0 \cap g \in S_1) \cup (\theta_p > 0 \cap g \in S_3)\} \tag{3.56}$$
$$\Psi_3 = \{[\theta_p, g] \mid (g \in S_3) \cap (\theta_p \leq 0)\}. \tag{3.57}$$

Suppose the pendulum, with the following controller: the input is chosen such that the system is critically damped towards $M$. Remark that this controller only affect the pendulum when the system is in mode $\Psi_{1,3}$. We choose $M = \{\theta_p \mid -c < \theta_p < c\}$.
Since the controller does not direct the pendulum into the equilibrium point, the gravitational force will always act on the pendulum. To eliminate this force, the controller gets an additional term $\pm mg_z lc$, where $\pm$ depends on the sign of the position of the pendulum. This can be written as $\mathrm{sgn}(\theta_p)$, and together, the controller becomes

$$u = -k_c \mathrm{dist}(x(t), M) - mg_z l\, c\, \mathrm{sgn}(\theta_p). \tag{3.58}$$

A MATLAB-simulation is done with $c = \varphi/4$, where all other constants remains similar to the previous simulations, given in Table 2.9. The results are given in Figure 3.7.

We observe a smooth, critical damped system, converging towards the neighbourhood of the origin.

This solution for a control scheme is a possibility when precision may be reduced, but where more stability is assured. Of course the choice of $M$, especially the size, is a trade-off between precision and stability. The larger $M$, the larger the region for local stability will hold, but the less precise the system is.
In general this strategy can be used when the desired state can be shifted away from the switch points of the hysteresis operator. Local stability is guaranteed with a proper controller, where the locality depends on the distance to the switch points of the hysteresis.
All the above control strategies which are discussed are relative low gain controllers. This are controllers which are easy to build, and cheap in use. However, if precision must be assured at all cost, other solutions must be sought. An investigation will be done in the next section.

## 3.7 Bang-bang controller

A complete other way of controlling a system subject to hysteresis, is the bang-bang strategy. The idea behind this bang-bang concept is to oscillate the input with sufficient high frequency from two outcomes, such that the 'average' of these inputs equals the desired inputs which is not directly reachable, for example due to the hysteresis. At least, with this kind of controller, we can let the input steer the system into a close neighbourhood.
In this section, this philosophy is worked out and a small, non-hysteretic example is given to illustrate this concept. Further, in this section, this idea is applied to a system subject to hysteresis. Finally, two simulations are done with the known example of the inverted pendulum with a slackness.

Suppose a dynamical system, which follows the dynamics $\dot{x} = Ax + Bu_1$ for a period of $\varepsilon/2$, and then switches to $\dot{x} = Ax + Bu_2$ for another period of $\varepsilon/2$. After this period, it switches back to the first described dynamics, and so on. This is called a bang-bang controlled system, since $\varepsilon$ is assumed to be small, while $u_1$ and $u_2$ forms together a discontinuous input. The system bangs from the first dynamics to the other. However, on average, the system will behave as it has an input $u = (u_1 + u_2)/2$. This will be stated in the next theorem.

**Theorem 3.17 (Bangbang controller).** *Suppose a simple bang-bang controller*

$$\dot{x} = \begin{cases} Ax + Bu_1 & \text{for} \quad (t \bmod \varepsilon) \in [0, \varepsilon/2) \\ Ax + Bu_2 & \text{for} \quad (t \bmod \varepsilon) \in [\varepsilon/2, \varepsilon) \end{cases} \tag{3.59}$$

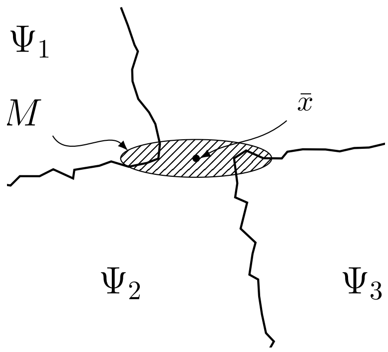



**Figure 3.6:** Sketch of the idea of a neighbourhood around the equilibrium point, where all modes are represented. In this illustration, the modes $\Psi_{1,2,3}$ corresponds to the description given in Example 3.8.

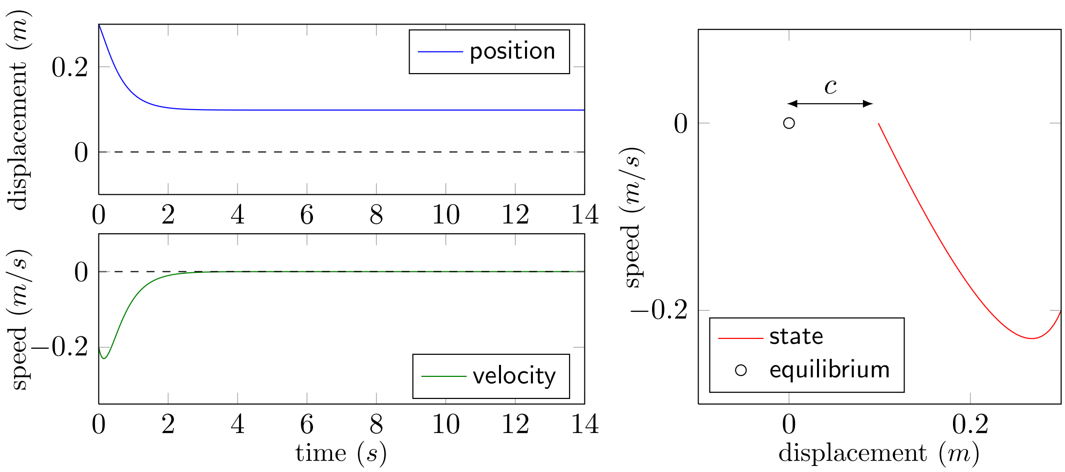



**Figure 3.7:** Simulation of an inverted pendulum with slackness in the pivot position. A critically damped controller is used, which steers the pendulum towards $x = \bar{x} + c$. Left: the position (blue) and velocity (green) of the pendulum, as function of time. Right: The phase plot (red) of the pendulum. The equilibrium (black) is shown at $(0, 0)$.

*for each period of length $\varepsilon$. When $\varepsilon$ goes to zero, the system* (3.59) *converges to*

$$\dot{x} = Ax + B\Big(\frac{u_1 + u_2}{2}\Big) \tag{3.60}$$

**Proof:** *According to the definition of a derivative, we see that*

$$\dot{x}(t) = \lim_{\varepsilon \to 0} \frac{x(t+\varepsilon) - x(t)}{\varepsilon} \tag{3.61}$$

*and by construction*

$$\dot{x}(t) = \lim_{\varepsilon \to 0} \frac{x(t+\varepsilon) - x(t+\varepsilon/2) + x(t+\varepsilon/2) - x(t)}{\varepsilon} \tag{3.62}$$

$$= \frac{1}{2}\lim_{\varepsilon \to 0} \frac{x(t+\varepsilon) - x(t+\varepsilon/2)}{\varepsilon/2} + \frac{1}{2}\lim_{\varepsilon \to 0} \frac{x(t+\varepsilon/2) - x(t)}{\varepsilon/2} \tag{3.63}$$

*These two terms are again the derivatives of $x(t)$, which is given in the first part of the interval, $[t, t+\varepsilon/2)$, by the first equation of* (3.59)*, and given by the second equation when it is in the second part of the interval. Therefore we state*

$$\dot{x}(t) = \frac{1}{2}\Big(Ax + Bu_2\Big) + \frac{1}{2}\Big(Ax + Bu_1\Big) = Ax + B\Big(\frac{u_1 + u_2}{2}\Big) \tag{3.64}$$

*which must be proven.* ■

In the above theorem, the input is composed by half of the time $u_1$, and the other half $u_2$. Of course, other input compositions are also possible. The average will be weighted by a factor which is equal to the partition of time. This is illustrated in the next, non-hysteretic example, Example 3.9.

**Example 3.9 (Controlling temperature, continued):**
As illustration, we look to our example of controlling the temperature of the boiler, Example 2.1. Suppose

$$\dot{T} = \begin{cases} -\dfrac{T}{20} + 0 & \text{for} \quad (t \bmod t_\varepsilon) \in [0, t_\varepsilon/4) \\ -\dfrac{T}{20} + 8 & \text{for} \quad (t \bmod t_\varepsilon) \in [t_\varepsilon/4, t_\varepsilon) \end{cases} \tag{3.65}$$

for each period of length $t_\varepsilon$. Now, we see by Theorem 3.17 that the system will converge to

$$\dot{T} = -\frac{T}{20} + \Big(\frac{1 \cdot 0 + 3 \cdot 8}{4}\Big) \tag{3.66}$$

for $t_\varepsilon$ sufficient small, which implies that the temperature will converge to a neighbourhood of $T = 120$. The smaller the $t_\varepsilon$, the more the system behaves like Equation (3.66). However, back to the reality of heating a boiler of a steam engine: it is very unrealistic for a train driver to switch each split second to the other heating policy.

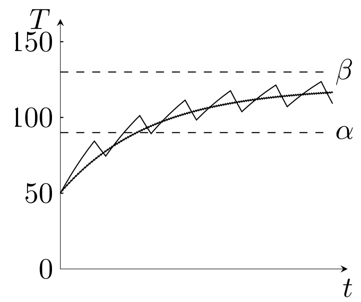



**Figure 3.8:** Temperature of the boiler, with two different chosen $t_\varepsilon$. The large sawtooth corresponds with a larger $t_\varepsilon$. This implies less switching, and less accurate in compare with the solution of Equation (3.66).

Remark that the whole hysteresis element of the dynamical system, described in equation (2.2), is neglected, and only the two possible outcomes $y_L = 0$ and $y_H = 8$

are used. In this example, no feedback is used to design this controller. The bang-bang-solution is also applied on the inverted pendulum problem, this is worked out in Example 3.10.

**Example 3.10 (Bang-bang controlled pendulum):** Suppose that the pendulum behaves exactly as stated in Example 2.3, but now a controller is applied, based on the bang-bang principle. As already stated in the theorem, the average must be the regular controller $u(t) = -kx(t)$.
In the first MATLAB-simulation, illustrated in Figure 3.9, we choose the controller as $u_1(t) = -kx(t) + 1$ for each $t$ where $(t \bmod \varepsilon) \in [0, \varepsilon/2)$ and $u_2(t) = -kx(t) - 1$ where $(t \bmod \varepsilon) \in [\varepsilon/2, \varepsilon)$. Clearly, the 'average' requirement is met. The period of the controller is chosen as $\varepsilon = 1/10$. The simulation results are given in Figure 3.9.
This academic example switches fast from input, such that the hysteron only switches from $S_1$ to $S_2$, the two sets in which the hysteron variable $g$ behaves. However, a bang-bang controller where the hysteron could be neglected must be much more rigorous. In fact, the disk must be oscillated very fast, from the situation that the pendulum is coupled at the right side, towards the situation that it is coupled at the left side (and vice versa). Mathematically speaking, the hysteron state $g$ must switch from $g(0) = -\varphi$ to $g(\varepsilon/2) = \varphi$ and back again to $g(\varepsilon) = -\varphi$. To achieve this, assuming that the input is constant, the input must be defined as

$$\pm 2\varphi = \int_0^{\varepsilon/2} \dot{g}(\tau) \mathrm{d}\tau = \int_0^{\varepsilon/2} \frac{u}{b} \mathrm{d}\tau \tag{3.67}$$

$$u = \pm \frac{2\varphi b}{\varepsilon/2} \tag{3.68}$$

Clearly, when the oscillations becomes faster, i.e. $\varepsilon$ becomes smaller, the input must be higher. A second MATLAB-simulation is done with the following controller

$$u_1(t) = -kx(t) + \frac{2\varphi b}{\varepsilon/2} \quad \text{for each } t \text{ where} \quad (t \bmod \varepsilon) \in [0, \varepsilon/2) \tag{3.69}$$

$$u_2(t) = -kx(t) - \frac{2\varphi b}{\varepsilon/2} \quad \text{for each } t \text{ where} \quad (t \bmod \varepsilon) \in [\varepsilon/2, \varepsilon). \tag{3.70}$$

The results are given in Figure 3.10 where again the switch period is chosen with $\varepsilon = 1/10$. Further, all physical constants and initial conditions remains the same.
We observe a much rougher behavior of the pendulum in compare to the first bang-bang controller, since the controller produces a much higher input for the system. However, it is more robust for distortions. In the previos simulation, there occured an overshoot. This will not happen with this controller, because it is not possible to stay in $S_2$ for longer than $\varepsilon$ time.

Actually, this solution needs some nuance in general. First of all, near to the equilibrium, the work what must be done is much higher then without such an oscillator. Furthermore, the ideal bang-bang controller oscillates infinitely fast. This means that $\varepsilon$ goes to zero, but looking to equation (3.68), the input will increase without bounds. This is clearly not possible with finite input.
An other disadvantage of this controller is the many collisions which take place by applying this controller. Obviously, this type of controller suffers from severe wearing. However, the position of the pendulum is quite precise, even when disturbance is added to the system.
With a bang-bang controller, there must be made a trade off in input and precision. Lower $\varepsilon$ requires higher input, but a shorter time of overshoot of the equilibrium. Other way round, a higher $\varepsilon$ has a lower input. But also the time that the system

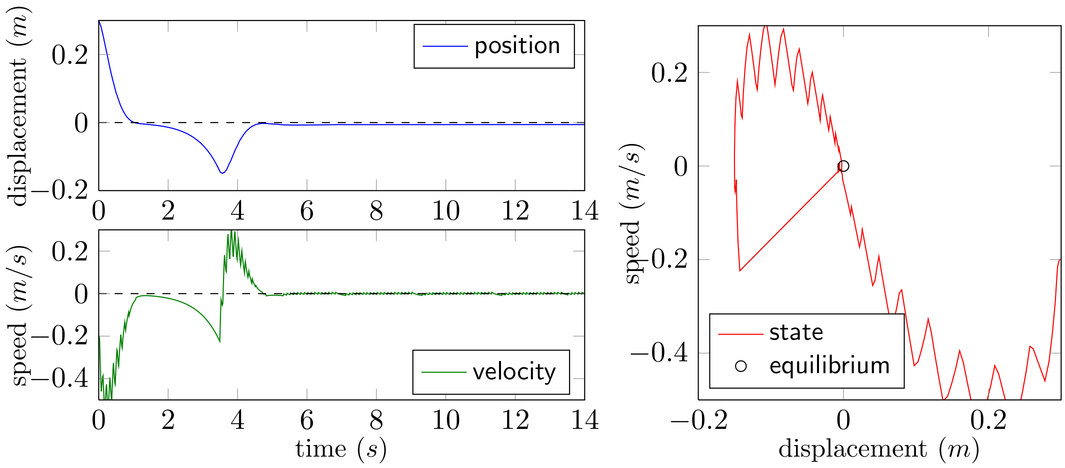



**Figure 3.9:** Simulation of an inverted pendulum with slackness in the pivot position. The underdamped controller is based on bang-bang, with $\varepsilon = 1/10$. The results can be compared with Figure 2.8. Left: the position (blue) and velocity (green) of the pendulum, as function of time. Right: The phase plot (red) of the pendulum. The equilibrium (black) is shown at $(0, 0)$.

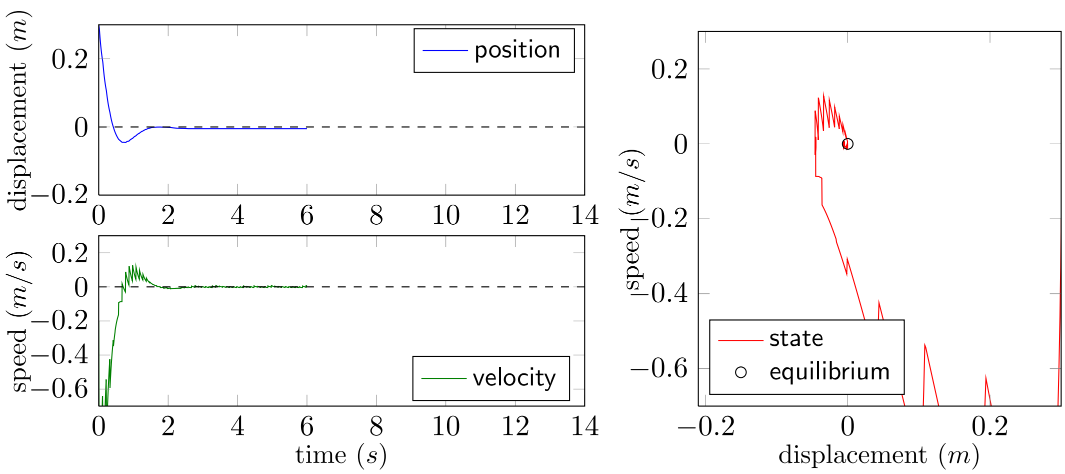



**Figure 3.10:** Simulation of the same example, with another bang-bangcontroller, described by equation (3.69)-(3.70). The time interval remains the same, $\varepsilon = 1/10$. The results can be compared with Figure 3.9. Left: the position (blue) and velocity (green) of the pendulum, as function of time. Right: The phase plot (red) of the pendulum. The equilibrium (black) is shown at $(0, 0)$.

steers in wrong directions becomes larger, so that precision becomes worse.

## 3.8 Switched control

In the previous section, we say that a bang-bang controller is a high gain controller. It has very high and rough input behavior. However, when the system is far away from its desired state, the bang-bang component of the controller,

$$\pm\frac{2\varphi b}{\varepsilon/2}, \tag{3.71}$$

is useless. Evenmore, in that situation it is even better to neglect this part at all. The relative larger the bang bang part of the controller, the more extreme the system will behave. For example, comparing Figure 3.9 with 3.10, we see that the second one has larger oscillations when the system is not close to its equilibrium.
Based on the idea of neglecting the bang-bang part of the controller, we describe in this section the design of a controller with different types of input, a so called switched controller. The bang-bang controller is already a special case of a swichted controller, but this controller depends on time and not the state wherein the system exists. The controller which directs to the target area is also a sort of switched controller, because the input switches when the system enters the target area.
Combining these things gives the following idea: Until the system is close enough to the desired state, regular, stabilizing control is applied to the system. In the neighbourhood of the desired state, the governing control becomes high gain. In the next example, this is elaborated with our pendulum. Also a simulation is done.

> **Example 3.11 (Switched control on inverted pendulum):** Suppose we are able to observe the hysteron state. Then we can make a controller which depend on this state, this is what we implement in the next MATLAB-simulation. Again, feedback control is used.
> Remark that $u = -k_c x$ makes the system critically damped. We decide to apply the following underdamped switched controller: when the hysteron state $g \in S_{1,3}$, then $u = -1.5k_c x$. In the case that $g \in S_2$, the controller has a ten times stronger gain: $u = -15k_c x$. This causes the hysteron to switch back to either $S_1$ or $S_3$, where the low gain control is again applied. Figure 3.11 shows the result of the simulation.
> We observe a good approach to the origin, and see that the pendulum will behave similar to the original example in Example **??** on page **??**. Only the amplitude is much smaller, since the controller gain is higher than original, which result in a shorter 'free fall'-modus.

A huge disadvantage of this type of controlling is, that it is nessecary to have information about the state of the hysteron. Most often, this is not known, and therefore this type of controlling is not possible on systems, subject to hysteresis. To circumvent this problem, one last idea is worked out.

Suppose a regular controller is applied to the hystesis, when the system is far away from the origin. Far away is defined as the norm of the state $x$ is much larger than the slackness, or hysteron state: $\|x\| \gg \|g\|$. Then the system will approximately behave as if there is no hysteresis. However, close to the origin, the hysteron will play a role in the behavior, and therefore, a bang-bang controller is applied in this case. This situation will be worked out in the well known example of the pendulum.

**Example 3.11 (continued):** Again switched controller is designed, which only depends on the position of the pendulum. When $|\theta_p| > c$, then the same low gain underdamped controller is applied as in Example 3.11. Otherwise, a bang-bang controller is applied as in Example 3.10. This is what we implement in the next MATLAB-simulation.
We observe a fast approach to the equilibrium. Close to the equilibrium, the pendulum will oscillate with a small amplitude. This is the result of the high gain, bang-bang controller, which is designed, to let the pendulum switch fast between the hysteron states. This type of controller makes the system robust for disturbances, since this disturbance is absorbed by the fast oscillations. In compare with the first switched controller, this controller keeps the pendulum closer to its equilibrium position.

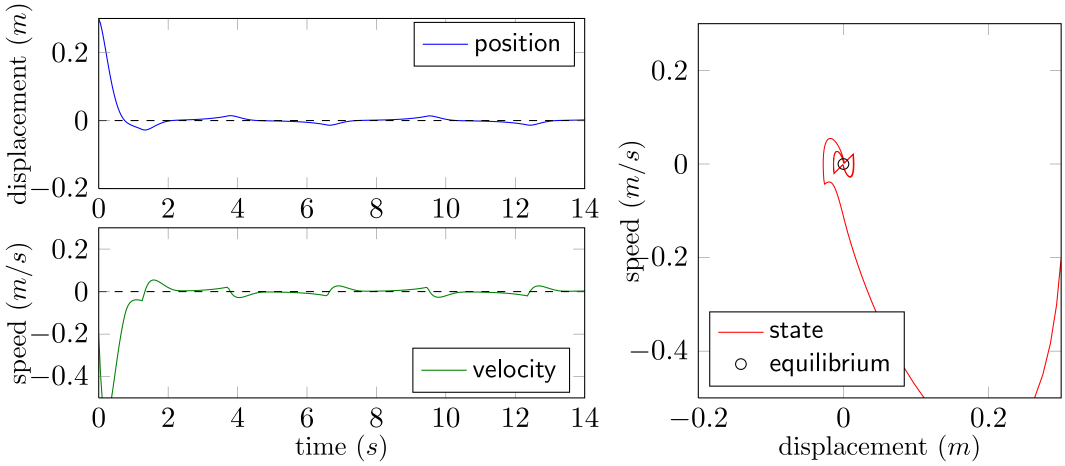



**Figure 3.11:** Simulation of an inverted pendulum with slackness in the pivot position. The controller exists of two parts, both are underdamped controllers. The controller which is applied when $g \in S_{1,3}$, has a gain of $k = 1.5k_c$. The underdamped controller, for $g \in S_2$ has a ten times higher gain, $k = 15k_c$. The results can be compared with Figure 2.8. Left: the position (blue) and velocity (green) of the pendulum, as function of time. Right: The phase plot (red) of the pendulum. The equilibrium (black) is shown at $(0, 0)$.

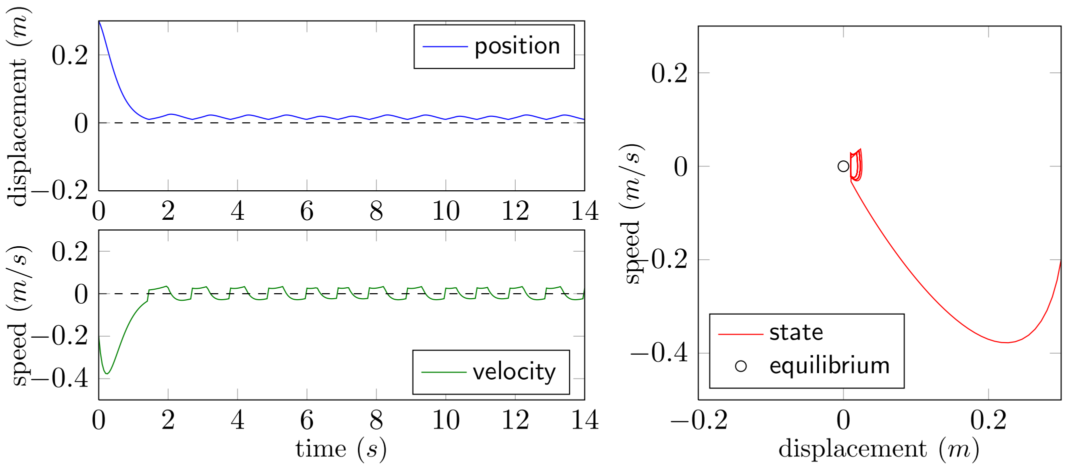



**Figure 3.12:** Simulation of an inverted pendulum with slackness in the pivot position. Again a switched controller is applied, an underdamped controller and a bang-bang controller. The controller which is applied far from the origin has a gain of $k = 1.5k_c$. The controller close to the origin behaves like the bang-bang principle. Left: the position (blue) and velocity (green) of the pendulum, as function of time. Right: The phase plot (red) of the pendulum. The equilibrium (black) is shown at $(0, 0)$.

# Chapter 4

# Conclusions

In the first part of this chapter the conclusions are discussed. The second part will contain some recommendations, and suggestions for further research.

**Conclusions** In this thesis the behavior of systems subject of hysteresis is explored. It gave insight where the difficulties arise, and shows that such systems are highly non-linear.

Controllability and stabilizability are investigated. Under assumptions, the controllers with fixed sign can steer a hysteretic system towards its equilibrium, without bothering the hysteresis. Fixed sign controllability in the discrete case is possible if and only if the $A$ matrix has no eigenvalues on the positive real axis. In the continuous case fixed sign controllability is possible if and only if $A$ has no eigenvalues on the whole real axis. Fixed sign stabilizability in the continuous case is guaranteed when the regular feedback matrix $(A + BN)$ has all eigenvalues on the negative real axis. However, it need feedback controller in the form of $u = \max(0, Nx)$.

Further, some notions of stability are given. Under assumptions, a system subject to hysteresis is practical-$\Omega$-stable and quasi-stable. These notions can be used in design of controllers, depending on the requirements of the system: If the system must be within certain bound for all time, then practical-$\Omega$-stability should be pursued. If the system must be very accurate, where time plays a secondary role, then quasi-stability is desirable.

Also other controllers are investigated: a bang-bang controller and a combination of bang-bang and a low gain: a switched controller. A bang-bang controller assures accuracy but is a trade off with the costs of high input. All these controllers are illustrated with an example. However, some numerical issues arise with a high frequency bang-bang controller and with a switched controller.

**Recommendations** First of all, in this thesis it is assumed that the hystertic element lies between controller and autonomous system. What if this element occurs in the feedback? Or what if this element appears twice? These questions are not handled here, but it is recommended to do so in further reseach. A paper with some illustrative examples about relay hysteretic controllers is written by Mahalanabis and Bhaumik [14]. However, also other recommendations must be taken into account.

Stability analysis of non-linear controller often uses Lyapunov second method. Khalil [9] discussed this for general non-linear systems. Johansson and Rantzer [8] and Heemels and De Schutter [6] use it to analyse stability of hybrid systems.

In this thesis no study on optimal control is done. However, in the development of controllers for systems subject to hysteresis, optimality is an aspect which is certainly interesting. Belbas and Mayergoyz wrote about it in [1], where they consider active hystersis. Optimal control can be defined as having the least cost (integrated input) to steer the state to a desired equilibrium. In this field, optimal control can also deal with the accuracy, the higher accurate, the better the control.

Robust control is also not mentioned in this report. Although the notion of local stability is made, it is not investigated how such a system can be made robust by certain controllers. An extensive survay of robustness of this particular kind of system is interesting because of the highly nonlinear behavior. This is already partly investigated by Valadkhan et al. [21].

In line with the recommendations about robustness, it should be mentioned that in reality, noise is an undesired property which must be taken into account. Only some system disturbances are mentioned in this thesis. Measurements, controllers and references are also prone to errors. Therefore, expanding to more types of disturbances, and investigate the behavior of this kind of systems is desirable.

# Appendix

## A Matlab code

The MATLAB-code which is used for the simulations in this master thesis is added as appendix. Of course, these scripts can be requested at the author.

**Listing A.1:** The main file `ExampleScript.m` is given here, which uses one of the three hysteron models. These three will (of course) give the same results, but uses an other way of modelling. This is described in the other listings.

```matlab
%% Time axis
global dt, dt = 0.001;
T = 14;
time = 0:dt:T;

%% Physical constants (according to Table 2.8)
m = 0.5;              %Mass of pendulum
l = 0.8;              %Length of pendulum
b = 2;                %Damping constant
Ip = m*l^2;           %Inertia of pendulum
gz = 9.81;            %Gravitational constant
phi = (pi/16);        %Slackness

%% Feedback gain for controller
Kc = 0.25*b^2/Ip + m*gz*l;      %To assure critical damping
K = Kc+2;                       %To assure an underdamped system.

%% Initial values
X = [0.3; -0.2];
u = 0; v = 0;
g = -phi;

%% Running for-loop
for i = 1:length(time)
    %Output of the controller, which depend on the position of the pendulum
    u(i) = controller(X(:,i),K);

    %Delay of the hysteron, as dynamical system.
%     % According to own equation 2.18
%     [v(i), g(i+1)] = piecewise(u(i),g(i));
%     % According the Duhemmodel
%     [v(i), g(i+1)] = duhem(u(i),g(i));
    % According the Relaymodel
    [v(i), g(i+1)] = relay(u,g);

    %Behavior of the pendulem, with a given input.
    X(:,i+1) = pendulum(X(:,i),v(i));
end

%% Plotting part
%Resize of the vectors, for plotting
X(:,end) = [];
plots(time, X, T)
```

**Listing A.2:** `pendulum.m` defines the behavior in the pendulum, as it is described in the examples. The used constants are also given in Table 2.9. It represents the system block, as sketched in Figure 2.6.

```
function [Xnew] = pendulum(X, u)
% X is de toestand van de pendule.
% u is de input van het systeem, gegeven in torque.
% Xnew is de nieuwe toestand van het systeem.

global dt
%% Dezelfde fysische constanten als het hoofdbestand: Table 2.8
m = 0.5;              %Mass of pendulum
l = 0.8;              %Length of pendulum
b = 2;                %Damping constant
Ip = m*l^2;           %Inertia of pendulum
gz = 9.81;            %Gravitational constant

%% Model van slechts de losse pendule, waarbij wrijving is meegenomen.
Xnew = X + ([0 1; gz/l -b/Ip] * X + [0; 1/Ip]*u)*dt;
end
```

**Listing A.3:** `controller.m` is build do model the control block, of the examples, as sketched in Figure 2.6.

```
function [u] = controller(X, k)
% Input X is de toestand van de pendule.
% Input k is de feedback-gain van de controller.
% u is de output van de controller, gegeven in torque.
u = - k * X(1);
end
```

**Listing A.4:** The function `piecewise.m` describes both the behavior of the slackness and the mapping of input to output of the hysteretic element. There are three different cases: left bound, right bound and the transient state.

```
function [v,gnew] = piecewise(u,g)
%v is de output van de hysteron. gnew is de nieuwe toestand, die wordt
%geupdate aan het einde. g moet worden meegegeven, om de toestand van het
%hysteron weer te geven. (gap) in meters.
%u is de input, in torque.
%v is de output, in torque.

global dt
b = 2;                %Damping constant
%De slackness phi is ingebakken in de hysteresis.
phi = (pi/16);        %Slackness

if g <= -phi          %Left bound
    g = -phi;
    v = u;
    gdot = max([0 u/b]);
elseif g >= phi       %Right bound
    g = phi;
    v = u;
    gdot = min([0 u/b]);
else %phi < g < phi %Transition state
    v = 0;
    gdot = u/b;
end

gnew = g + gdot*dt;
end
```

**Listing A.5:** Similar to `piecewise.m`, `duhem.m` describes three different cases. However, the mapping of input to output is handled separately, and modelled according the Duhem model, as described in Section 2.3.

```
function [v,gnew] = duhem(u,g)
```

```
%v is de output van de hysteron. gnew is de nieuwe toestand, die wordt
%geupdate aan het einde. g moet worden meegegeven, om de toestand van het
%hysteron weer te geven. (gap) in meters.
%u is de input, in torque.
%v is de output, in torque.

global dt
b = 2;              %Damping constant
%De slackness phi is ingebakken in de hysteresis.
phi = (pi/16);      %Slackness

%Relay ontwikkeling, los van de mapping van u naar v.
if (g <= -phi)      %Left bound
    g = -phi;
    gdot = max([0 u/b]);
elseif (g >= phi)   %Right bound
    g = phi;
    gdot = min([0 u/b]);
else %phi < g < phi %Transition state
    gdot = u/b;
end

%Duhem model
if (u <= 0)
    v = (1 - sign(g+phi)) * u;
elseif (u >= 0)
    v = (1 + sign(g-phi)) * u;
else
    v = 0;
end

gnew = g + gdot*dt;
end
```

**Listing A.6:** As third way of modelling the hysteron, `relay.m` models this according the formal description of a relay hysteron.

```
function [v,gnew] = relay(U,G)
%v is de output van de hysteron. gnew is de nieuwe toestand, die wordt
%geupdate aan het einde. G moet worden meegegeven, om de toestand van het
%hysteron weer te geven. (gap) in radialen.
%G is gehele vector van hysteron toestand.
%U is de gehele input, in torque.
%v is de output, in torque.
g = G(end);         %Laatste entry van toestand
u = U(end);         %Laatste entry van input

global dt
b = 2;              %Damping constant
%De slackness phi is ingebakken in de hysteresis.
phi = (pi/16);      %Slackness

%Relay ontwikkeling, los van de mapping van u naar v.
if (g <= -phi)      %Left bound
    g = -phi;
    gdot = max([0 u/b]);
elseif (g >= phi)   %Right bound
    g = phi;
    gdot = min([0 u/b]);
else %phi < g < phi %Transition state
    gdot = u/b;
end

% In de vergelijking wordt dit tijdstip genoemd als $t_1$.
lastup = find(G >= phi, 1, 'last');
lastdown = find(G <= -phi, 1, 'last');
% Een marge om nul heen, vanwege numerieke problemen
lastzero = find(abs(U) < 0.001, 1, 'last');
% Dit zijn overigens de indices die bij de tijdvector horen. Als dit event
```

```
% nog niet het geval is geweest, dan wordt er een lege matrix meegegeven.


% Vanaf de laatste entry waar g = phi zat, moeten ALLE entries van de input
% groter zijn dan 0, dus I moet bestaan uit 1-en.
Iup = (U(lastup:end) >= 0);
% Vanaf de laatste entry waar g = -phi zat, moeten ALLE entries van de input
% kleiner zijn dan 0, dus I moet bestaan uit 1-en.
Idown = (U(lastdown:end) <= 0);
% Vanaf dat de laatste input = 0, moeten ALLE waarden van G tussen
% -phi en phi zitten, dus I moet bestaan uit 1-en.
Izero = (G(lastzero:end)-phi < 0) & (G(lastzero:end)+phi > 0);

% Als alle entries na de switch groter zijn dan 0, (dus I moet bestaan
% uit 1-en.) dan output = input.
if (sum(Iup) == length(Iup)) && ~isempty(lastup)
    v = u;
% Analoog, alleen symmetrisch, output = input.
elseif (sum(Idown) == length(Idown)) && ~isempty(lastdown)
    v = u;
% Als er een input dicht genoeg bij 0 zit, dan moeten na die input ALLE
% entries van de switch tussen -phi < G < phi zitten. Dan output = 0.
elseif ((sum(Izero) == length(Izero)) && ~isempty(lastzero))
    v = 0;
else % Er zijn restgevallen: bijvoorbeeld bij initializatie.
    v = 0;
end

gnew = g + gdot*dt;
end
```

**Listing A.7:** The file plots.m is purely to plot the simulation results in the way how it is represented in the report.

```
function plots(time, X, T)
%Plot of the position of the pendulum, in function of time
subplot(2,5,1:3)
plot(time, X(1,:))
set(gca,'XTickLabel', [])
ylabel('displacement (m)')
hold on
plot([0 T], [0 0], ':k')     %Equilibrium
legend('position', 'Location', 'NorthEast')
xlim([0 T])
ylim([-0.2 0.5])

%Plot of the velocity of the pendulum, in function of time
subplot(2,5,6:8)
plot(time, X(2,:), 'color', [0 0.5 0])
xlabel('time (s)')
ylabel('speed (m/s)')
hold on
plot([0 T], [0 0], ':k')     %Equilibrium
legend('velocity', 'Location', 'SouthEast')
xlim([0 T])
ylim([-0.5 0.3])

%Phaseplot of the pendulum, position to velocity
subplot(1,5,4:5)
plot(X(1,:),X(2,:), 'r')
xlim([-0.2 0.3])
xlabel('postion (m)')
ylim([-0.5 0.3])
ylabel('speed(m/s)')
hold on
plot(0,0, 'ko')                 %Equilibrium in phaseplot
legend('state', 'equilibrium', 'Location', 'SouthWest')
end
```

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