16
Boundary layers

16.1 Introduction
When might we not be able to construct a regular perturbation expansion for a function in terms of a parameter $\epsilon \to 0$? Or, if we have one, where might it not be valid? One thing that might go wrong is that either the function we are approximating, or the approximation itself, may have singularities. Another is that the approximation may slowly drift away from the true solution, as we saw for the second term of the small-amplitude regular expansion for the pendulum. A third possibility is that our function oscillates very rapidly, with a period of, say, $O(\epsilon)$: we look at this case in Chapter 23. A fourth possibility is that the function changes rapidly in a very small layer, say of width $O(\epsilon)$, but is smooth elsewhere. Such a small layer is known as a boundary layer if attached to the boundary of the solution interval or domain, and an interior layer if it is internal; see Figure 16.1.

16.2 Functions with boundary layers; matching
Some functions come with built-in boundary layers. A prototype example, which crops up all over the place in applications, is

$$f(x; \epsilon) = e^{-x/\epsilon} \quad \text{for} \quad 0 < x < 1, \quad \epsilon \to 0.$$  

This function starts off with a value of 1 at $x = 0$, and becomes negligibly small, certainly smaller than any power of $\epsilon$, by the time $x \gg O(\epsilon)$. All its effort is concentrated in a boundary layer of thickness $O(\epsilon)$ near the origin. This example is rather trivial, but it is fairly clear that if $f$ is a bit
more complicated, say
\[ f(x; \epsilon) = e^{-x/\epsilon} g(x) + h(x), \]
where \( g(x) \) and \( h(x) \) are \( O(1) \) functions, then we don’t need to know all the details of \( g \) and \( h \) to have a pretty good idea of what \( f \) does. When \( x = O(1) \), the term \( e^{-x/\epsilon} \) is so small that we can usually forget about it, and we have the outer expansion
\[ f(x; \epsilon) \sim h(x) + \text{exponentially small correction} \]
(the exponentially small correction often goes by the catch-all name of \textit{transcendentally small terms}). When \( x \) is small, however, we expect \( g(x) \) and \( h(x) \) to be close to their initial values \( g(0) \) and \( h(0) \), so that
\[ f(x; \epsilon) \sim g(0)e^{-x/\epsilon} + h(0), \]
albeit here it is not quite so obvious how big the error is.

The real point of this discussion is not to tell us how to expand functions that we already know. It is that we can often describe a function with a boundary layer using two expansions, an outer expansion valid away from the boundary layer and an \textit{inner expansion} valid in the boundary layer. In an application, the full function may be the solution of some horrendously difficult problem, but if we can identify where the boundary layers are we may be able to formulate simpler problems for the inner and outer expansions and thereby obtain a good description of the full solution without actually having to find it as a whole.

Before we plunge into a series of examples, we should first look a little more closely at the question of how we ‘join up’, or \textit{match}, the inner and outer expansions. We’ll do this first assuming we know the full function, so that we are just verifying that we can do it. Later, we

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1 The Navier–Stokes equations spring to mind: the viscous boundary layer in high Reynolds number flow is an early and classic example of the technique in action.

Notice that the limit as \( x \to 0 \) of the outer solution, \( h(0) \), is not in general equal to \( f(0; \epsilon) \). It is the job of the boundary layer to accommodate this discrepancy.
will use the matching to convey information between the two regions so as to complete the solution. For example, we may have undetermined constants as the result of solving a differential equation, and if so we fix these by matching.

16.2.1 Matching

There are various ways of joining together inner and outer expansions, and it is in the nature of the subject that no way is universal; there are examples for which any method fails. However, the Van Dyke rule, which we now discuss, is as robust as any and it certainly works for all the problems treated in this book.

Let us return to the example we have just discussed, but with a slightly more complicated function

\[ f(x; \varepsilon) = e^{-x/c} g(x; \varepsilon) + h(x; \varepsilon), \]

where \( g(x; \varepsilon) \) and \( h(x; \varepsilon) \) have regular expansions

\[ g(x; \varepsilon) \sim g_0(x) + \varepsilon g_1(x) + \cdots, \quad h(x; \varepsilon) \sim h_0(x) + \varepsilon h_1(x) + \cdots, \]

valid in the whole domain, say the interval \([0, 1]\). For example, take

\[ f(x; \varepsilon) = e^{-x/c} (1 + x) + x + e^{\varepsilon x}, \]

so that here

\[ g(x; \varepsilon) = 1 + x, \quad h(x; \varepsilon) = x + e^{\varepsilon x} \sim 1 + x + \varepsilon x + \frac{1}{2} \varepsilon^2 x^2 + \cdots. \]

The function \( f(x; \varepsilon) \) is plotted as the solid curve in Figure 16.3. Now for any fixed value of \( x > 0 \), the exponential term \( e^{-x/c} \) tends to zero so

\[ 2 \text{ A popular alternative is matching via an 'intermediate region' between the boundary layer and the outer solution; see Exercise 3. Much cruder is 'patching', in which we simply equate the values of the inner and outer expansions at a set value of (say) } x: \text{ this cannot inform us about the structure of the problem but it can be a useful part of a numerical attack.} \]
fast as $\epsilon \to 0$ that we can neglect it by comparison with any power of $\epsilon$.

The outer expansion for this example is therefore

$$f(x; \epsilon) \sim 1 + x + \epsilon x + \frac{1}{2} \epsilon^2 x^2 + \cdots,$$

and it is valid provided that $x \gg O(\epsilon)$. However, it does not give a good picture of what happens near $x = 0$; and indeed, its limit as $x \to 0$, which is 1, is not equal to $f(0, \epsilon) = 2$. This is the dashed curve in Figure 16.3.

We can investigate the behaviour near the origin more closely by rescaling $x$ in the boundary layer, writing $x = \epsilon X$; the variable $X$ is often known as a boundary layer or inner variable. This gives

$$f(x; \epsilon) = F(X; \epsilon) = g(\epsilon X) e^{-X} + h(\epsilon X).$$

Now it should be safe to construct a regular expansion of $g(\epsilon X)$ and $h(\epsilon X)$, to give

$$F(X; \epsilon) \sim e^{-X} \left( g(0) + \epsilon X g'(0) + \cdots \right) + h(0) + \epsilon X h'(0) + \cdots.$$

$$= F_0(X) + \epsilon F_1(X) + \cdots.$$

This is the inner expansion. For our example, we have

$$F(X; \epsilon) = e^{-X}(1 + \epsilon X + \epsilon^2 X \epsilon X) + \cdots \sim 1 + e^{-X} + \epsilon (X e^{-X} + X) + \epsilon^2 X + \cdots.$$

This is the dotted curve in Figure 16.3.

How should we achieve the joining of the inner and outer expansions?

**Van Dyke's matching principle.** Van Dyke's matching rule is a way of achieving the joining up. It is stated as follows.

The $m$-term inner expansion of the $n$-term outer expansion

matches with

the $n$-term outer expansion of the $m$-term inner expansion.
What on earth does this tell us? It says that we must do the following.

1. Construct $n$ terms of the outer expansion in terms of the outer variable $x$. That is, expand up to the first $n$ of the gauge functions (for example, powers of $\epsilon$).
2. Rewrite this expansion in terms of the inner variable $X$.
3. Expand again in terms of the gauge functions for the inner expansion. (These are often, but not always, the same as the outer gauge functions.)
4. Retain the first $m$ terms.

This constructs the first line of the matching principle above. Then, proceed the other way round:

1. Construct $m$ terms of the inner expansion in terms of the inner variable $X$.
2. Rewrite this expansion in terms of the outer variable $x$.
3. Expand again in terms of the gauge functions for the outer expansion.
4. Retain the first $n$ terms.

That gives the third line of Van Dyke's rule. Finally, these two expansions should match; that is, they should represent the same function. Notice that you simply swap the positions of 'm-term inner' and 'n-term outer' in stating the two parts of the rule, so it does express a kind of commutativity.

Let's see how this works for the example above. Recall that the outer expansion is

$$f(x; \epsilon) \sim f(x; \epsilon) \sim 1 + \epsilon x + \frac{1}{2} \epsilon^2 x^2 + \cdots$$

and the inner expansion is

$$F(X; \epsilon) \sim 1 + e^{-X} + \epsilon(Xe^{-X} + X) + \epsilon^2 X + \cdots.$$ 

Start, as always, with the easiest problem.

**One-term outer and inner expansions, $m = n = 1$.** The one-term ($n = 1$) outer expansion is

$$1 + x.$$ 

In inner variables, this is

$$1 + \epsilon X.$$ 

Expanded to one term ($m = 1$), which means that we truncate it by leaving out all smaller terms, we have

$$1.$$
Going the other way, the one-term \((n = 1)\) inner expansion is

\[1 + e^{-x}.
\]

In outer variables, this is

\[1 + e^{-x/e}\]

which, expanded to one term, is

\[1,
\]

because the exponential is small. The two expansions do indeed match.

To see this, let's do two terms in each expansion:

\[
\begin{align*}
1 &+ x + \epsilon x & \text{two-term outer} &\ldots \\
1 &+ \epsilon X + \epsilon^2 X^2 & \ldots \text{ in inner variables} &\ldots \\
1 &+ \epsilon X & \ldots \text{ expanded to two terms} &\ldots \\
1 &+ e^{-X} + \epsilon (X e^{-X} + X) & \text{two-term inner} &\ldots \\
1 &+ e^{-x/e} + x(1 + e^{-x/e}) & \ldots \text{ in outer variables} &\ldots \\
1 &+ x & \ldots \text{ expanded to two terms} &\ldots 
\end{align*}
\]

Again, the two sides of Van Dyke agree.

16.3 Examples from ordinary differential equations

For most of us, the first encounter with boundary layers comes via an ordinary differential equation, and there are many fascinating problems arising in this area. Boundary layers commonly occur when a small parameter in a problem multiplies the highest derivative, because then that derivative can become large without imbalancing the equation. Typically, the outer expansion, in which the higher derivative term is neglected, fails to satisfy one or more of the boundary conditions. Rescaling in the boundary layer allows us to rectify this situation (and should lead to a simpler 'inner' problem). A rather similar situation was discussed earlier, in the context of the quadratic equation \(\epsilon x^2 + x - 1 = 0\); there dropping the term of highest degree lost us one of the roots.

Our first example is a first-order differential equation where, having constructed the inner and outer expansions, they match automatically. In the second example, a second-order equation, we have to go through the matching process in order to determine some of the constants in the solution, so it plays a vital role.
A first-order equation. Consider the first-order equation
\[ \varepsilon \frac{dy}{dx} + y = \sin x, \quad x > 0, \quad y(0) = 1. \]

Or a sensible guess.

It is easy enough to solve by an integrating factor:
\[ y = \frac{\sin x - \varepsilon \cos x}{1 + \varepsilon^2} + \frac{\varepsilon}{1 + \varepsilon^2} e^{-x/\varepsilon}, \]

and this solution can be used to verify all the results of the approximate expansion. For the outer expansion, we try a regular perturbation series
\[ y \sim y_0 + \varepsilon y_1 + \cdots, \]

to find that
\[ \varepsilon \left( \frac{dy_0}{dx} + \varepsilon \frac{dy_1}{dx} + \cdots \right) + y_0 + \varepsilon y_1 + \cdots = \sin x. \]

Successive terms are just read off:
\[ \text{at } O(1), \quad y_0 = \sin x, \]
\[ \text{at } O(\varepsilon), \quad y_1 = \frac{dy_0}{dx} = -\cos x \]
and so on. However, this expansion does not satisfy the initial condition, the danger signal being the \( \varepsilon \) multiplying the highest derivative, as a consequence of which we never have to solve a differential equation but merely need to differentiate functions that are already known.

Now for the inner expansion. We don’t, at this stage, know how big it should be, although we can have a pretty good guess, as it will be determined by a balance between the omitted highest derivative and some other term. Still, suppose that it is a region near \( x = 0 \) of size \( O(\delta) \), where \( \delta \ll 1 \). So, write
\[ x = \delta X, \quad y(x) = Y(X). \]

Then the rescaled (inner) problem is
\[ \varepsilon \frac{dY}{\delta dX} + Y = \sin \varepsilon X, \quad X > 0, \quad Y(0) = 1. \]

Now, in the rescaled differential equation the term \( Y \) is a priori \( O(1) \) and \( \sin \varepsilon X \) is \( O(\varepsilon) \). So the only way to get a balance is to take \( \delta = \varepsilon \), leaving
\[ \frac{dY}{dX} + Y = \sin \varepsilon X \sim \varepsilon X - \cdots. \]

Solving this by a regular expansion, the first two terms in the inner expansion are
\[ Y \sim e^{-X} + \varepsilon(X - 1 + e^{-X}) + \cdots. \]
Now for the matching, beginning with the one-term outer expansion. This is \( \sin x \), which in inner variables is \( \sin \epsilon X \sim 0 + O(\epsilon) \). So, the one-term inner of the one-term outer is zero. Now dig out the the one-term inner, which is \( e^{-X} = e^{-x/\epsilon} \) in outer variables. The one-term outer is also zero, and so matching works at this order. If we take two terms, the two-term outer is \( \sin x - \epsilon \cos x \), which to two terms in the inner variable is \( \epsilon (X - 1) \) and matches with what is left of the two-term inner \( e^{-X} + \epsilon (X - 1 + e^{-X}) \) after it has been written in outer variables and expanded to two terms (so the exponentials both go). Higher-order matching can be carried out in a similar fashion.

**A second-order equation.** In second-order problems, the matching can convey useful information from the outer expansion to the inner one or vice versa. Suppose for example that

\[
\frac{d^2 y}{dx^2} + \frac{dy}{dx} = \frac{1}{1 + x^2},
\]

with \( y(0) = 0 \) and \( y \to 1 \) as \( x \to \infty \), a two-point boundary value problem on an infinite interval. We can solve it explicitly, with some work, but that is weightlifting.

Expanding in powers of \( \epsilon \), the leading-order outer problem is

\[
\frac{dy_0}{dx} = \frac{1}{1 + x^2},
\]

from which

\[
y_0 = \tan^{-1} x + c_0,
\]

where \( c_0 \) is a constant. Now, we can satisfy the boundary condition at infinity by choosing \( 1 = \frac{1}{2} \pi + c_0 \), so that \( c_0 = 1 - \frac{1}{2} \pi \), but then \( y \) does not vanish at \( x = 0 \). However, rescaling \( x = \epsilon X \) near \( x = 0 \) gives the inner problem

\[
\frac{d^2 Y}{dX^2} + \frac{dY}{dX} = \frac{\epsilon}{1 + \epsilon^2 X^2},
\]

with \( Y(0) = 0 \). The leading-order solution is

\[
Y_0(X) = C_0 (1 - e^{-X})
\]

where \( C_0 \) is still unknown. However, matching with the outer solution is achieved at this order if \( C_0 = y_0(0) = c_0 = 1 - \frac{1}{2} \pi \): the matching is essential to specify the inner solution fully. Higher-order terms in the expansion will throw up further undetermined constants, which will be determined by higher-order matching.

An important practical point to note is that the inner problem is effectively solved on an infinite domain. Suppose that the outer problem has a
boundary condition specified at $x = 1$, say $y(1) = 1$. In inner variables, this translates into a condition at the far end $X = 1/\epsilon$. However, this equals infinity for all practical purposes, and the condition at $X = 1/\epsilon$ is replaced by a matching condition at infinity (in the inner region).

We have just scratched the surface of the many fascinating examples that have been devised and investigated for ordinary differential equations alone. For more details, see books such as [27, 33, 36, 49]. We now return to an earlier case study, before moving on to look at some partial differential equations.

## 16.4 Case study: cable laying

Recall that in our case study of laying an undersea cable (see Section 4.3), we wrote down a model in which the angle $\theta$ between the cable and the horizontal satisfies

$$
\epsilon \frac{d^2 \theta}{ds^2} - F^* \sin \theta + (F_0 + s) \cos \theta = 0,
$$

where $F_0$ is an unknown constant (equal to the dimensionless vertical force on the sea bed at the point where the cable touches down), $F^*$ is a known dimensionless constant and $\epsilon$ is a small dimensionless constant measuring the relative importance of the cable rigidity and the cable weight. The boundary conditions for the problem are

$$
\theta = 0, \quad \frac{d\theta}{ds} = 0 \quad \text{at} \quad s = 0,
$$

and $\theta$ is prescribed at $s = \lambda$.

This problem is ideally suited to a boundary layer expansion, with a small parameter multiplying the highest derivative. The leading-order outer solution $\theta_0(s)$ satisfies

$$
\tan \theta_0 = \frac{s + F_0}{F^*},
$$

but it does not satisfy the conditions at $s = 0$. Before investing too much energy in it, let us look at the possibility of a boundary layer near $s = 0$. Clearly $\theta$ is small in such a layer, and a little playing around, starting with the obvious guess that the boundary layer is where $s = O(\epsilon^{1/2})$, suggests the scalings

$$
s = \epsilon^{1/2} \xi, \quad \theta = \epsilon^{1/2} \phi, \quad F_0 = \epsilon^{1/2} f_0,
$$

following which the leading-order term in a regular expansion for $\phi$ satisfies

$$
\frac{d^2 \phi_0}{d\xi^2} - F^* \phi_0 + s + f_0 = 0.
$$
Because there can be no exponentially growing term, the two boundary conditions at \( \xi = 0 \) tell us both \( \phi_0 \) and \( f_0 \):

\[
\phi_0(\xi) = \frac{\xi}{F^*} - \frac{1}{(F^*)^{3/2}} \left( 1 - e^{-\xi(F^*)^{1/3}} \right), \quad f_0 = -\frac{1}{(F^*)^{1/2}}. \tag{16.1}
\]

It is easy to see that this matches with the outer solution since, substituting for \( F_0 \) and writing \( s = e^{1/2\xi} \) in our expression for \( \theta_0 \), we see that the inner limit of the outer solution is

\[
e^{1/2\xi} \frac{\xi + f_0}{F^*},
\]

which is just the same as the outer limit of the inner solution, obtained by neglecting the exponential term in (16.1).

We have also learned that \( F_0 \) is small, so away from the boundary layer the outer solution satisfies

\[
\tan \theta_0 = \frac{s}{F^*}.
\]

See the exercises at the end of the chapter for a demonstration that the solution of this equation is a catenary, as we might expect if the bending stiffness of the cable is negligible.

### 16.5 Examples for partial differential equations

Although ordinary differential equations lead to many interesting boundary layers, the technique has its greatest impact when applied to partial differential equations, simply because they are so much more difficult. The original boundary layer was Prandtl’s analysis of the high-Reynold’s-number flow of a viscous fluid past a flat plate, and fluid mechanics remains a prolific source of these problems. However, we use simpler examples from heat flow and potential theory to illustrate the ideas involved.

#### 16.5.1 Large-Peclet-number advection–diffusion past an infinite flat plate

Suppose that a liquid flows with velocity \((U, 0)\) past a flat plate along the positive \(x\) axis, that the temperature at infinity is zero and that the plate is heated to a temperature \(T(x)\). Finding the heat transfer from the plate to the fluid is a prototype problem for many practical situations. Recall from Chapter 3 that the relevant dimensionless model for the temperature \(u(x, y)\) is

\[
\text{Pe} \left\frac{\partial u}{\partial x} \right = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},
\]

where \(\text{Pe}\) is the Peclet number.
with the boundary condition
\[ u(x, 0) = T(x), \quad y = 0, \quad x > 0, \]
and the condition that \( u \to 0 \) at infinity. By symmetry, we need solve only for \( y > 0 \).

Suppose that the Peclet number is large (advection-dominated heat transfer), so that we can write
\[ \text{Pe} = \frac{1}{\epsilon^2}, \]
where \( 0 < \epsilon \ll 1 \). Thus
\[ \frac{\partial u}{\partial x} = \epsilon^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \]
with
\[ u(x, 0) = T(x), \quad y = 0, \quad x > 0, \quad u \to 0 \text{ at infinity.} \]

What is the structure of the problem?

First we try a regular expansion in powers of \( \epsilon \), \( u \sim u_0 + \epsilon u_1 + \ldots \). The lowest-order equation is
\[ \frac{\partial u_0}{\partial x} = 0, \]
which, with the fact that \( u \) vanishes at upstream infinity, means that \( u_0 = 0 \). This in turn means that there can be no ‘source’ term for \( u_1 \), and it too vanishes identically, as do all the higher-order terms in the expansion. As in the ordinary-differential-equation examples, the regular perturbation expansion fails to satisfy both boundary conditions (at infinity and on the \( x \)-axis). The danger signal is, as before, that the small parameter \( \epsilon^2 \) multiplies the highest derivatives in the advection–diffusion equation.

We rectify this situation with a boundary layer near the plate; it is known as a thermal boundary layer. The scaling we need is \( y = \epsilon Y \), \( u(x, y) = U(x, Y) \), leading directly to
\[ \frac{\partial U}{\partial x} = \epsilon^2 \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial Y^2}. \]
The first term in a regular perturbation, \( U_0(x, Y) \), satisfies
\[ \frac{\partial U_0}{\partial x} = \frac{\partial^2 U_0}{\partial Y^2}, \quad U_0(x, 0) = T(x), \quad U_0(x, Y) \to 0 \text{ as } Y \to \infty \]
(the last condition is a simple matching condition with the outer solution: again, note how the matching replaces a boundary condition far away from the boundary in the inner region). This is a parabolic equation for \( U_0 \), in which \( x \), which measures distance along the plate, plays the role of time and \( Y \) is the space variable; with the initial condition \( U(0, Y) = 0 \),
also obtained by matching back to the outer expansion, its solution can be written down as an integral using Duhamel’s principle. When \( T(x) = 1 \) there is a similarity solution \( U_0(x, Y) = F(Y/\sqrt{2x}) \), where

\[
\frac{d^2 F}{dz^2} + z \frac{dF}{dz} = 0, \quad z = \frac{Y}{\sqrt{2x}},
\]

so that

\[
U_0(x, Y) = \int_{Y/\sqrt{2x}}^{\infty} e^{-s^2/2} \, ds.
\]

This formula (and the Duhamel solution for non-constant plate temperature) shows that the thermal boundary layer grows as the square root of distance along the plate.

Further aspects of this problem, including a direct comparison with the exact solution when \( T(x) = 1 \), are dealt with in Exercise 7. A more complicated example, large-Peclet-number flow past a cylinder, is the subject of Exercise 8; it explains how a balance between conduction and advection leads to the amplification of conduction known as wind-chill.

### 16.5.2 Traffic flow with small anticipation

In Chapter 8 we looked at the simple model

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U(\rho))}{\partial x} = 0
\]

(16.2)

for the density \( \rho(x, t) \) of traffic travelling along a road. We saw that shocks, described by curves \( x = S(t) \), can form and that their speed is given by the Rankine–Hugoniot relation

\[
\frac{dS}{dt} = \frac{[\rho U(\rho)]^+}{[\rho]^+}.
\]

We also suggested that if drivers anticipate the traffic density, rather than simply responding to its local value, the model

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U(\rho))}{\partial x} = \epsilon \frac{\partial}{\partial x} \left( \rho \frac{\partial \rho}{\partial x} \right)
\]

(16.3)

might be appropriate. In Exercise 5 on p. 113 you showed that travelling waves of this equation moving with speed \( V \), in which \( \rho \) changes from \( \rho_- \) at \( \xi = x - Vt = -\infty \) to \( \rho_+ \) at \( \xi = \infty \), also lead to the Rankine–Hugoniot condition in the form

\[
V = \frac{[\rho U(\rho)]_{-\infty}^{\infty}}{[\rho]_{-\infty}^{\infty}}.
\]
We can now tie these results together using matched expansions for (16.3) rather than Rankine–Hugoniot for (16.2).

The idea is to treat the shock as an interior layer – a boundary layer that is not fixed onto a boundary – for equation (16.3). A regular expansion of the solution to (16.3) away from the shock (the outer expansion) simply leads to (16.2) at leading order. There will be a shock at an as yet unknown location \( x = S(t) \).\(^3\) Near this location, introduce the inner variable
\[
x = S(t) + \epsilon X, \quad \text{so that} \quad \frac{\partial}{\partial x} \leftrightarrow \frac{1}{\epsilon} \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial t} \leftrightarrow \frac{\partial}{\partial t} - \frac{1}{\epsilon} \frac{dS}{dt} \frac{\partial}{\partial X}.
\]

With \( \rho(x, t) = R(X, t) \), the inner problem is
\[
\frac{\epsilon}{\partial t} \left( \frac{dS}{dt} \frac{\partial R}{\partial X} \right) + \frac{\partial (RU(R))}{\partial X} = \frac{\partial}{\partial X} \left( R \frac{\partial R}{\partial X} \right).
\]

Once again, in the spirit of matched expansions, we solve this equation for \(-\infty < X < \infty\), with matching conditions at \( X = \pm\infty \). When we construct the leading-order term \( R_0 \) in a regular expansion, the time derivative does not feature. That is, time only appears as a parameter, through \( dS/dt \); the solution is ‘slowly varying’ in \( t \) on the inner scale, although on the outer \( (O(1)) \) scale \( t \) is fully involved.

The final piece of the jigsaw is the matching procedure: we need to impose \( R_0 \to \rho_{\pm} \) as \( X \to \pm\infty \), where \( \rho_{\pm} \) are the limiting (leading-order) outer values of \( \rho \) on either side of the shock. Putting the whole lot together, this gives precisely the Rankine–Hugoniot condition for \( dS/dt \); we see that we can interpret the inner layer as a smoothed-out shock.

### 16.5.3 A thin elliptical conductor in a uniform electric field

Sometimes it is the geometry of the solution domain, rather than the differential equation itself, that leads to a matched-expansion problem. As a simple example, suppose that, in two dimensions, a perfect conductor in the shape of the ellipse
\[
x^2 + \frac{y^2}{2\epsilon^2} = 1
\]
is placed in a uniform electric field \((E, 0)\), for which the potential (without the ellipse) is \( \phi(x, y) = -Ex \). We want to calculate the electric potential when the ellipse is present.

\(^3\) Technically, we should expand \( S(t) \) in terms of \( \epsilon \), but we are not going to calculate to the order of accuracy that would warrant this step.
The problem to solve is $\nabla^2 \phi = 0$ outside the ellipse, with $\phi = 0$ on the ellipse and $\phi \sim -Ex + O(1)$ at infinity. As it happens, it can be solved explicitly by conformal maps, but suppose that we did not know this. What can we do when $\epsilon \ll 1$?

What should we expect? When $\epsilon = 0$, the ellipse is a conducting plate from $(−1, 0)$ to $(1, 0)$. In this case, we should see the field lines bent towards it, because it is a short circuit for the field, and in particular the field should be very high (indeed, singular) at the ends. In short, the plate acts as a lightning conductor does, collecting the field at one end and ejecting it at the other. When the ellipse is thin but not a plate, we expect to see something similar.

First, construct a regular expansion valid away from the ellipse and, in particular, away from its tips $x = \pm 1$. The leading-order problem in this expansion is to solve $\nabla^2 \phi_0$ away from the slit from $(−1, 0)$ to $(1, 0)$, with $\phi_0 = 0$ on this slit and $\phi_0 \rightarrow -Ex$ at infinity. The solution is found by standard complex-variable methods to be

$$\phi_0(x, y) = -E \Re (z^2 - 1)^{1/2},$$

where $z = x + iy$ and the branch cut for the square root is taken along the slit, so that $(z^2 - 1)^{1/2} \sim z$ at infinity. This is a splendid approximation (it is the exact potential for a zero-thickness conductor) but it is singular at $x = \pm 1$. For example, as $z \rightarrow 1$, $\phi \sim -E \sqrt{2} \Re (z - 1)^{1/2}$ and hence $E = -\nabla \phi$ is infinite.

To investigate further, look near the right-hand end by setting $x = 1 + \epsilon^2 X$, $y = \epsilon^2 Y$; see Figure 16.4. In these inner variables, the tip of the ellipse becomes, approximately, the parabola $Y^2 = -4X$ (the factor 4 is the reason for the factor 2 in the equation of the ellipse). The inner problem for $\Phi(X, Y) = \phi(x, y)$ is to solve Laplace’s equation outside this curve, with $\Phi = 0$ on the approximate parabola and, from matching, $\Phi \sim -E \sqrt{2} \Re Z^{1/2} + o(1)$ at infinity, where $Z = X + iY$. This latter is a matching condition with the inner limit of the outer solution $\phi_0$. There are various ways to solve this problem; the use of parabolic coordinates, as introduced for heat transfer from a flat plate, is one and conformal

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*Note that the condition $\phi = 0$ on the ellipse has been linearised onto the $x$-axis, just as in water-wave problems.*

You should derive them by putting $x = 1 + \delta X$, $y = \delta Y$ and looking for balances in the equation of the ellipse. Note that $x$ and $y$ are scaled in the same way, so that Laplace’s equation is not altered. There is no reason for it to be changed; only a long-thin geometry, or some other external reason for differential scaling, would have that effect.

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*4 Aeroplanes have small spikes in strategic places to help lightning on its way after it has struck the fuselage. Passengers inside are protected by the Faraday cage effect of the metal skin of the aircraft.*
mapping is another. They all lead to the solution
\[ \Phi_0(X, Y) = -E\sqrt{2}(Z + 1)^{1/2} - 1 \]
and we see that the apparent singularity is indeed resolved by the inner region, since the branch-point for \((Z + 1)^{1/2}\) is safely inside the conductor.

Notice that this procedure would work for any conductor whose shape is \(y^2 = \epsilon^2 f(x)\), provided that the ends are approximately parabolic. We only see the effect of the details of the shape at \(O(\epsilon)\) in the outer expansion. Expanding to this order also brings in eigensolutions with \((z^2 - 1)^{-1/2}\) singularities at \(z = \pm 1\). Although ostensibly worrying, they simply tell us that the outer expansion breaks down near the tips, the symptom being that when \(|z^2 - 1| = O(\epsilon)\) the second term in the outer expansion, which is \(O(\epsilon(z^2 - 1)^{-1/2})\), is the same size as the first term, \(-E(z^2 - 1)^{1/2}\). There is no contradiction, and the constants that multiply the eigensolutions can be determined by matching with the inner region.

16.6 Exercises

1 A simple expansion near a singularity. Consider the function
\[ f(x; \epsilon) = \frac{1}{x + \epsilon} \]
as \(\epsilon \to 0\). If \(x = O(1)\), expand by the binomial theorem to show that
\[ f(x; \epsilon) \sim \frac{1}{x} - \frac{\epsilon}{x^2} + \cdots . \]
Clearly this expansion is invalid near \(x = 0\), as the first term is singular and the second term is larger than the first. Rescale \(x = \epsilon X\) to find a valid approximation for small \(x\). (This technique is useful for integrals of the form \(\int_0^1 g(x)/(x + \epsilon)\) \(dx\).

2 Expanding a function. Find inner and outer expansions, correct to \(O(\epsilon^2)\), for the function
\[ f(x; \epsilon) = \frac{e^{-x/x}}{x} + \frac{\sin \epsilon x}{x} - \epsilon \coth \epsilon x. \]

3 Matching by intermediate regions. The idea behind this matching principle is to choose a range of values of the independent variable(s) that is large compared with the boundary layer but small compared with the outer region. For example, in the problem described in subsection 16.2.1, the intermediate region might be \(x = O(\epsilon^{1/3})\). Then both inner and outer expansions are written in terms of an intermediate variable \(x = e^{1/2 \xi}\), re-expanded as asymptotic series in this
new variable, and compared; they should be the same. Carry out this procedure for the example of subsection 16.2.1,

\[ f(x; \epsilon) = e^{-x/\epsilon}(1 + x) + x + e^{\epsilon x}, \]

for which the outer expansion is

\[ f(x; \epsilon) \sim 1 + x + \epsilon x + \frac{1}{2} \epsilon^2 x^2 + \cdots \]

and the inner expansion is

\[ F(X; \epsilon) \sim 1 + e^{-X} + \epsilon(Xe^{-X} + X) + \epsilon^2 X + \cdots. \]

Set \( x = \epsilon^{1/2} \xi \), \( X = \epsilon^{-1/2} \xi \), and show that the intermediate expansion of both functions is

\[ 1 + \epsilon^{1/2} \xi + \epsilon^{3/2} \xi + \cdots. \]

4 A two-point boundary value problem. Use matched asymptotic expansions to find an approximate solution to the two-point boundary value problem

\[ \epsilon \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0, \quad 1 < x < 2, \]

\[ y(1) = 0, \quad y(2) = 1, \quad 0 < \epsilon \ll 1. \]

How can you tell that there is a boundary layer at \( x = 1 \) but not at \( x = 2 \)? What happens if \( \epsilon \) is small and negative?

5 An artificial example. Find an approximate solution to

\[ \frac{d^2 u}{dx^2} + \frac{d u}{dx} = \frac{u + u^3}{1 + 3u^2}, \quad u(0) = 0, \quad u(1) = 1. \]

First find the outer solution: which boundary condition will it satisfy and why? Then find the solution in the boundary layer near \( x = 0 \) and carry out the matching. (The right-hand side of this example is selected so that it (a) gives an easy solution to the outer problem and (b) is uniformly Lipschitz in \( u \), and there is no question of blow-up. I very much doubt that the full equation can be solved explicitly, but the approximation tells you all about the structure of the solution.)

You may need to convince yourself by drawing a graph that the equation \( u + u^3 = a \) has a unique real root for each \( a \).

6 Cable laying with small bending stiffness. In Section 16.4, we derived the equation

\[ \tan \theta_0 = \frac{s}{F*} \]

for the leading order gradient of a cable with small bending stiffness. Remembering that \( \tan \theta_0 = dy/dx = y' \) and that

\[ \frac{dy}{dx} = (1 + (y')^2)^{1/2}, \]
show that the solution consistent with \( y(0) = 0 \) and \( y'(0) = 0 \) (because \( \theta_0(0) = 0 \)) is

\[
y = F^* \left( \cosh(x/F^*) - 1 \right).
\]

Deduce that the ship’s dimensionless position is given by

\[
x^* = F^* \cosh^{-1}(1 + 1/F^*)
\]

and that the tensioner angle \( \theta^* \) and dimensionless thrust \( F^* \) are related by

\[
\tan^2 \theta^* = \frac{1 + 2F^*}{(F^*)^2}.
\]

7 Large-Péclet-number flow past a flat plate heated to a constant temperature. Suppose that \( u(x, y) \) satisfies

\[
\frac{\partial u}{\partial x} = \epsilon^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),
\]

with

\[
u(x, 0) = 1, \quad y = 0, \quad x > 0, \quad u \to 0 \text{ at infinity}.
\]

Introducing parabolic coordinates \( \xi, \eta \) satisfying

\[
x + iy = -(\xi + i\eta)^2, \quad \eta > 0, \quad -\infty < \eta < \infty,
\]

show that the curves \( \eta = \text{constant} \) are parabolas wrapped around the positive \( x \)-axis, that the curves \( \eta = \text{constant} \) are parabolas wrapped around the negative \( x \)-axis and that the two families of curves are orthogonal. Show that the problem becomes

\[
-\epsilon^2 \left( \frac{\partial u}{\partial \xi} \right)^2 + \eta \frac{\partial u}{\partial \eta} = \epsilon^2 \left( \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} \right), \quad \xi > 0,
\]

with \( u(0, \eta) = 1 \) and \( u \to 0 \) at infinity. Further, show that the solution takes the form \( u(\xi, \eta) = f(\xi) \) and find \( f \). When \( \epsilon \) is small, eliminate \( \eta \) to show that \( \xi \sim \epsilon y/2\sqrt{x} \) and hence confirm the correctness of the thermal boundary layer solution of subsection 16.5.1.

Now return to the problem of subsection 16.5.1, with \( u = T(x) \) on \( y = 0, \ x > 0 \), where \( T(x) \) is smooth and \( T(0) \neq 0 \). Show that, in addition to the thermal boundary layer described in the text, there is a small region centred at the tip of the plate \((0, 0)\) in which both \( x \) and \( y \) are \( O(\epsilon) \) and for which the leading-order problem is a version of the first part of this question.

8 Wind-chill. Consider the large-Péclet-number version of the advection–diffusion problem of subsection 3.1.1, steady heat transfer from a circular cylinder in a potential flow with velocity \( u \),
with $\mathrm{Pe} = 1/\epsilon^2$, $\epsilon \ll 1$. In plane polar coordinates $r, \theta$, we have $u = \nabla (\cos \theta (r - 1/r))$, and the scaled temperature $T(r, \theta)$ satisfies

$$\cos \theta \left(1 - \frac{1}{r^2}\right) \frac{\partial T}{\partial r} - \frac{\sin \theta}{r} \left(1 + \frac{1}{r^2}\right) \frac{\partial T}{\partial \theta}$$

$$= \epsilon^2 \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2}\right)$$

with $T = 1$ on $r = 1$ and $T \to 0$ at infinity.

Show that the only solution for the first term in a regular expansion in the outer region (away from the cylinder) that is consistent with the condition at infinity is $T_0 = 0$. Deduce that there must be a boundary layer on the cylinder. Define an inner variable $R$ by $r = 1 + \delta R$, where $\delta$ is small. Show that a consistent balance can be achieved in the partial differential equation if $\delta = \epsilon$.

With this scaling, write $u(r, \theta) = U(R, \theta)$ in the boundary layer, and show that the first term $U_0(R, \theta)$ of the inner expansion satisfies

$$2R \cos \theta \frac{\partial U_0}{\partial R} - 2 \sin \theta \frac{\partial U_0}{\partial \theta} = \frac{\partial^2 U_0}{\partial R^2},$$

with $U_0(0, \theta) = 1$ and $U_0 \to 0$ as $R \to \infty$.

Show that there is a similarity solution in the form

$$U_0(R, \theta) = F(R g(\theta))$$

and, by finding the differential equation that it satisfies, that

$$g(\theta) = \frac{\sin \theta}{\sqrt{1 + \cos \theta}}.$$ 

Show also that $F(\xi)$ satisfies $F'' + \xi F' = 0$, and find it.

Notes: (i) The differential equation for $g(\theta)$ is first-order and its solution contains an arbitrary constant, so that $1 + \cos \theta$ is replaced by $1 + \epsilon \cos \theta$. The choice $\epsilon = 1$ is motivated by matching with an $O(\epsilon) \times O(\epsilon)$ region around the upstream stagnation point in which the full problem must be solved (but in a simplified geometry): it says that as we approach this point, the solution depends on $y/\sqrt{2X}$ in coordinates $x = 1 + \epsilon X$, $y = \epsilon Y$ centred there.

(ii) There is also a small region near the downstream stagnation point in which the full problem must be solved. This is succeeded by a thin wake of hot liquid that carries the heat away from the cylinder.

(iii) The total heat transfer from the cylinder is, in dimensionless terms, $O(1) \times O(1/\epsilon)$ (length $\times$ heat flux). This is large: wind-chill!

9 Potential outside an ellipse. Show that the conformal map $z = \xi + 1/\xi$ maps circles $|\xi| = r, r \geq 1$, into ellipses in the $z$-plane, that $|\xi| = 1$ maps to the slit from $-1$ to $1$ along the real axis and that $\xi \sim z$ at infinity. Show also that the real part of $W(\xi) = -E(\xi - 1/\xi)$ compares with Milne-Thomson's circle theorem for potential flow.
where $\zeta = \xi + i\eta$ is a harmonic function that vanishes on $|\xi| = 1$ and tends to $-E\xi$ at infinity. Hence find the exact solution to the problem of a conducting ellipse in a uniform electric field. Also, find the circle that maps onto the thin ellipse of the text, and hence confirm the accuracy of the asymptotic approximation given there.

10 Logarithms. Logarithms pose considerable difficulties in matching. The following example shows why. Let

$$f(x; \varepsilon) = 1 + \frac{\log x}{\log \varepsilon}, \quad x > 0.$$ 

Show that its one-term outer expansion for $x = O(1)$ is $f \sim 1$.

Write $x = \varepsilon X$ to focus on the boundary layer near $x = 0$. Show that the one-term inner expansion for $F(X; \varepsilon) = f(x; \varepsilon)$ is $F \sim 2$. Deduce that Van Dyke's matching rule fails when matching the one-term inner and outer expansions.

Show that the rule works if, when matching, we take terms involving $\log \varepsilon$ together with those of $O(1)$.

(The practical cure for this failure is exactly as in the exercise, to take $\log \varepsilon$ terms together with $O(1)$ terms when matching. After all, what is $\log 10^4$?)

'Instead of considering 10 as large, let's consider 10 as small.'