Weakly nonlinear dynamics of dunes

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ABSTRACT: Weakly nonlinear analyses have proved their validity in the field of morphodynamic instability to describe the evolution of finite amplitude perturbations of the bed topography. In a recent study, Colombini and Stocchino (2008) analyzed theoretically the case of dunes and antidunes that develop, under suitable conditions, in an infinitely wide open channel with an erodible bottom composed by uniform sediments. By introducing a slow timescale in the analysis, they derived an amplitude equation of the Landau-Stuart (LS) type, which describes the nonlinear evolution of a linearly unstable perturbation in the neighbourhood of its marginal conditions. The analysis of the steady solutions of the amplitude equation shows that, for values of the ratio of the shear velocity to the depth-averaged velocity of practical interest, dune bifurcation is supercritical, whereas antidune bifurcation is subcritical. In the former case a stable equilibrium solution is achieved, which compares satisfactorily with the dune heights observed in laboratory channels. Introducing also a slow spatial scale, an amplitude equation of the Ginzburg-Landau (GL) type is eventually derived, which describes the temporal as well as the spatial modulation of marginally unstable dunes. The weakly nonlinear dynamics of a narrow spectrum of unstable waves centered around the critical wavenumber is then analysed, whereby Landau theory is limited to the temporal evolution of the critical mode. Moreover, it is possible to study the stability of GL solutions against general perturbations in contrast to the LS theory where only the stability against perturbations with exactly the critical wavenumber can be analyzed. Periodic solutions of the GL amplitude equation can either be stable, which means that a periodically modulated pattern will emerge, or unstable. In fact, the group velocity of the unstable wavepacket depends on the wavenumber; therefore, local convergence and divergence of the perturbations occurs, possibly causing the periodic solution to become unstable. This depends on the coefficients of the Ginzburg-Landau equation, which in turn are related to the relevant flow and sediment parameters, namely the Froude number and the ratio of grain size over flow depth.

1 INTRODUCTION

Dunes appear in the so-called lower flow regime corresponding to small values of the Froude number and are characterized by downstream propagation and by being almost out of phase with respect to water-surface gravity waves. On the contrary, antidunes occur in the upper flow regime, i.e. for values of the Froude number close to unity, and typically propagate upstream, being almost in phase with free surface oscillations. In both cases, the main geometrical characteristics of the bedforms scale with the mean flow depth. The idea that bedform formation in rivers can be interpreted in terms of an instability process of the system composed by the flow and the erodible bed dates back to the sixties, when the first seminal studies on this subject were published (Kennedy 1963, Reynolds 1965). This research field is still quite active (ASCE 2002) and several morphodynamic patterns have been investigated making use of techniques imported from the field of hydrodynamic stability. Linear analyses allow for the definition of unstable regions in the parameter space where bedforms are expected to form (Engelund 1970, Fredsøe 1974, Coleman & Fenton 2000, among others) and to an indication on the wavelength and celerity of the most unstable disturbances. No information is gathered on bedform amplitude at a linear level, however. More recently, Paarlberg et al. (2007) used a parametrization of the separation streamline to avoid modelling the flow and sediment transport in the separation zone itself and successfully used this concept in a model to compute dynamic roughness due to dunes (Paarlberg et al. 2009). Extending previous linear studies (Colombini 2004, Colombini & Stocchino 2005), Colombini & Stocchino (2008) presented a temporal weakly non linear analysis of dunes and antidunes. This bifurcation analysis of the Landau-Stuart am-

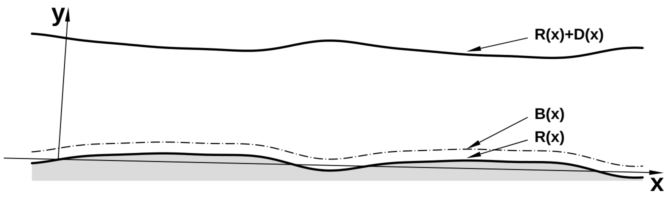

Figure 1. Sketch of flow configuration

plitude equation has shown that for the realistic values of the parameters, dune instability is supercritical whereas antidunes are subcritical. In the former case, an equilibrium amplitude is obtained, which compares satisfactorily against experimental observations. In the present contribution, the work of Colombini & Stocchino (2008) is extended to describe the spatial modulations of wave packets (which is a natural extension of the theory).

2 FORMULATION OF THE PROBLEM

Let us consider a uniform turbulent free surface flow in a infinitely wide straight channel. The triplet composed by the fluid density ρ, the mean friction velocity u_f^* and depth D^* of the unperturbed uniform flow has been used for nondimensionalization. In the following, variables with a star superscript are to be intended as dimensional variables.

Moreover, we define a nondimensional conductance coefficient C as the ratio between the unperturbed depth-averaged velocity $\overline{U}^*$ and the mean friction velocity u_f^*, which can be related to the flow depth and the sediment diameter d_s^* through the Keulegan equation (ASCE 1963) for fully rough turbulent flow:

$$C = \frac{\overline{U}^*}{u_f^*} = \frac{1}{\kappa} \ln \left(\frac{11.09 D^*}{2.5 d_s^*} \right), \tag{1}$$

where κ is the Von Kármán constant, taken as 0.4, and the roughness height has been set equal to $2.5 d_s^*$ after (Engelund and Hansen 1967).

The unsteady Reynolds and continuity equations are written in dimensionless form as:

$$U_{,t} + UU_{,x} + VU_{,y} + P_{,x} - \frac{SC^2}{F^2} - T_{xx,x} - T_{xy,y} = 0,$$
$$V_{,t} + UV_{,x} + VV_{,y} + P_{,y} - \frac{C^2}{F^2} - T_{xy,x} - T_{yy,y} = 0,$$
$$U_{,x} + V_{,y} = 0, \tag{2}$$

where $\mathbf{U} = (U, V)$ is the local velocity vector averaged over turbulence, P is the pressure and $\mathbf{T} = \{T_{ij}\}$ is the 2-D Reynolds stress tensor. Moreover, S is the mean bed slope, assumed to be small, and $F = \overline{U}^* / \sqrt{gD^*}$ is the Froude number.

A sketch of the coordinate system adopted is shown in figure 1, where the flow is bounded between the two lines $y = R(x,t)$ and $y = R(x,t) + D(x,t)$, with D the local flow depth. The lower boundary is set at the reference level R, where the velocity is assumed to vanish.

The differential system (2) is associated with an appropriate set of kinematic and dynamic boundary conditions at the domain boundaries. The Reynolds stresses are modelled through a Boussinesq closure that implies the evaluation of an algebraic eddy viscosity (ν_t), based on the mixing length approach.

Regarding the sediment dynamics, we consider bedload to be the only transport mechanism, so that the Exner equation reads:

$$F R_{,t} + Q_0 \Phi_{,x} = 0, \qquad Q_0 = C \frac{d_s \sqrt{(s-1) d_s}}{(1 - p_s)}, \tag{3}$$

where Φ is the dimensionless form of the sediment transport capacity per unit width and s and p_s are relative density and porosity of the sediment, respectively.

The function Φ is known to depend on a dimensionless form of the bed shear stress, namely the Shields stress θ_b. Results are only moderately affected by the choice of a particular form for the function Φ. In the following, the classical Meyer-Peter & Müller (1948) formula:

$$\Phi = A_m (\theta_b - \theta_c)^{\frac{3}{2}} \qquad \theta_b \geq \theta_c, \tag{4}$$

has been employed, where θ_c is the critical Shields stress for incipient motion. The values of θ_c and A_m have been set equal to 0.0495 and 3.97, respectively, in accordance with the corrections proposed by Wong & Parker (2006) in their revisitation of the work of Meyer-Peter & Müller (1948). In addition, the effect of gravity on the grain motion is included by setting the critical Shields stress θ_c equal to:

$$\theta_c = 0.0495 - \mu (S - R_{,x}), \tag{5}$$

where μ is a dimensionless constant set equal to 0.1 after (Fredsøe 1974).

Finally, the transformation $(x, y, t) \rightarrow (\xi, \eta, \tau)$ is introduced that maps the flow domain into a rectangular domain.

3 LINEAR THEORY

In this section we briefly summarize the essential steps of the linear theory, referring for a detailed description to Colombini & Stocchino (2005). The analysis is performed in terms of normal modes, which implies that a generic function is expanded as:

$$G(\xi, \eta, \tau) = G_0(\eta) + \epsilon G_1(\xi, \eta, \tau), \tag{6}$$

where ϵ is a small parameter.

Moreover, we set

$$G_1(\xi, \eta, \tau) = A G_{11}(\eta) \exp[ik(\xi - \omega \tau)] + c.c., \tag{7}$$

where A, k and ω are the amplitude, wavenumber and complex celerity of the bed perturbation, respectively and $c.c.$ stands for complex conjugate. Expanding each variable in the governing equations, boundary conditions and turbulent closure model and collecting terms at the same order, we are left at $O(\epsilon)$ with the following differential problem

$$\mathcal{L}_{11}\mathbf{Z}_{11} = D_{11}\mathbf{D}_{11} + R_{11}\mathbf{R}_{11}, \tag{8}$$

where the unknown vector is

$$\mathbf{Z}_{11} = (U_{11}, V_{11}, T_{t11}, T_{n11})^T \tag{9}$$

and, for convenience, the equations have been rewritten in terms of the tangential (T_t) and normal (T_n) stresses acting on surfaces at constant η. In (8) the linear differential operator $\mathcal{L}_{11}$ and the vectors $\mathbf{D}_{11}$ and $\mathbf{R}_{11}$ read, respectively:

$$\mathcal{L}_{11} = \begin{pmatrix} d/d\eta & ik/2 & -1/(2\nu_{T0}) & 0 \\ ik & d/d\eta & 0 & 0 \\ U_0^\omega - 4k^2\nu_{T0} & -U_0' & d/d\eta & ik \\ 0 & U_0^\omega & ik & d/d\eta \end{pmatrix} \tag{10}$$

$$\mathbf{D}_{11} = \begin{pmatrix} 0 \\ ikU_0'\eta \\ U_0^\omega U_0'\eta - 2k^2\eta(1-\eta) - 1 \\ ik\eta - 2ik(1-\eta) \end{pmatrix} \tag{11}$$

$$\mathbf{R}_{11} = \begin{pmatrix} 0 \\ ikU_0' \\ U_0^\omega U_0' - 2k^2(1-\eta) \\ ik \end{pmatrix} \tag{12}$$

where $U_0^\omega = -ik(U_0 - \omega)$ and primes stand for derivatives with respect to η. The amplitude of depth and bed perturbations, namely D_{11} and R_{11}, are treated as parameters to be determined.

The solution of the linear differential system (8) reads:

$$\mathbf{Z}_{11} = c_{11}^{(1)}\mathbf{Z}_{11}^{(1)} + c_{11}^{(2)}\mathbf{Z}_{11}^{(2)} + D_{11}\mathbf{Z}_{11}^{(D)} + R_{11}\mathbf{Z}_{11}^{(R)}. \tag{13}$$

Thus, $\mathbf{Z}_{11}$ is expressed as a linear combination of two linearly independent solutions of the homogeneous initial value problem

$$\mathcal{L}_{11}\mathbf{Z}_{11}^{(1,2)} = 0, \tag{14}$$

which satisfy the boundary conditions at the lower boundary, plus particular solutions of the non-homogeneous differential systems:

$$\mathcal{L}_{11}\mathbf{Z}_{11}^{(D)} = \mathbf{D}_{11}, \qquad \mathcal{L}_{11}\mathbf{Z}_{11}^{(R)} = \mathbf{R}_{11}, \tag{15}$$

again satisfying the lower boundary conditions. Without loss of generality, the constants $c_{11}^{(1)}$ and $c_{11}^{(2)}$ are chosen so as to represent the amplitude of the perturbed tangential and normal stresses at the reference level, respectively.

The imposition of the linearized boundary conditions at the free surface plus the linearized Exner equation leads to the the algebraic system:

$$\mathbf{U}_{11} \cdot \mathbf{C}_{11} = \{0\}, \tag{16}$$

where the array $\mathbf{U}_{11}$ is equal to:

$$\begin{pmatrix} \begin{bmatrix} V_{11}^{(1)} & V_{11}^{(2)} & V_{11}^{(D)} + U_0^\omega & V_{11}^{(R)} + U_0^\omega \\ T_{t11}^{(1)} & T_{t11}^{(2)} & T_{t11}^{(D)} & T_{t11}^{(R)} \\ T_{n11}^{(1)} & T_{n11}^{(2)} & T_{n11}^{(D)} + S_0^{-1} & T_{n11}^{(R)} + S_0^{-1} \end{bmatrix}_1 \\ \begin{bmatrix} T_{t11}^{(1)} & T_{t11}^{(2)} & T_{t11}^{(D)} & T_{t11}^{(R)} - \frac{\omega F_0}{Q\theta_{r0}} - \frac{ik\mu}{\theta_{r0}} \end{bmatrix}_{\eta_b} \end{pmatrix}$$

and

$$\mathbf{C}_{11} = (c_{11}^{(1)}, c_{11}^{(2)}, D_{11}, R_{11})^T, \tag{17}$$

A nontrivial solution of the above system is found for those particular values of ω that make the determinant of the matrix associated to the above system vanish. The behaviour of these eigenvalues has been extensively investigated by Colombini & Stocchino (2005) and is briefly summarized in the following.

Three separate eigenvalues display unstable regions in the $(k - F)$ space (see Figure 2): two of them can be readily associated with the formation of dunes and antidunes, while the third describes the instability of fast sediment waves that appear at high Froude numbers (i.e. $F \geq 2$) associated with the presence of roll-waves. The antidune mode is characterized by a small negative celerity (upstream propagation) while the dune mode propagates downstream (positive celerity). The free surface and the bed oscillations are found to be approximately in phase for antidunes and out of phase for dunes, with a small lag coherent with the corresponding direction of migration.

For each value of the coefficient C, two critical points can be identified in the stability plot, say (k_{cd}, F_{cd}) and (k_{ca}, F_{ca}), which are circled in the close-up pictures of figure 2. They identify the onset of instability for each mode, since, as the Froude number equals F_{cd} (F_{ca}), the basic plane bed solution loses stability towards periodic perturbations characterized by wavenumber k_{cd} (k_{ca}), which represent the bedform. The critical Froude number for roll-wave instability F_{cr} is found in the long wave limit $k_{cr} \to 0$.

4 WEAKLY NONLINEAR THEORY

We intend to investigate the weakly nonlinear evolution of the perturbations of the flow-bed system in a neighbourhood of the points (k_{cd}, F_{cd}) and (k_{ca}, F_{ca}) shown in figure 2. We then define:

$$F = F_c(1 + \epsilon^2 F_2), \qquad k = k_c(1 + \epsilon k_1), \tag{18}$$

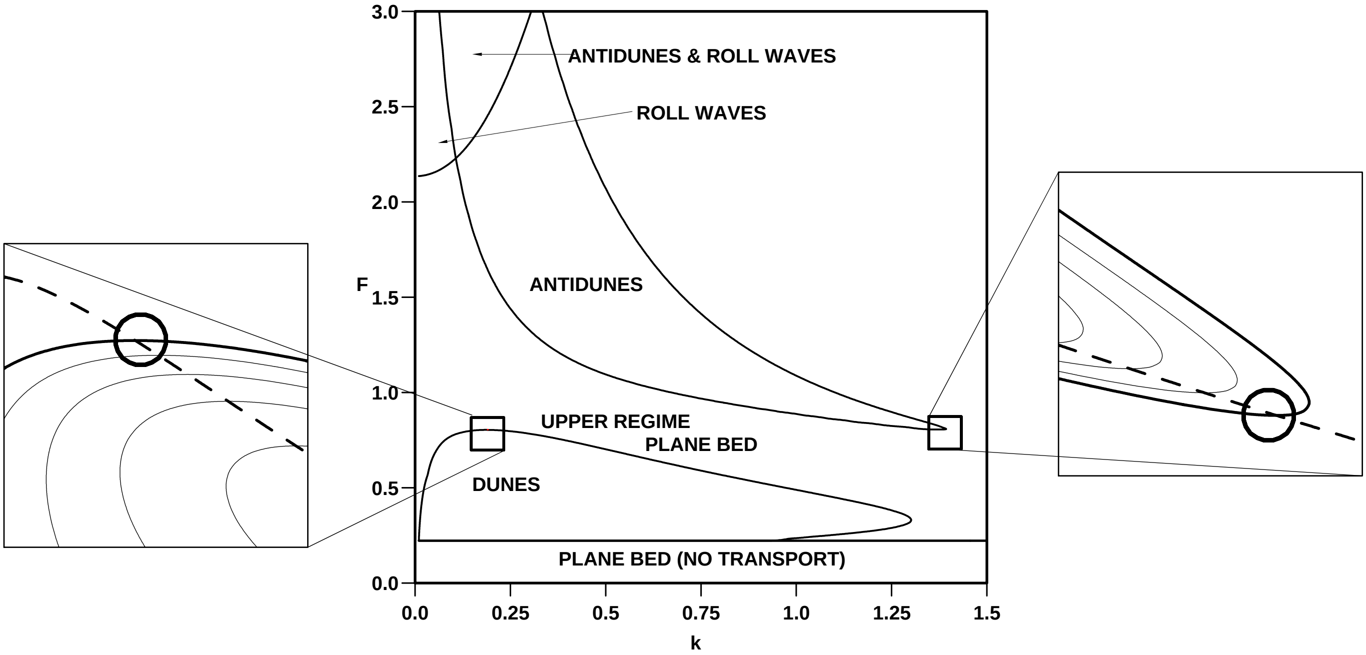

Figure 2. Regions of instability for dunes, antidunes and roll waves; $C = 20$, $f = 0.02$. The dashed lines in the close-up pictures correspond to the lines of maximum growth rate.

where the subscript c indicates either of the critical points, whereas F_2 and k_1 are dummy parameters that define the extension of the neighbourhood in the F and k directions, respectively.

In order to investigate the modulation of a basic critical wave with wavenumber k_c and celerity ω_c we employ a multiscale perturbation technique and define a slow time scale T and a slow spatial scale X such that:

$$T = \epsilon^2 \tau, \qquad X = \epsilon(\xi - c_g \tau) \tag{19}$$

where c_g is the group velocity of the wave packet. Derivatives with respect to τ and ξ become, respectively:

$$\frac{\partial}{\partial \tau} \to \frac{\partial}{\partial \tau} - \epsilon c_g \frac{\partial}{\partial X} + \epsilon^2 \frac{\partial}{\partial T}. \tag{20}$$

$$\frac{\partial}{\partial \xi} \to \frac{\partial}{\partial \xi} + \epsilon \frac{\partial}{\partial X} \tag{21}$$

We then expand the solution in the form:

$$G(\xi, \eta, \tau, X, T) = G_0 + \epsilon G_1 + \epsilon^2 G_2 + \epsilon^3 G_3, \tag{22}$$

and collect terms at the various order of approximation in ϵ.

4.1 $O(\epsilon^1)$

At the linear level, the structure of the solution is analogous to (7):

$$G_1 = A(X,T) G_{11} E_1 + c.c., \tag{23}$$

where, in general:

$$E_n = \exp[n i k_c (\xi - \omega_c \tau)], \tag{24}$$

and the complex function $A(X,T)$ is now a slowly varying function (in time and space) to be determined. The differential system (8) is recovered, with $\omega = \omega_c$ and $k = k_c$. As expected no information is gathered on the amplitude A at this level of approximation.

4.2 $O(\epsilon^2)$

The structure of the solution at second order reads:

$$G_2 = \{A^2 G_{22} E_2 + c.c.\} + |A|^2 G_{20} + F_2 G_{20F} + \\ + \{(A_{,X}\, G_{21X} + A_2 G_{11}) E_1 + c.c.\} \tag{25}$$

Four separate differential problems are then obtained at this order, namely:

$$\mathcal{L}_{22} \mathbf{Z}_{22} = D_{22} \mathbf{D}_{22} + R_{22} \mathbf{R}_{22} + \mathbf{P}_{22}, \tag{26}$$

$$\mathcal{L}_{20} \mathbf{Z}_{20} = S_{20} \mathbf{S}_{20} + \mathbf{P}_{20}, \tag{27}$$

$$\mathcal{L}_{20} \mathbf{Z}_{20F} = S_{20F} \mathbf{S}_{20} + \mathbf{P}_{20F}, \tag{28}$$

$$\mathcal{L}_{11} \mathbf{Z}_{21X} = D_{21X} \mathbf{D}_{11} + \mathbf{P}_{21X}, \tag{29}$$

where the linear differential operator $\mathcal{L}_{2n}$ is obtained from $\mathcal{L}_{11}$ by substituting k with nk_c. Applying the same substitution to $\mathbf{D}_{11}$ and $\mathbf{R}_{11}$ leads to the vectors $\mathbf{D}_{2n}$, $\mathbf{R}_{2n}$. Finally, the vectors $\mathbf{P}_{2q}$ are lengthy functions of η produced by nonlinear interactions, parameter perturbations and multiple scales. Space does not allow here for a more detailed description of all the quantities appearing in (26-29), some of which can be found in Colombini & Stocchino (2008).

The solution of the above differential problems proceeds by expanding the $\mathbf{Z}_{2q}$ as linear combinations of two linearly independent solutions of the homogeneous problems plus an appropriate number of particular solutions of the non-homogeneous systems derived by (26-29).

The imposition of the boundary conditions at the free surface plus the Exner equation eventually produces, for each problem, a non-homogeneous algebraic system of the kind:

$$\mathbf{U}_{2q} \cdot \mathbf{C}_{2q} = \mathbf{U}_{2q}^{(P)}, \tag{30}$$

the solution of which allows for the determination of the unknowns constants $\mathbf{C}_{2q}$.

Moreover, the strict analogy between the $21X$ system (29) and the linear system (8), ultimately leads to:

$$\mathbf{U}_{11} \cdot \mathbf{C}_{21X} = c_g \mathbf{U}_{21X}^{(1)} + \mathbf{U}_{21X}^{(2)}, \tag{31}$$

The homogeneous part of the algebraic system (31) admits of a non-trivial solution so that a solvability condition has to be satisfied, namely the determinant of the matrix obtained by substituting the right-hand side of (31) into the last column of $\mathbf{U}_{11}$ must vanish. Having set:

$$\beta_i = \det(\mathbf{U}_{11}^{(i)}), \tag{32}$$

where the array $\mathbf{U}_{11}^{(i)}$ is obtained by replacing the last column of $\mathbf{U}_{11}$ with the vector $\mathbf{U}_{21X}^{(i)}$, the following relationship is readily obtained:

$$\beta_1 c_g + \beta_2 = 0 \tag{33}$$

which allows for the determination of the group velocity c_g.

Finally, note that in (25) a second, unknown amplitude function A_2 appears. The latter is introduced as a consequence of the non-uniqueness of the solution as provided by the solvability condition we used for the determination of the group velocity c_g. It should be noted that A_2 is unimportant for the subsequent analysis and can be left undetermined.

4.3 $O(\epsilon^3)$

At third order the spatial dependence of the fundamental is reproduced and therefore we can write:

$$G_3 = G_{31} E_1 + c.c.. \tag{34}$$

and the related differential system reads:

$$\mathcal{L}_{11}\mathbf{Z}_{31} = D_{31}\mathbf{D}_{11} + R_{31}\mathbf{R}_{11} + + A_{,T}\,\mathbf{P}_{31}^{(1)} + |A|^2 A\mathbf{P}_{31}^{(3)} + A_{,XX}\,\mathbf{P}_{31}^{(4)} \tag{35}$$

where the vectors $\mathbf{P}_{31}^{(1,3,4)}$ are functions of η expressed in terms of products of basic, leading and second order components of the perturbations.

Once the particular solutions $\mathbf{Z}_{31}^{(P1,P3,P4)}$ of the non-homogeneous differential systems:

$$\mathcal{L}_{11}\mathbf{Z}_{31}^{(P1,P3,P4)} = \mathbf{P}_{31}^{(1,3,4)}, \tag{36}$$

are obtained, the boundary conditions at the free surface and the Exner equation can be cast in a similar way as (31) to give:

$$\mathbf{U}_{11} \cdot \mathbf{C}_{31} = AF_2\mathbf{U}_{31}^{(2)} + |A|^2 A\mathbf{U}_{31}^{(3)} + + A_{,T}\,\mathbf{U}_{31}^{(1)} + A_{,XX}\,\mathbf{U}_{31}^{(4)} \tag{37}$$

where the first term on the right-hand side is generated by the boundary conditions and by the Exner equation.

As before, since the homogeneous part of the system (37) admits of a non-trivial solution, a solvability condition has to be imposed. Having set:

$$\delta_i = \det(\mathbf{U}_{11}^{(i)}), \tag{38}$$

where the array $\mathbf{U}_{11}^{(i)}$ is obtained by substituting the vector $\mathbf{U}_{31}^{(i)}$ into the last column of $\mathbf{U}_{11}$, we find:

$$\delta_1 A_{,T} + \delta_2 F_2 A + \delta_3 |A|^2 A + \delta_4 A_{,XX} = 0, \tag{39}$$

that, after some manipulations, takes the form of the complex Ginzburg-Landau equation (CGLE):

$$A_{,T} = \alpha_1 F_2 A + \alpha_2 |A|^2 A + \alpha_3 A_{,XX}\,. \tag{40}$$

5 ANALYSIS OF GINZBURG-LANDAU EQUATION

Setting $\alpha_3 = 0$ in (40), a Landau-Stuart equation is recovered, which has been extensively studied for the case of dunes and antidunes by Colombini & Stocchino (2008). We briefly summarize in the following their main results. If the real part of the cubic coefficient α_2 is negative, the bifurcation is termed supercritical and an equilibrium amplitude is eventually attained as $T \to \infty$:

$$A_e = \sqrt{-\frac{\alpha_1^r F_2}{\alpha_2^r}}. \tag{41}$$

For realistic values of the parameter, namely the nondimensional conductance coefficient C, dune instability is found to be supercritical, whereas antidunes are consistently subcritical. In the latter case, no information is gathered on the amplitude at the present level of approximation.

We then limit our analysis of the CGLE to the case of dunes, and, firstly, bring (40) into standard form by means of a suitable transormation. Substitution of

$$A = A_e \exp(i\frac{\alpha_1^i}{\alpha_1^r}T')A'(X',T') \tag{42}$$

where

$$X' = \sqrt{\frac{\alpha_1^r F_2}{\alpha_3^r}} X \qquad\qquad T' = \alpha_1^r F_2 T \tag{43}$$

yields the rescaled CGLE (primes are dropped for convenience)

$$A_{,T} = A + (1+ib)A_{,XX} - (1+ic)|A|^2 A. \tag{44}$$

with

$$b = \frac{\alpha_3^i}{\alpha_3^r} \qquad\qquad c = \frac{\alpha_2^i}{\alpha_2^r} \tag{45}$$

Note that $b > 0$ and $c < 0$ in the whole range of conductance coefficients C considered. This follows from an analysis of the coefficients α_i from equation (40) where dune data from various experiments are used.

We consider periodic plane-wave solutions of the form

$$A = P\exp(i\Theta), \qquad \Theta = KX - \Omega T \tag{46}$$

where $P(X,T)$, $\Theta(X,T) \in \mathbb{R}$. By substituting (46) into (44) and collecting the real and imaginary parts, we obtain

$$P^2 = 1 - K^2, \qquad \Theta = KX - (bK^2 + cP^2)T \tag{47}$$

This gives for every choice of K the amplitude and shift in phase (with respect to the critical wave) of the dunes. Due to the signs of b and c, it turns out that the nonlinear dunes propagate slower when compared to the linear theory.

Now, periodic solutions of the type (46) need not to be stable. The stability can be studied by considering perturbations of P and Θ:

$$A = [P + \rho(X,T)]\exp[i(\Theta + \theta(X,T))] \tag{48}$$

Substitution of (48) into (46) leads after linearization to a system of partial differential equations for ρ and θ. We then set:

$$\rho(X,T) = \varrho\exp[i(lX - \lambda T)], \tag{49}$$
$$\theta(X,T) = \vartheta\exp[i(lX - \lambda T)], \tag{50}$$

and end up with the algebraic homogeneous system:

$$(M - i\lambda I)\begin{pmatrix}\varrho \\ P\vartheta\end{pmatrix} = 0 \tag{51}$$

for some matrix M. Stability of (46) is then reduced to an eigenvalue analysis of M. This sets a condition on K (the generalized Eckhaus criterion):

$$K^2 < \frac{1+bc}{3+2c^2+bc} \tag{52}$$

as long as the Benjamin-Feir-Newell criterion

$$1 + bc > 0 \tag{53}$$

holds (for details on this stability analysis, see Schielen et al. (1993)). For the present case, the latter criterion is never satisfied in the whole range of parameters investigated. This means that none of the periodic solutions of the type (46) is stable, the Stokes wave $(K = 0)$ being the last to become unstable. For the special case of the Stokes wave, it can be shown that for the eigenvalues of M holds:

$$\lambda^2 + 2i\lambda(1+l^2) - (1+b^2)l^4 - 2l^2(1+bc) = 0, \tag{54}$$

By imposing that the imaginary parts of both eigenvalues of M for $K = 0$ (i.e. the Stokes wave) are negative the following condition on the wavenumber l is found:

$$l \geq l_c = \sqrt{\frac{-2(1+bc)}{1+b^2}}. \tag{55}$$

which implies that longer perturbations of the Stokes wave are the most unstable (Stuart & Di Prima 1978). In this case however, the Stokes wave itself is also unstable.

Considering a slowly modulated Stokes wave, a physical interpretation of this stability criterion (53) can be found in Schielen et al. (1993). Since $c < 0$ for the present case, the dispersion relation (47) implies a negative nonlinear correction of the (positive) bedforms celerity that is maximum at the top of the envelope, so that the bedforms at either side of it propagate faster. Therefore the bedforms on the downstream side lengthen, whereas the waves at the upstream side are shortened. This is associated to a positive variation of the group velocity c_g with the wavenumber k that implies an accumulation of energy at the top of the envelop, a necessary condition for instability to occur.

A natural question to ask is then what pattern of dunes will evolve in the case of an unstable Stokes wave? Due to the instability of the Stokes wave, the Ginzburg-Landau equation does not allow for a pure periodic pattern of dunes. Instead, it is more likely that a quasi-periodic pattern will occur. Solutions of the Ginzburg Landau then describe the *envelope* of the dune-evolution (because the amplitude equation depends on a slow time and spatial variable). To see whether this indeed occurs, a spectral analysis of (44) can be performed. In Doelman (1991), it is shown that, depending on the values of the coefficients, chaotic solutions (for the envelope) are possible, through a classical scenario of period doublings. This however, is not of particular importance for field-studies of dunes. Also quasi-periodic solutions for the envelope already give a very irregular sequence in the amplitude of dunes.

6 CONCLUSIONS

The linear analysis of dunes, leading to regions of instability as depicted in figure 2 can be extended to a weakly nonlinear analysis in a straightforward way. For dunes, this leads after tedious calculations to a nonlinear amplitude equation of the Ginzburg Landau type. The bifurcation for dunes at F_{cd} is supercritical. Hence, starting with a flat bed, for decreasing Froude number, dunes start to evolve, and their nonlinear evolution can adequately be described by the solutions of the Ginzburg-Landau equation. Simple periodic solutions turn out to be unstable, however. This suggests that more complicated behaviour of dunes will emerge: the moving dunes show an increase and decrease in amplitude as they evolve in time and space. Their migration speed is also slightly less than predicted by linear theory.

Although not shown in this paper, a similar analysis can be done for anti-dunes. In that case, however, it turns out theat the bifurcation is subcritical. This means that the flat bed looses stability for $F > F_{ca}$, but the bifurcating solution is also unstable. In that case the perturbation analysis of section 4 must be extended to fifth order, which eventually yields a quintic version of the CGLE. Besides antidunes, there are other phenomena in nature that exhibit a similar behaviour (Plane Poiseuille flow and Taylor flow, for instance). Eckhaus (1989) have studied the general case of degenerate modulation equations while Doelman & Eckhaus (1991) have looked at periodic and quasi-periodic solutions. To find the quintic coefficient however, requires an even larger number of computations due to nonlinear interactions.

The weakly nonlinear analysis of dunes presented herein follows closely the weakly nonlinear analysis of bars developed by Schielen et al. (1993). This is slightly remarkable because a depth-averaged model is used in the latter, whereas depth plays an essential role in the dynamics of dune. However, in both cases, the main difficulty in the analysis is the determination of the nonlinear coefficient of CGLE. Once determined, the analysis of the Ginzburg-Landau equation itself follows rather straightforwardly. The nonlinear corrections of the shape lead for dunes as well as for bars to steeper fronts and weaker slopes at the lee sides. Furthermore, both bedforms decelerate with respect to the linear theory. A spectral analysis will reveal more dynamic behaviour in the evolution of the envelope and hence of the individual dunes.

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