Nonlinear Stability Analysis and Pattern Formation in Morphological Models

Nietlineaire Stabiliteitsanalyse en de Formatie van Patronen in Morfologische Modellen

(Met een samenvatting in het Nederlands)

Proefschrift

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Chapter 1

General Introduction

In 1937 the Dutch poet H. Marsman wrote the following poem (on the left, the original Dutch text, on the right, my translation):

\[
\begin{align*}
\text{Denkend aan Holland} & \quad \text{Thinking of Holland} \\
\text{zie ik brede rivieren} & \quad \text{I see wide rivers} \\
\text{traag door oneindig} & \quad \text{going slowly through ne’erending} \\
\text{laagland gaan,} & \quad \text{lowlands,} \\
\end{align*}
\]

If Marsman would have been an applied mathematician (with a poetic tendency), he might have written

\[
\begin{align*}
\text{Thinking of Holland} & \\
\text{I see rivers with a large width-to-depth ratio} & \\
\text{as well as a small Froude number} & \\
\text{flow through infinite domains,} & \\
\end{align*}
\]

and after having done some analysis, he would have decided that life was pretty interesting: the rivers he describes show all kinds of pattern formation on the bed and, if the conditions are right, they may even start to meander. Fortunately, Marsman wasn’t a mathematician and besides, he lived too early: the tools that are needed to describe the phenomena of pattern formation have been developed in the late sixties and the years after. Nevertheless, he had in some sense a prospective view: wide rivers are most interesting from the pattern formation point of view.

Of course, there is no need to think only of Holland to see wide rivers with interesting behaviour: there exist many rivers and channels all over the world whose characteristics are such, that the bed (and consequently, the flow field) shows spatially and temporally evolving patterns. A way to get insight in this kind of structures is to consider the channels and rivers in idealized situations as straight or sinusoidal reaches with (nearly) flat bottoms and study their behaviour with respect to small perturbations of the flow-field and bed-level. The determination of the conditions under which these simple states loose their stability properties and the description of the temporal and spatial evolution of the
resulting patterns are in some sense what this thesis is about: Nonlinear stability analysis, applied to morphological problems. We will consider both items in the sections below, starting with the former.

1.1 Nonlinear stability analysis

Many physical problems can be modelled in terms of a nonlinear partial differential equation, depending on one or more control parameters (denoted by \( R \)), for a vector of unknowns \( \Phi(t, x, y) \) which depends on time \( t \), and on the spatial variables \( x \in \mathbb{R}^n \) and \( y \in \Omega \subset \mathbb{R}^n \). One is often interested in stability properties of a solution with a simple temporal and spatial structure which denotes some kind of relevant physical situation. This solution is named basic state and for many situations, it is important to know under what circumstances this solution is stable or unstable. Many experiments in physics, chemistry and biology indicate that when the control parameter (which can be any relevant parameter in the model) is increased above some critical value, the basic state looses its stability properties and instead, a pattern with a more complicated spatial and temporal structure evolves. This can be periodic or quasi-periodic, depending on the control parameter; there is numerical evidence that even chaotic solutions are possible. As examples, we mention here the three most classical (nonlinear) stability problems in fluid dynamics: Poiseuille flow (the flow between two parallel plates, where the Reynolds number is used as control parameter, see Stewartson and Stuart (1971), Davey et al. (1974), Holmes and Wood (1986)), Taylor-Couette flow (between two rotating cylinders where the ratio of angular velocity is used as control parameter, see DiPrima and Swinney (1981) Chossat and Iooss (1985)), and Rayleigh Bénard convection (a fluid, heated from below and cooled from above; the Rayleigh number is used as control parameter, see Bénard (1900), Newell and Whitehead (1969), Brand et al. (1986), Schopf and Zimmerman (1989)). Besides these classical examples, Kuramoto (1984) has used the techniques, which will very briefly be described below, to analyse chemical processes.

Usually, the analysis starts with a linear stability analysis of the basic state, which is to say that we consider (infinitesimally) perturbations, i.e. we consider

\[
\Phi = \phi_b + \varepsilon e^{ikx+\omega t}U(y)
\]

(1.1.1)

with \( \phi_b \) the basic state. The function \( U(y) \) describes the structure in the bounded direction. In the situation of Chapter 2, \( U(y) \in \Omega \subset \mathbb{R}^n \) and turns out to have a simple harmonic structure, with wavenumber \( p\pi, p \in \mathbb{N} \). The wave number \( k \in \mathbb{R} \) and \( \omega \in \mathbb{R} \) are arbitrary and \( 0 < \varepsilon \ll 1 \). The linear stability analysis typically results in an eigenvalue problem with eigenvalues \( \omega(k, R) \) and eigenfunctions \( U(y) \), both depending on \( p, k \) and \( R \). The eigenfunctions are determined by considering appropriate boundary conditions. The basic solution is said to be stable with respect to perturbations with wave number \( k \), if for a fixed \( R \) the real part of the eigenvalue \( \omega(k, R) \) is negative. Conversely, the basic solution is unstable if the real part of the eigenvalue \( \omega(k, R) \) is positive. Hence, we can derive the so-called neutral stability curve \( Re(\omega(k, R)) = 0 \) which divides the \((k, R)\)-space into two areas. In one part, the basic state is stable, in the other part, the basic state is unstable (see figure 1.1). Actually, the graph of the neutral stability curve depends on \( p \),
the wavenumber in the transverse direction. We will only consider the situation for \( p = 1 \); for an explanation, we refer to Chapter 2. For the type of problems that are considered in this thesis, the neutral stability curve is locally a parabola with minimum \((k_c, R_c)\).

For \( R > R_c \), there is a bounded continuum of eigenvalues \( \omega \) with positive real part. They correspond with (infinitely many) waves which are (linearly) unstable perturbations of the basic state. These perturbations grow (from linear point of view) exponentially in time.

The object of the nonlinear stability analysis is then to describe the nonlinear evolution of the linearly unstable waves (and thus of the basic state). In this thesis, we restrict ourselves to the case where \( R \) is close to \( R_c \). In that case, these waves are slow modulations of the critical wave at \( k = k_c, R = R_c \). By now, it is well known that the evolution of the perturbations of the basic state is completely determined by an amplitude function \( A(\xi, \tau) \) with \( \xi \) and \( \tau \) rescaled slow spatial and time coordinates. This is to say that we can write:

\[
\begin{align*}
\Phi - \phi_b &= \varepsilon A(\xi, \tau) U_c(y) e^{ik_c x + \omega_c t} + c.c. + \text{h.o.t.} \\
\xi &= \varepsilon(x + \nu_k t) \\
\tau &= \varepsilon^2 t
\end{align*}
\]

where \( \nu_k \), the group velocity, defined as \( \partial \omega / \partial k \), evaluated at critical conditions \( k_c \) and \( R_c \), and with \( U_c(y) \) the eigenfunction evaluated at critical conditions. (Throughout this thesis, \( c.c. \) denotes complex conjugated and \( \text{h.o.t} \) denotes higher order terms). It turns out that \( A \) satisfies an amplitude equation, known as the Ginzburg-Landau equation, which in first order reads:

\[
A_\tau = r A + a A_{\xi \xi} + b |A|^2 A
\]

where \( r \) is defined by \( R - R_c = r \varepsilon^2 \), and \( a \) and \( b \) coefficients which can be calculated explicitly for a given problem.

The theory can also be applied on problems with more than one unbounded direction:

![Figure 1.1: A typical example of a neutral stability curve.](image-url)
In that case, the neutral stability curve becomes a surface and one can derive the multidimensional equivalent of the Ginzburg-Landau equation. For more details about general aspects of the Ginzburg-Landau equation, we refer to Eckhaus (1992) and the references cited there.

The Ginzburg-Landau equation can be considered as a generic model equation, describing the nonlinear evolution of patterns which evolve in the case that the basic state of a given physical system becomes unstable. Mathematically, it is a widely studied subject.

1.1.1 A short story about the history of the Ginzburg-Landau equation

The Ginzburg-Landau equation has, more or less independently, been derived by various authors. The relevant papers are Newell and Whitehead (1969) (who derived a Ginzburg-Landau equation with real coefficients for the case of Bénard convection, Segel (1969) (who treats the same problem), Stewartson and Stuart (1971) (they studied the problem of Poiseuille flow and consequently, they derived a Ginzburg-Landau equation with complex coefficients) and finally DiPrima et al. (1974) (they studied hydrodynamical problems in a more general setting). In 1965, the stationary equivalent of the equation was posed (not derived) by L.D. Landau and V.L. Ginzburg in a paper on superconductivity, see Landau and Ginzburg (1965). Until the early eighties, it seems that there was no name associated with the equation; either one paraphrased the equation or used words as ‘amplitude equation’ or ‘modulation equation’. In the early eighties, the names of Ginzburg and Landau became associated with the equations, although formally, they did not derive a Ginzburg-Landau equation in their paper! However, the ‘right’ name, i.e. something like the Segel-Newell-Whitehead-Stewartson-Stuart-DiPrima-Eckhaus equation is a bit unpractical in daily use, which might be the reason that nowadays, the term Ginzburg-Landau equation is widely accepted.

1.2 River morphology: a short survey

Recently, Seminara (1995) wrote an excellent review paper about this subject. We briefly discuss some items that are of relevance in the context of this thesis.

Our interest in river morphology has to do with patterns, originating on the erodible bottom. The mechanism that gives rise to these patterns comes from the interaction of sediment with the flow in the channel. The movement of sediment particles is driven by hydrodynamical forces and often leads to some kind of equilibrium profile. This however, is already a subtle (philosophical?) point. Intrinsically, natural rivers are self adjusting to their environment, which itself is never in equilibrium. The flow conditions and the production of sediment are strongly affected by the climate and the change of seasons. Furthermore, small changes in the flow field affect on their turn the cross-section of the river. Hence, one can hardly speak of an equilibrium configuration of rivers. Nevertheless, it is often useful (if not the only working method) to think of the evolution of a pattern as a perturbed state, relatively to some ideal equilibrium state of the system, viewed on
some specific temporal and spatial scale.

The choice of scaling is very important for the character of the pattern on one hand, and for the mathematical model describing those patterns on the other hand. Usually, one distinguishes between the following scales (see Schielen and Mosselman (1995)):

- **Micro scale.** Phenomena with amplitudes on the order of several hundreds times the sediment size, e.g. ripples.
- **Meso scale.** Phenomena with amplitudes on the order of the river depth, e.g. dunes, antidunes.
- **Macro scale.** Phenomena with amplitudes on the order of the river width, e.g. alternating bars, meanders, braiding rivers.
- **Mega scale.** Then we are considering the evolution of the system as a whole, where we for instance allow the banks to evolve. The scale belonging to this kind of phenomena can be denoted by \( h/i \), where \( h \) is the river depth and \( i \) is the inclination of the river in its natural environment. Considering phenomena on the mega scale, we should think of for instance several meander bends.

The morphological aspects of this thesis are mainly restricted to the macro scale and more specifically to alternating bars. At some points, we briefly consider some aspects related to meandering.

Alternating bars are the result of an instability mechanism related to uniform flow in a straight channel with an erodible bottom, where small perturbations in the form of three dimensional sand waves, characterised by oblique alternating fronts, turn out to be unstable. In a first approximation, they can be described as the composition of a sine-function in the longitudinal direction and half a cosine-function in the transversal direction (see figure 2.3 (a)), such that the throughs and crests are alternating with respect to the sidewalls. They are sometimes called free bars, to emphasize that they propagate through the channel with a certain velocity and to distinguish them from point bars which are stationary. We will come back to the point bars later on. At this point however, we want to remark that the travelling and stationary character of the free bars and the point bars respectively, is not the only difference between them. Free bars are the result of an instability mechanism, point bars are (as we will indicate in Chapter 3) ‘forced’ by the geometry of the channel.

The wavelength of the free bars is of the order of the river width. To study the alternating bars, one usually makes two assumptions which simplify the problem considerably. One assumes that the channel conditions are such that the channel is wide enough for side wall effects to be neglected and furthermore, one assumes that the flow conditions are such that flow separation downstream of the oblique bars does not play a rôle. Taking this into account, one can assume that the velocity field can be averaged over the depth and thus consider a two dimensional model. The equations are often denoted by shallow water equations or Saint Venant equations. They can be derived from the Navier Stokes equations, see for instance Pedlovsky (1987). During the last decades, there have appeared many papers about alternating bars in straight channels with nonerodible side walls. Here, we mention
the contributions of Hansen (1967), Callander (1969), Engelund and Skovgaard (1973), Parker (1976), Fredsoe (1978) and Olesen (1983) because of the great leap forward that is made in those papers. Most of these papers are dealing with a linear stability analysis of the bar pattern and use a two dimensional model. Thus, they are able to predict an initial wavelength and velocity of the bars but, by construction, they can not pass judgement about the amplitude.

The first attempts to predict the amplitude development of the bars has been made by Colombini et al. (1987), who derived the Landau equation for the time dependent amplitude. The Landau equation is actually the space-independent analogue of the Ginzburg-Landau equation; in the derivation of this equation, one does not take into account that a whole continuum of waves becomes unstable for $R > R_c$ but considers merely perturbations with a fixed wave number $k_0$ in the neighbourhood of $k_c$. Using this method one can, by construction, only consider existence and stability of spatially periodic solutions near criticality. Later, the study of Colombini et al. (1987) was extended by Tubino (1991), who incorporated varying discharge in the model (which leads to a Landau equation with slightly varying coefficients) and by Schielen et al. (1993) (see Chapter 2 of this thesis), who derived the more general Ginzburg-Landau equation for the space and time development of the amplitude. Very recently, Komarova and Newell (1995) wrote an interesting paper which can be viewed as an important contribution to the nonlinear theory of alternating bars. In their paper they show that in Chapter 2, we have neglected the influence of the so called mean flow. Mathematically, the mean flow can be considered as the $k = 0, p = 0$-mode; this mode turns out to be neutrally stable for all values of $R$ ! Hence, it should be taken into account in the nonlinear theory. Up to this moment however, we have the impression that despite the substantial contribution of Komarova and Newell (1995), the questions related to the incorporation of the mean flow have not completely been solved.

After having fully understood the bar problem in straight channels with nonerodible side walls, one can make the next, in some sense natural, step in the analysis of river morphological problems, which is dropping the assumption of nonerodibility. This would allow the river to meander freely. The ultimate goal would then be to derive and analyse a model which takes into account erosion and thus, would predict the evolution of the planform of the river as a whole, in time and space. This however, appears to be a very difficult problem, although in the last decades a lot of progress has been made, see for instance Ikeda et al. (1981), Parker et al. (1982) and Seminara and Tubino (1994).

Let us expand a little bit on the subject of meandering. For a long time, it has been thought that the formation of alternating bars is a necessary condition for rivers to meander. A possible mechanism would be that the flow starts to meander as a result of the formation of the bars. This causes erosion of the side walls which transforms the initially straight channel into a sinusoidal one. A problem herewith is, that the bars themselves are often not stationary, but travel through the channel with a velocity that might be too high to start the meander formation. In those cases, we get an ‘uniform’ sedimentation of the walls, which makes the river wider. As a result, the bars slow down and the formation of the meander process might start by the aforementioned process. There are however, also other mechanisms proposed which take as starting point a steady deformation of the bed.
1.3 On the use of model problems

The most important one is described by Struiksma et al. (1985). There, it is shown that under certain conditions, there may be perturbations of the cross section possible which cause a steady oscillation of the bed pattern which starts at the location of the perturbation with a finite amplitude and which is damped in the streamwise direction. For a more elaborate overview and discussion of the main mechanisms, see Seminara (1989).

In trying to understand the problem of meandering, one can make an intermediate step: study the pattern formation in a (regular) slightly curved channel with fixed boundaries. This is a natural step, in the light of the following reasoning. In nature, rivers are never straight and even artificially straightened rivers and channels cannot be considered perfectly straight. There are always small imperfections in the side walls. One can model this situation by considering a river with slightly curved boundaries, which has also a small ‘amplitude’. This is to say that the deviation with respect to a well chosen reference level is small. It turns out that a flat bed and a uniform flow can no longer serve as a basic solution of the model that describes the above mentioned situation. Instead, we observe that there will develop a pattern of so-called point bars, a stationary bar pattern with a complicated cross-section. Then, one can again consider the stability of this basic state, which may lead to a situation where migrating and nonmigrating bars coexist and interact.

The paper of Blondeaux and Seminara (1985) is one of the first on this subject. After that, Seminara and Tubino (1989) and Tubino and Seminara (1990) has extended the analysis. Once we understand what influence these curved boundaries have on the dynamical situation we have set yet another step in the direction of gaining understanding in the complex process of river meandering.

1.3 On the use of model problems

The considerations made in the previous two sections have paved the way of a possible working method. Keeping in mind the theory of linear and nonlinear stability analysis, together with the knowledge of amplitude equations, we can study the problem of bed formation in straight channels, make a small detour to curved channels to explore the dynamical situation, and finally use the explorations to tackle the problem of free meandering. This however is a too optimistic view on the problem. Already for the curved channels, the situation turns out to be fairly complex, and we run into all kinds of trouble when we try to perform a linear and nonlinear stability analysis. Part of the problem is that the basic state itself, belonging to a curved channel, is already far from trivial.

Therefore, we propose a different method of analysing, meant to circumvent the problems related to the construction of the physical model and the associated basic state and thus, concentrate on the linear and nonlinear stability analysis itself. That is another point that we want to make: the concept of reducing complicated problems into simple models which contain only the most essential features of the related physical problem. In the analysis of these problems, we can fully concentrate on the basic mechanisms of instability, meanwhile keeping the calculations as limited as possible. Once the model problem is fully understood, we can return to the ‘real’ problem, taking into account all the aspects that we learned from the analysis of the model problem and that, some way or another, will reveal
themselves in the physical problem. After this has been done, there is only one thing left: translating the mathematical-physical results back to ‘the real world’. This last step is necessary in the sense that it makes the mathematical results accessible for those who are involved in the river engineering. From mathematical point of view, this might not be the most interesting part of the work, but on the other hand, is might be the most productive one. It is work on the boundary of several sciences. Steps in this direction, and applied on the bar-pattern in rivers, have been made by Schielen and Mosselman (1995).

1.4 Outline of the contents

The contents of this thesis can roughly be divided into two parts, one with a physical character and one with a more mathematical character. The physical part encloses Chapter 2 and Chapter 3, up to and including section 3.4. The mathematical part starts at Chapter 3, section 3.5 and covers the remaining Chapters.

In Chapter 2, we apply the theory of modulation equations on the problem of bed deformation in straight channels. We will show that under certain conditions, a flat bed is unstable with respect to small perturbations and a pattern of alternating bars starts to develop. Using Ginzburg-Landau theory, we can describe the spatial and temporal evolution of the bars.

In Chapter 3, we consider ‘more realistic’ (but still idealized) situations: slightly curved channels with erodible bottom and fixed side walls. After having derived a basic state (that is, a simple stationary solution, bounded in up- and downstream direction), we are in principle ready for a linear and nonlinear stability analysis. Rather than doing this, we focus our attention on the derivation of a model which contains the essential features of curved channels. We do this in the hope that we can clearly see the influence of periodic boundaries on the linear and nonlinear analysis.

In Chapter 4, we make a thorough analysis of the model problem, where the ‘usual’ Ginzburg-Landau theory acts as a guide-line. It turns out that there is a rich world of modulation equations, which differ sometimes significantly from the Ginzburg-Landau equation. When they differ, and what modulation equation appears depends on the relation between two small parameters. One measures the distance of a bifurcation parameter above some critical value for which the basic state becomes linearly unstable, the other measures the ‘amplitude’ of the periodic boundaries.

In Chapter 5, we analyse solutions of the various equations that we derived in Chapter 4. Again, we are guided by (the solutions of) the ‘usual’ Ginzburg-Landau theory and we study the influence of small periodic effect on those solutions. First, we look at constant, and simple space periodic, stationary solutions and consider the stability properties of these solutions. Actually, we consider finite-dimensional Galerkin approximations of solutions. Finally, in Chapter 6, we study once more the equations that we derived in Chapter 4. Now however, we take a different viewpoint, and use a Hamiltonian approach. More than the (stationary) solutions itself, we try to gain information about their topological structure.
Chapter 2

On the Nonlinear Dynamics of Free Bars in Straight Channels\footnote{The contents of this Chapter appeared in \emph{Journal of Fluid Mechanics} (1993), \textbf{252}, pp. 325-356, with A. Doelman and H.E. de Swart as co-authors.}

2.1 Introduction

The bed of most natural rivers consists of material which may be transported by currents. The interactions between river flow and the erodible boundaries result in the formation of morphological features such as bars, bends and meanders. Observations indicate that bars often occur as a series of propagating waves in the downstream direction with an alternating transversal amplitude structure. Typical wave lengths are of the order of the river width, the waves travel several meters each day and their amplitudes are approximately twenty percent of the undisturbed water depth. On the other hand river meanders have much larger characteristic scales than the bars. A better understanding of the behaviour of these phenomena is of interest, both from a practical and theoretical point of view.

In general the dynamics of morphological systems is rather complicated due to the strong feed-back between currents and the various bed forms. Therefore, it becomes worthwhile to consider simplified models in which a particular phenomenon may be investigated in isolation. The motivation is that these problems are more easy to deal with, whereas they yield information on the fundamental physical mechanisms. In many of such studies, including the present one, it is assumed that only the bottom is erodible and the river is modelled as an infinitely long straight channel. This eliminates the process of free river meandering, which is believed to be a secondary response on the formation of bars on the bottom or which may be generated by forcing mechanisms (Blondeaux and Seminara (1985), Crosato (1990)).

The early studies on the dynamics of river bars were based on a linear instability theory, see Callander (1969), Engelund and Skovgaard (1973), Parker (1976), Fredsoe (1978), Olesen (1983). They investigate the behaviour of small perturbations on a basic state describing a uniform current over a flat bottom. The result was a selection of the most unstable wave length, for which alternate bars start to develop, if the width-to-depth ratio
becomes sufficiently large. These theories, however, only describe the initial stage of the evolution of the bars. If the amplitude becomes finite the linear theory is no longer valid because nonlinear terms become important.

The nonlinear evolution of bars was investigated by Colombini et al. (1987) and Fukuoka (1989). They applied a weakly nonlinear theory for perturbations which grow on a time scale which is large compared to the typical period of the waves. The result of their two time-scale analysis was a so-called Landau equation describing the time evolution of the wave amplitude. It was demonstrated that all non-transient solutions of this equation are periodic and represent a finite-amplitude periodic alternate bar pattern. They only considered the case where the wavenumber is fixed in the neighbourhood of the critical wavenumber for which instability first occurs. This choice is disputable since in fact all waves in a narrow spectrum, centred around the critical wavenumber, are unstable. Due to the dispersion properties of this wave group, modulations will also occur on a spatial scale. If this effect is included in the weakly nonlinear theory, a modified amplitude equation is found which is called the Ginzburg-Landau equation. Since the group velocity varies with the wavenumber we may expect this equation to describe local convergence and divergence of the perturbation energy, which may cause the periodic solutions obtained from the Landau theory to become unstable (Lighthill (1978)). As a result bar patterns with a more complex temporal and spatial behaviour may be expected. Furthermore, it is possible to determine the stability of its solutions against general perturbations (in contrast to the Landau theory, where one can only study the stability of periodic solutions against perturbations with exactly the same wavenumber). The Ginzburg-Landau theory is therefore an essential extension of the Landau theory, and the aim of the present paper is to investigate the possible modified behaviour of bar patterns as described by the Ginzburg-Landau equation.

The Ginzburg-Landau equation has been derived for many physical systems such as Rayleigh-Bénard convection (Newell and Whitehead (1969)), Poiseuille flow (Stewartson and Stuart (1971)), or more recently, wind-driven water waves (Blennerhasset (1980)), chemical processes (Kuramoto (1984)), binary fluid convection (Schopf and Zimmerman (1989)) etc. For morphological systems it has, to our knowledge, not been done before. Therefore in this paper a derivation of the Ginzburg-Landau equation will be presented for a simple morphological model and the behaviour of its solutions will be studied. The basic model, introduced in section 2, describes the interaction between currents, forced by the inclination of a straight channel with fixed banks, and an erodible bottom. The sediment is assumed to be uniform and non-cohesive and is transported as bedload. The model has a basic state corresponding to an uniform flow over a flat bottom. From the linear stability analysis of section 3, a minimum width-to-depth ratio of the channel is obtained at which the basic state becomes unstable. This yields a critical wavenumber of the bed form perturbation which first starts to grow. For slightly larger values of the width-to-depth ratio a weakly nonlinear theory will be applied in section 4 which results in the Ginzburg-Landau equation. This equation describes the nonlinear dynamics of the envelope amplitude of a packet of marginally unstable free bars, and the coefficients are presented in terms of the original morphological parameters, which are the friction coefficient, width-to-depth ratio and the sediment transport coefficients. In section 5 solutions of the Ginzburg-Landau equation and
the corresponding bed profiles are discussed. The analysis of the mathematical properties of the Ginzburg-Landau equation has been the subject of many papers. Results of Keefe (1985), Doering et al. (1989) and Doelman (1989) demonstrate the possibility of periodic, quasi-periodic and chaotic solutions. However, for the present morphological model a less rich behaviour is found for realistic choices of the parameters. It will be demonstrated that the periodic alternate bar pattern, obtained by Colombini et al. (1987) using Landau theory, can be unstable in the case the bed is dune-covered. The subsequent dynamical behaviour is investigated by using a spectral model of the Ginzburg-Landau equation. The results show that quasi-periodic bar patterns are found instead. A discussion of these results and some conclusions are presented in the final section.

We would like to emphasize that the main purpose of this paper is the derivation and the analysis of the modulation equation. To simplify our presentation we have considered a model for the process of river bed evolution which only has the very essential features necessary to describe the main characteristics of this process. Therefore we did not consider important effects in our model such as the variation of the drag coefficient (Einstein (1950), Colombini et al. (1987)), the variation of the bed slope coefficient (Sekine and Parker (1992)), the effect of secondary currents (Rozovskij (1957)) etc.. This probably means that the predictions based on our analysis will only give crude indications of what happens in nature. However, comparisons with earlier studies on the linear and nonlinear analysis have been made and our results were found to be in good agreement with these studies. Furthermore, our model problem can be extended to a more realistic model which can be studied by exactly the same techniques as our model. We based our analysis on the simple model to minimize the amount of mathematical computations and to facilitate the presentation. The (weakly) nonlinear pattern formation in a realistic model will also be governed by the same modulation equation (only the coefficients will be in a different range). This modulation equation is the most general tool in nonlinear stability theory: by its mathematical nature it governs all possible pattern generating processes (Newell (1974)). We therefore emphasize in this paper the structure which is present in morphological processes, a structure which enables us to analyze and predict patterns of non-periodic (or even chaotic) kind.

2.2 The Model

We consider a depth-averaged shallow water flow in a straight, infinitely long channel, having a uniform (mild) slope, which we denote by $i_0 \ll 1$. Further, we consider the banks to be non-erodible, whereas the bottom consists of non-cohesive sediment, which we assume to be transported as bedload. A situation sketch is presented in figure 2.1. It is clear that for using a depth-averaged model the channel width should be much larger than the undisturbed water depth. This also motivates the neglect of horizontal diffusion of momentum, since this effect is confined to thin boundary layers along the side-walls which are not of interest for the present analysis. Furthermore, it should be demonstrated a posteriori that the effect of flow separation, caused by the resulting bed forms, may be ignored. This implies that the bar amplitudes should be much smaller than their characteristic horizontal scales. This appears to be the case for the present features as was already found by
The equations of motion then become:

\[
\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} + g \nabla \zeta = \mathbf{F} \tag{2.2.1}
\]

\[
\frac{\partial (\zeta - z_b)}{\partial t} + \nabla \cdot (\mathbf{U}(\zeta + h_* - z_b)) = 0 \tag{2.2.2}
\]

\[
\frac{\partial z_b}{\partial t} + \nabla \cdot \mathbf{S} = 0 \tag{2.2.3}
\]

As shown in figure 2.1, we use an orthogonal coordinate system, where we assume that the \(x\)-direction coincides with the streamwise direction. In (2.2.1)-(2.2.3) \(\mathbf{U} = (u, v)\) is the depth-averaged velocity in \(x\) and \(y\)-direction, \(\zeta\) is the elevation of the disturbed free surface with respect to the undisturbed water depth \(h_*\), and \(z_b\) is the disturbed bed-level with respect to the undisturbed water depth. The nabla-operator is defined as \(\nabla = (\partial/\partial x, \partial/\partial y)\).

The vector \(\mathbf{F}\) represents the forcing and friction mechanisms, which we model as

\[
\mathbf{F} = \left(-C \frac{u \sqrt{u^2 + v^2}}{\zeta - z_b + h_*} + i_0 g, -C \frac{v \sqrt{u^2 + v^2}}{\zeta - z_b + h_*}\right) \tag{2.2.4}
\]

Note that this means that we model the bottom stress in the direction of the depth-averaged velocity.

In (2.2.1)-(2.2.4), \(C = g/C_f^2\), the drag coefficient, where \(g\) is acceleration due to gravity and \(C_f\) is the Chézy-coefficient. We assume that the bottom evolution only depends on the local flow parameters. The volumetric sediment flux \(\mathbf{S}\), with components \(S_x\) and \(S_y\) in the \(x\)- and \(y\)-direction respectively, is modelled as

\[
\mathbf{S} = \sigma |\mathbf{U}|^b \left(\frac{\mathbf{U}}{|\mathbf{U}|} - \gamma \nabla z_b\right) \tag{2.2.5}
\]
for some \( b > 0 \) and \( \gamma > 0 \); \( \sigma \) depends on the bed porosity and on the sediment properties. Note that \( b > 3 \) corresponds to a dune-covered bed. Many transport formulas are of this type, see the review in van Rijn (1989). Note that we omit the effect of secondary flow on the direction of the sediment transport. It is noted by Parker and Johannesson (1989) that in the case of a straight channel this is a secondary effect which may be neglected. Since our aim is to consider a simple conceptual model, which only contains the essential physical mechanisms responsible for morphological instabilities, we take constant values for the parameters \( b, C \) and \( \gamma \). In fact, accurate values for \( b \) are unknown and typical choices range between 2 and 7, see the review by van Rijn (1989). A physically more realistic choice for the drag coefficient would be the parameterization suggested by Einstein (1950), however we consider the variation on the local water depth to be a higher-order effect. Finally, a constant \( \gamma \) corresponds to the simple expressions derived by Bagnold (1956) and Engelund (1974), in which it represents the inverse of the dynamic coefficient of Coulomb friction. More general expressions for \( \gamma \), including dependence on the bottom stress, are discussed by Sekine and Parker (1992). We have taken \( \gamma \) of order 1 throughout our analysis.

We close the model by the following boundary conditions

\[
v = 0, \quad S_y = 0\quad \text{on } \Gamma
\]

where we denote by \( \Gamma \) the walls, i.e. \( y = 0 \) and \( y = y^* \). These boundary conditions represent the assumption that the walls are impermeable for water as well as sediment. From (2.2.5) and (2.2.6) it easily follows that \( \partial z_b / \partial y = 0 \) on \( \Gamma \).

Note that the model allows for a steady uniform flow in the downslope direction, of which the magnitude is determined by a balance between forcing and dissipation (see 2.4). If we denote the corresponding variables with an asterix, it follows

\[
\begin{align*}
u^*_s &= \frac{i_0 g h_s}{C}, & v_* &= 0, & \zeta_* &= 0, & z_{bs} &= 0
\end{align*}
\]

(2.2.7)

The next step in the analysis is to make the variables dimensionless. This is done in the following way: we substitute:

\[
\begin{align*}
U &= (u, v) = u_* \hat{U}, & z_b &= h_s \hat{z}_b, & X &= (x, y) = y_*(\hat{X}),
\end{align*}
\]

\[
\begin{align*}
\zeta &= \frac{u^*_s \hat{\zeta}}{g}; & t &= \frac{y_s h_s \hat{t}}{\sigma u^*_s} = \hat{T}
\end{align*}
\]

(2.2.8)

The scaling for \( \zeta \) is motivated by the fact that the pressure force tends to zero if \( u_* \) tends to zero, independent of the value for \( C \). Thus there must be a balance between the advection terms and the pressure gradient. The scaling for the time means that we scale with the morphological time scale. This should be clear since we are interested in bed form instabilities which occur on this time scale.

Substitution of (2.2.7) in (2.2.1)-(2.2.3) yields the following model for the scaled quantities (hats are dropped for convenience):

\[
\begin{align*}
\kappa \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial \zeta}{\partial x} &= -CR( -1 + \frac{u \sqrt{u^2 + v^2}}{F^2 \zeta + 1 - z_b} )
\end{align*}
\]

(2.2.9)
\[
\frac{\kappa}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial y} = -CR\left(\frac{v \sqrt{u^2 + v^2}}{F^2 \zeta + 1 - z_b}\right)
\tag{2.2.10}
\]

\[
\kappa F^2 \frac{\partial \zeta}{\partial t} - \kappa \frac{\partial z_b}{\partial t} + \frac{\partial}{\partial x}\{u(F^2 \zeta + 1 - z_b)\} + \frac{\partial}{\partial y}\{v(F^2 \zeta + 1 - z_b)\} = 0
\tag{2.2.11}
\]

\[
\frac{\partial z_b}{\partial t} + \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} = 0
\tag{2.2.12}
\]

Here

\[
S = (S_x, S_y) = |U|^b\left(\frac{U}{|U|} - \frac{\gamma}{R} \nabla z_b\right)
\tag{2.2.13}
\]

and

\[
R = \frac{y_s}{h_*}, \quad F = \frac{u_*}{\sqrt{gh_*}}, \quad \kappa = \frac{y_s}{u_*T}
\tag{2.2.14}
\]

which are the width-to-depth ratio of the channel, the Froude number and the ratio of the time scale of flow adaption and the morphological time scale, respectively.

We shall now simplify the model (2.2.9)-(2.2.12) by applying two approximations. First it is observed that the flow responds to a change in the bottom by generating travelling gravity waves. This adjustment process is controlled by dissipation of which the characteristic time scale is assumed to be larger than (or of the same order as) the advective time scale \(y_s/u_s\), but small compared to the morphological time scale \(T\). This implies that \(\kappa\) in (2.2.9-2.2.12) is very small. We may therefore omit the time derivatives in the flow equations, which means that the flow instantaneously adapts to the evolution of the bed (quasi-stationary approach).

The second approximation is related to the observation that many river flows have a small Froude number. Thus we shall neglect in eq. (2.2.9)-(2.2.12) all terms containing the parameter \(F\), which means that we apply the rigid-lid approximation. An additional motivation for this step follows from the results of the linear theory to be discussed in the next section. It appears that in the limit \(F \to 0\) the calculations simplify considerably whereas essential physical mechanisms determining bed form instabilities are still included. On the other hand it implies that the possible applicability of our model to laboratory experiments, where Froude numbers are often of order 1 or larger, is limited.

The above considerations lead to the following model, which we shall use as starting point for the subsequent analysis:

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial \zeta}{\partial x} = -CR\left(-1 + \frac{u \sqrt{u^2 + v^2}}{1 - z_b}\right)
\tag{2.2.15}
\]

\[
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial y} = -CR\left(\frac{v \sqrt{u^2 + v^2}}{1 - z_b}\right)
\tag{2.2.16}
\]
\[ \frac{\partial}{\partial x} \{ u(1 - z_b) \} + \frac{\partial}{\partial y} \{ v(1 - z_b) \} = 0 \]  
(2.2.17)

\[ \frac{\partial z_b}{\partial t} + \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} = 0 \]  
(2.2.18)

with \( R, S_x \) and \( S_y \) as in (2.2.13)-(2.2.14).

It is readily observed that a basic state of the scaled model is given by

\[ (u_0, v_0, \zeta_0, z_{b0}) = (1, 0, 0, 0) \]  
(2.2.19)

### 2.3 Linear Theory

In this section we study the linear stability of the basic state \( \phi_0 = (u_0, v_0, \zeta_0, z_{b0}) \) defined in (2.2.19). Although this kind of analysis is quite standard we shall present results in some detail, since the nonlinear theory, developed in section 4, is based on the linear analysis. We study the stability of the basic state by considering the evolution of small perturbations. Thus we substitute in (2.2.15)-(2.2.18)

\[ \phi = \phi_0 + \phi' \]  
(2.3.1)

and neglect nonlinear terms to find four linear partial differential equations. They may be symbolically written as

\[ \mathcal{L} \phi' = 0 \]  
(2.3.2)

where the elements of the \( 4 \times 4 \) matrix \( \mathcal{L} \) contain known partial derivatives with respect to \( x, y \) and \( t \):

\[ \mathcal{L} = \begin{pmatrix} \frac{\partial}{\partial x} + 2CR & 0 & \frac{\partial}{\partial y} & CR \\ 0 & \frac{\partial}{\partial x} + CR & 0 & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial t} + \frac{\gamma}{R}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \\ b_3 \frac{\partial}{\partial x} & b_3 \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial t} \end{pmatrix} \]  
(2.3.3)

The channel geometry allows for travelling wave solutions in the \( x \)-direction with unknown lateral structure. Thus we substitute in (2.3.2)

\[ \phi' = f(y) e^{ikx+\omega t} + c.c. \text{ with } f(y) = (\hat{u}(y), \hat{v}(y), \hat{\zeta}(y), \hat{z}_b(y)) \]  
(2.3.4)

where \( k \) is a real-valued wavenumber, \( \omega \) the complex frequency, \( c.c. \) denotes complex conjugate and \( f(y) \) represents the amplitudes. We reduce the four equations to one equation for \( \hat{z}_b \) by Gauss-elimination, resulting in a fourth order differential equation

\[ Lz_b(y) \equiv b_3 \frac{d^4 z_b}{dy^4} + (\omega a_2 + b_2) \frac{d^2 z_b}{dy^2} + (\omega a_1 + b_1) z_b(y) = 0 \]  
(2.3.5)

with boundary conditions \( dz_b/dy = d^3 z_b/dy^3 = 0 \) on \( \Gamma \) (i.e. for \( y = 0, y = 1 \)).

In (2.3.5)

\[ b_3 = -\frac{\gamma}{R}(ik + 2CR), \]
\[ b_2 = 2i k^3 \left( \frac{\gamma}{R} \right) + k^2 (-1 + 3C \gamma) + i k CR (3 - b), \]
\[ b_1 = -i k^5 \left( \frac{\gamma}{R} \right) + k^4 (-C \gamma + b) - i k^3 CR b, \]
\[ a_2 = 2RC + ik, \quad a_1 = -ik^3 - k^2 CR. \]

Equation (2.3.5) is an eigenvalue problem which admits solutions of the form \( z_b(y) = e^{\mu y} \). Substitution of this expression in (2.3.5) gives
\[ \mu^4 b_1 + \mu^2 (w a_2 + b_2) + (w a_1 + b_1) = 0 \quad (2.3.6) \]
an equation which is quadratic in \( \mu \), with roots \( \mu_1, \mu_2 \). The solution of (2.3.5) then reads:
\[ z_b(y) = A_1 e^{\mu_1 y} + A_2 e^{\mu_2 y} + A_3 e^{\mu_2 y} + A_4 e^{\mu_2 y} \]
and applying the boundary conditions yields the eigenfunctions
\[ z_b(y) = A \cos(p\pi \cdot y) \quad p = 1, 2 \ldots \quad (2.3.7) \]
where \( A \) is an arbitrary amplitude. Thus in (2.3.6) the eigenvalues should have the value \( p\pi i \). Then by back-substitution into the four algebraic equations relating the amplitudes of the perturbation we recover for the vector \( f(y) \) the following:
\[ f(y) = A \begin{pmatrix} \alpha_u \\ \alpha_v \\ \alpha_z \\ \cos(p\pi y) \end{pmatrix} \equiv A \cdot \Phi \quad \text{with} \quad \alpha_u = \frac{-k^2 \gamma - ik - \omega - \pi^2 \gamma}{(b-1)i k}, \]
\[ \alpha_v = \frac{-(b-1)i k}{\pi p}, \quad \alpha_z = \frac{(ik+CR) \alpha_v}{\pi p}. \quad (2.3.8) \]

Hence, the lateral structure of the wave solution is now known and can be denoted with \( \phi' = A \cdot \Phi \cdot E \) where \( E = e^{ikx+\omega t} \).

Finally, the frequency \( \omega \) is related to the wavenumber \( k \) by a dispersion relation, as follows from (2.3.6):
\[ \omega = \frac{p^4 \pi^4 b_3 - p^2 \pi^2 b_2 + b_1}{p^2 \pi^2 a_2 - a_1} \quad (2.3.9) \]
It is obvious from (2.3.4) that the stability of the basic state is determined by the real part of \( \omega \). If the real part \( \omega_r \) of \( \omega \) is smaller than 0, perturbations of the kind (2.3.4) decay exponentially in time, i.e., the basic state is stable. Accordingly, if \( \omega_r \) is greater than zero, we are dealing with exponentially growing solutions, i.e., an unstable basic state. Hence, \( \omega_r = 0 \), the so-called neutral curve, is a separatrix between the exponentially growing and decaying solutions of the linear problem. Using (2.3.9) we derive a formula describing the neutral curve:
\[ \lambda = \frac{-X(X+1)^3 \delta}{\delta (X+1)(X+2)^2 - X(2X+1)} \quad (2.3.10) \]
with
\[ \lambda = \frac{C^2 R^2}{p^2 \pi^2}, \quad X = \frac{k^2}{p^2 \pi^2}, \quad \delta = \frac{\gamma C}{\beta}, \quad \beta = b - 1 \quad (2.3.11) \]
In figure 2.2 we have plotted the neutral curve for a fixed value of $\delta$ which lies in the physically interesting area. We observe that for a width-to-depth ratio $R$ smaller than a certain critical value $R_c$, all perturbations decay exponentially. For $R > R_c$ a range of linear waves have positive growth rate ($\omega_r > 0$) and consequently the basic state is unstable. Note also that the wave with wave number $k = 0$, is marginally stable ($\omega_r = 0$) for all values of $R$. This observation comes from Komarova and Newell (1995), see also section 2.6.1. We expect interesting behaviour of the perturbations if $R$ passes through $R_c$, i.e. when the basic state looses its stability properties. It turns out that varying $R$ is a good starting point for studying the dynamics of our underlying system.

Let us finally make an observation about the position of the minima of the neutral curves as a function of $p$ in more detail. If we denote by $(k_c(p), R_c(p))$ the minimum as function of $p$, it is clear from (2.3.10) that holds:

$$(k_c(p), R_c(p)) = p \cdot (k_c(1), R_c(1)).$$

This means that increasing $p$ leads to a higher minimum of the neutral curve and a larger critical wavenumber. Thus, the $p = 1$-mode (the situation of alternate bars) is the first mode that becomes unstable. Therefore, we may restrict our attention on the case $p = 1$.

From now on a subscript ‘$c$’ means the evaluation of operators and functions at critical conditions, i.e., at $k = k_c, R = R_c$. So for instance, $\Phi_c = \Phi|_{k=k_c, R=R_c}$. The linear solution of the problem at critical conditions then reads:

$$\phi_t = \phi'_c = A\Phi_c E_c$$

where $\Phi$ follows from (2.3.8) and $E_c = e^{ik_c x + \omega_c t}$. In order to obtain a better understanding of the neutral curve we perform a perturbation analysis with respect to the small parameter
\( \delta = \gamma C/\beta \). For realistic choices of the morphological parameters, typical values for \( \delta \) range between 0.0001 and 0.01. Although this expansion is in principle unnecessary, it provides predictions concerning the bar formation which depend explicitly (instead of implicitly) on the model parameters \( b, C \) and \( \gamma \). Therefore, this approach yields a better insight in the fundamental behaviour of the model. Then, using (2.3.10) two physically interesting roots of the denominator appear:

\[
X_1 = 4\delta + O(\delta^2),
\]

\( \text{(2.3.13)} \)

\[
X_2 = \frac{2}{\delta} - \frac{9}{8} + O(\delta),
\]

\( \text{(2.3.14)} \)

i.e., there are no destabilizing waves with wave numbers outside the interval described by these two asymptotes. The asymptotic expressions for \( k_c \) and \( R_c \) read

\[
k_c = \sqrt{2\pi} \delta^{\frac{1}{4}} \left\{ 1 + \frac{19}{4} \delta + O(\delta^2) \right\},
\]

\( \text{(2.3.15)} \)

\[
R_c = \frac{\pi \gamma}{\beta} \delta^{-\frac{1}{2}} \left\{ 1 + 2\delta^{\frac{1}{2}} + 4\delta + O(\delta^2) \right\}.
\]

\( \text{(2.3.16)} \)

and we derive the asymptotic expression for \( \omega_c \) from (2.3.9)

\[
\omega_c = -ik_c \left\{ 1 + \beta \delta^{\frac{1}{4}} - 5\beta \delta + O(\delta^2) \right\}
\]

\( \text{(2.3.17)} \)

Note that \(|i\omega_c|\) is the wave frequency at critical conditions. Asymptotic expressions for \( \alpha_u, \alpha_v, \alpha_\zeta \) are given in Appendix A.

The physical mechanism of instability can be understood from a closer investigation of the perturbation equations (2.3.1)-(2.3.2). In order to obtain unstable bedform disturbances the morphological system should allow for convergence of sediment at the wave crests. As described by the evolution equation for the bottom this requires a phase difference between the divergence of the sediment transport and the bottom disturbances which is between \( \pi/2 \) and \( 3\pi/2 \). Here only the transport corresponding to \( \gamma = 0 \) needs to be considered, since the down slope term in the total transport always causes damping of the perturbations. It follows straightforward from (2.3.2)-(2.3.3) that in the cases \( CR = 0 \) (no bottom friction), \( b = 1 \) or \( v' = 0 \) (a one-dimensional model) this phase difference is exactly \( \pi/2 \) or \( 3\pi/2 \). This implies that perturbations will decay by diffusive mechanisms, due to the down-slope correction term in the volumetric sediment transport. Some of these results follow directly from combination of the two linearized continuity equations in (2.3.2)-(2.3.3), yielding

\[
\frac{\partial z'_b}{\partial t} + b \frac{\partial z'_b}{\partial x} - \frac{\gamma}{R} \nabla^2 z'_b = \beta \frac{\partial v'}{\partial y}
\]

\( \text{(2.3.18)} \)

This is an advection-diffusion equation which, in the absence of forcing on the right-hand side, describes decaying travelling wave solutions. Obviously, necessary conditions for the occurrence of unstable free bars are a two-dimensional model and the presence of bottom friction. On the other hand the latter mechanism also causes direct damping of the perturbations. This implies that the energy flux from the basic flow to the perturbations should exceed the dissipation of energy due to bottom friction. The energy gain is provided for
by advection terms in the equations of motion, which in the linear theory are proportional to the wavenumber $k$. Thus instabilities cannot occur in the limit $k \rightarrow 0$, i.e., long waves are stable. On the other hand, the diffusive terms in the sediment continuity equation are proportional to $\gamma k^2/R$ and thus cause stabilization of the short waves. The combined effect of bottom friction and the down-slope correction term in the volumetric sediment transport results in a critical width-to-depth ratio below which all waves are stable. The long- and short-wave cut-offs are described by the asymptotic results (2.3.13) and (2.3.14).

In figure 2.2 only the long-wave asymptote of the neutral curve is visible. We remark that the stabilization of short waves in the model of Colombini et al. (1987) cannot be explained in this sense, since they neglect longitudinal bed slope corrections in the parameterization of the sediment transport. On the other hand, they allow for a drag coefficient depending on the local bed roughness as well as for free surface effects, which are important in case the Froude number is not small. Both mechanisms appear to cause an effective damping of perturbations with large wave numbers.

For realistic values of the morphological parameters we find critical width-to-depth ratios between 20 and 30, critical wave-lengths between 5 and 15 channel-widths and wave periods between 5 and 15 morphological time scale units. We have compared these predictions with those of previous linear stability studies concerning free bars in straight channels. In particular the results of Engelund and Skovgaard (1973), Fredsoe (1978) and Colombini et al. (1987) are relevant because their models include the important effect of bed slope correction terms in the sediment transport. In all of these studies it is demonstrated that the agreement of the theoretical findings with observations is quite satisfactory. At this point we recall that our model contains three major simplifications compared to the other models, i.e., free surface effects, the variation of the drag coefficient with the local water depth and the presence of a critical shear stress for erosion are neglected. Despite these rather rigorous simplifications the results of our linear stability analysis are in good agreement with those of the previous studies. In particular it follows from Colombini et al. (1987) that in the low Froude number regime their critical width-to-depth ratio $\beta_c \equiv R_c/2$ ranges between 10 and 15, whereas their critical wave-lengths are 6 to 12 river width units.

2.4 Weakly nonlinear theory

In the previous section it has been demonstrated that for width-to-depth ratios larger than a critical value $R_c$ the basic state of our model, i.e., a uniform flow over a flat bed, is unstable. The linear theory shows that in this case a spectrum of wave-like perturbations with exponentially growing amplitudes will develop. However, this description is only valid in the initial growth stage, where the wave amplitudes are infinitesimally small. If we want to describe the nonlinear dynamic behaviour, we must take into account the nonlinear interactions between the various (linearly-unstable) wave components. A detailed analysis is possible if the width-to-depth ratio $R$ is only slightly larger than the critical value $R_c$. Therefore, we restrict ourselves to width-to-depth ratios $R$, for which holds:

$$R = R_c(1 + \hat{r} \varepsilon^2) = R_c + r \varepsilon^2$$

where $\varepsilon \ll 1$, $\hat{r} = O(1)$ and $r = O(R_c)$ (2.4.1)

From mathematical point of view parameter $\varepsilon$ should be ‘small enough’ in order to be able to perform a perturbation analysis. The nonlinear theory described below has been
applied to various classical hydrodynamic stability problems (see for instance Newell and Whitehead (1969) Stewartson and Stuart (1971)) and has been tested experimentally (see for instance Drazin and Reid (1981) for an overview). In these experiments it has been found that the predictions based on the (weakly) nonlinear theory might be valid for $R/R_C$ considerably larger than 1, i.e. physically, the theory seems to be valid for $\varepsilon$ of an order 1 magnitude.

Because the neutral curve can be approximated by a parabola near its minimum, it means that we consider wave numbers $k$, for which

$$|k - k_c| = O(\varepsilon), \quad \varepsilon \ll 1$$ ...

(2.4.2)

The unstable waves are thus limited to a narrow spectrum around the critical wavenumber $k_c$. Besides, the waves are marginally unstable: they grow on a time scale which is large compared to the typical wave periods. The consequences of these assumptions can be seen by expanding the complex frequency $\omega$ in a Taylor-series, near $(k_c, R_c)$:

$$\omega = \tau_r (R - R_c) + \frac{1}{2} \tau_{k^2} (k - k_c)^2 + \ldots +$$

$$\omega_c + i [\nu_k (k - k_c) + \nu_r (R - R_c) + \frac{1}{2} \nu_{k^2} (k - k_c)^2 + \ldots]$$ ...

(2.4.3)

where

$$\tau_r + i \nu_r = \left( \frac{\partial \omega}{\partial R} \right)_c, \quad i \nu_k = \left( \frac{\partial \omega}{\partial k} \right)_c, \quad \tau_{k^2} + i \nu_{k^2} = \left( \frac{\partial^2 \omega}{\partial k^2} \right)_c$$ ...

(2.4.4)

where 'c' means, as before, evaluation at critical conditions. Here $\tau_r$ measures the growth rate of the marginally unstable waves, whereas $\nu_r$ gives the frequency shift of the perturbation with respect to the basic wave. Furthermore, $\nu_k$ is the group velocity of the wave packet; $\tau_{k^2}$ and $\nu_{k^2}$ will be interpreted later on. Note that $\omega_c$ in (2.4.3) is an imaginary number, as defined in (2.3.17).

Substitution of (2.4.3) in the linear solution of the bottom perturbation (2.3.4) yields

$$z_b \sim e^{ik_xt + \omega t} + \ldots + c.c$$

$$= e^{ik_c x + \omega_c t} \cdot e^{i(k-k_c)(x+\nu_k t)} \cdot e^{(\tau_r (R - R_c) + \frac{1}{2} \tau_{k^2} (k - k_c)^2) t} + \ldots + c.c.$$ ...

(2.4.5)

$$= A(\xi, \tau) E_c + \ldots + c.c.$$

(2.4.6)

This describes (the modulation of) a basic critical wave with wavenumber $k_c$ and frequency $\omega_c$. The modulation is at slow temporal and spatial scales, which are described by the coordinates

$$\tau = \varepsilon^2 t \quad \text{and} \quad \xi = \varepsilon (x + \nu_k t)$$ ...

(2.4.7)

where (2.4.1) and (2.4.2) are used. Note that we thus have introduced a long time scale $\tau$ and a long spatial scale $\xi$ which is a slow, moving coordinate, travelling with the group velocity. This is due to the fact that the envelope of a linear dispersive wave packet travels with this velocity. This behaviour suggests the use of multiple scale analysis in order to
obtain an evolution equation for the amplitude $A(\xi, \tau)$.

Note also that the proposed scaling has its influence in the set of equations (2.2.15)-(2.2.18):

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \varepsilon \nu k \frac{\partial}{\partial \xi},$$
$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \xi}$$

Since we expect that solutions of the full system will be small for $R$ close to $R_c$ we expand

$$\phi = \phi_0 + \lambda \phi_1 + \lambda^2 \phi_2 + \ldots + c.c.$$ (2.4.9)

where $\lambda = \varepsilon^\nu$ for some $\nu$ to be determined. Substituting (2.4.8)-(2.4.9) into equations (2.2.15)-(2.2.18) yields, by construction, at zeroth order the basic solution:

$$\phi_0(x, y, t, \xi, \tau) = A(\xi, \tau) \Phi_c E_c$$ (2.4.10)

(compare to (2.3.12)) for some, up till now, unknown function $A$ (see below for a more detailed discussion). We have to analyse the higher order systems in order to determine the behaviour of $A(\xi, \tau)$. The nonlinear interactions generate at the second order second harmonics and residual components, however, no extra information on $A$ can be obtained.

In third order, the fundamental wave is reproduced by interaction of the fundamental wave itself and the components produced at the second order. This $\lambda^3$-component should balance the linear growth of the fundamental wave, which is of order $\varepsilon^2 \lambda$. Hence we choose $\lambda = \varepsilon$, i.e., $\nu = 1$. Based upon the observed interaction-mechanism between the various harmonics components and on scaling (2.4.7) we propose the expansion

$$\phi(x, y, t, \xi, \tau) = \phi_0 + \varepsilon \phi_01 + \varepsilon^2 \phi_02 + \ldots + c.c.$$ (2.4.11)

with $\phi_{pq} = \phi_{pq}(\xi, \tau, y)$ describing the response in $O(\varepsilon^q, E_p^c)$. In this expansion, we have neglected the influence of the so called mean flow (which is related to the observation of Komarova and Newell (1995), that the wave with wave number $k = 0$ is marginally stable for all values of $R$; see also section 2.6.1).

We substitute (2.4.11) into equations (2.2.15)-(2.2.18) (and take into account (2.4.1) and (2.4.8)) and compare equal orders of $\varepsilon^q E_p^c$. In general we will find a set of linear ordinary differential equations of the type

$$\mathcal{L}_{pc} \phi_{pq} = f_{pq} \quad \text{where} \quad \mathcal{L}_{pc} = \mathcal{L}_p \mid_{k = k_c}$$ (2.4.12)

with

$$\mathcal{L}_p = \begin{pmatrix}
  p(ik) + 2CR & 0 & p(ik) & CR \\
  0 & p(ik) + CR & \frac{\partial}{\partial y} & 0 \\
  p(ik) & \frac{\partial}{\partial y} & 0 & -p(ik) \\
  p(bik) & \frac{\partial}{\partial y} & 0 & p\omega - \frac{b}{R}(\{p(ik)\}^2 + \frac{\partial^2}{\partial y^2})
\end{pmatrix}$$ (2.4.13)
and \( f_{pq} \) are inhomogeneous terms which depend on \( \xi, \tau \) and \( y \). They are generated by the nonlinear interactions between the various harmonics and due to the coordinate transformations and the introduction of the ‘new’ control parameter \( r \). Note that \( \mathcal{L}_1 \) coincides with the linear operator defined in (2.3.3). This means that \( \mathcal{L}_1c \) is singular. Thus, a priori, (2.4.12) will not be solvable for a general inhomogeneous term \( f_{1q} \) : one has to apply a solvability criterion (or Fredholm alternative) in order to solve (2.4.12) (there are no problems for \( p \neq 1 \)). Applying such a condition at the level \( p = 1, q = 3 \), results in the desired balance between the linear growth and the influence of the nonlinear interactions and yields a nonlinear evolution equation which governs the behaviour of \( A(\xi, \tau) \). This equation is known as the Ginzburg-Landau modulation equation.

We are now going to discuss the equations of the type (2.4.12) for the several values of \( p \) and \( q \). We start with \( p = 0, q = 0 \), which leads to the recovering of the basic solution \( \phi_0 \). The case \( p = 1, q = 1 \) has already been discussed; we recovered the linear problem and, since \( f_{11} = 0 \) and \( \mathcal{L}_c \) does not depend on \( \xi \) and \( \tau \) we introduced the amplitude function \( A(\xi, \tau) \). Note thus that \( \phi_{11} = A(\xi, \tau)\Phi_c \), where \( \Phi_c \) follows from (2.3.12). The next case we consider is the \( O(\varepsilon^2) \) residual balance, i.e., \( p = 0, q = 2 \). Note that \( f_{02} \) is nonzero since it is generated by interactions between the \( \varepsilon E_c \)-mode and its complex conjugate. Because \( \phi_{11} \) is already known, it follows that \( f_{02} = |A|^2 \cdot \tilde{f}_{02} \), for some \( \tilde{f}_{02}(y) \) which can be calculated explicitly.

It turns out that we have

\[
f_{02} = |A|^2 \begin{pmatrix} c_1 \sin^2(\pi y) + c_2 \cos^2(\pi y) \\ c_3 \sin(2\pi y) \\ c_4 \cos(2\pi y) \\ c_5 \cos(2\pi y) \end{pmatrix}
\]  \hspace{1cm} (2.4.14)

Making use of the structure of \( \mathcal{L}_c \) and \( f_{02} \), we find the solution for \( \phi_{02} \):

\[
\phi_{02} = |A|^2 \begin{pmatrix} u_{02s} \sin^2(\pi y) + u_{02c} \cos^2(\pi y) \\ v_{02} \sin(2\pi y) \\ \zeta_{02} \cos(2\pi y) \\ \zeta_{02} \cos(2\pi y) \end{pmatrix}
\]  \hspace{1cm} (2.4.15)

The coefficients \( u_{02c}, u_{02s}, v_{02}, \zeta_{02}, \zeta_{02} \) are functions of \( (\alpha_u)_c, (\alpha_v)_c, (\alpha_\xi)_c, k_c \) and \( R_c \), which are known as asymptotic series in the small parameter \( \delta \), defined in (2.3.11). The final results are presented in Appendix A.

The next case we consider is \( p = 2, q = 2 \), i.e the second order and second harmonic system. This systems is forced by nonlinear interactions between the \( \varepsilon E_c \)-mode and itself. The analysis of this case is similar to the previous case, i.e., the first order field \( \phi_{11} \) generates the inhomogeneous term \( f_{22} \), which can now be written as

\[
f_{22} = A^2 \begin{pmatrix} d_1 \sin^2(\pi y) + d_2 \cos^2(\pi y) \\ d_3 \sin^2(\pi y) \\ d_4 \sin^2(\pi y) + d_5 \cos^2(\pi y) \\ d_6 \sin^2(\pi y) + d_7 \cos^2(\pi y) \end{pmatrix}
\]  \hspace{1cm} (2.4.16)
The solution of (2.4.12) with \( p, q = 2 \) is of the form

\[
\phi_{22} = A^2 \begin{pmatrix}
    u_{22s} \sin^2(\pi y) + u_{22c} \cos^2(\pi y) \\
    v_{22s} \sin^2(\pi y) \\
    \zeta_{22s} \sin^2(\pi y) + \zeta_{22c} \cos^2(\pi y) \\
    z_{22s} \sin^2(\pi y) + z_{22c} \cos^2(\pi y)
\end{pmatrix}
\]

and substitution results in 7 algebraic equations for the 7 unknowns in (2.4.16). The asymptotic results are given in Appendix A.

Now, consider the first harmonic response of the second order system, i.e., the case \( p = 1, q = 2 \). This results in the equation

\[
\mathcal{L}_{1c} \phi_{12} = f_{12}
\]

(2.4.18)

Calculations show that \( f_{12} \) can be written as follows:

\[
f_{12} = i \left( \frac{\partial \mathcal{L}_1}{\partial k} \right) c \frac{\partial \phi_{11}}{\partial \xi} = i \frac{\partial A}{\partial \xi} \left( \frac{\partial \mathcal{L}_1}{\partial k} \right) c \Phi_c
\]

(2.4.19)

Note that in this case \( f_{12} \) is generated due to the coordinate transformation introduced in (2.4.7) and therefore contains only first order derivatives with respect to the slow spatial coordinate \( \xi \), i.e. \( f_{12} = \frac{\partial A}{\partial \xi} f_{12} \), for some \( f_{12} \). We now make the following observation: consider the general linear problem (2.3.2)-(2.3.3), which in the context of this section can also be written as

\[
\mathcal{L}_1 \Phi = 0
\]

(2.4.20)

Differentiation of (2.4.20) with respect to \( k \) and evaluation in \( k = k_c, R = R_c \) yields:

\[
\left( \frac{\partial \mathcal{L}_1}{\partial k} \right)_c \Phi_c + \mathcal{L}_{1c} \left( \frac{\partial \Phi}{\partial k} \right)_c = 0
\]

(2.4.21)

This observation can be used to find a solution for the problem \( \mathcal{L}_{1c} \phi_{12} = f_{12} \). Equation (2.4.18) can be reformulated as

\[
\mathcal{L}_{1c} \phi_{12} = -i \frac{\partial A}{\partial \xi} \mathcal{L}_{1c} \left( \frac{\partial \Phi}{\partial k} \right)_c
\]

(2.4.22)

with the solution

\[
\phi_{12} = \phi_{12p} + \phi_{12h}
\]

(2.4.23)

where

\[
\phi_{12p} = -i \frac{\partial A}{\partial \xi} \left( \frac{\partial \Phi}{\partial k} \right)_c \text{ and } \mathcal{L}_{1c} \phi_{12h} = 0.
\]

Note that \( \phi_{12p} \) is a known particular solution, whereas the homogeneous solution is given by \( \phi_{12h} = A_2(\xi, \tau) \Phi_c \), with a second, unknown amplitude function \( A_2 \). This second amplitude function is a result of the application of the Fredholm alternative where the solvability condition on this level is automatically satisfied by \( f_{12} \) (essentially this is a consequence of our
28 Chapter 2. On the Nonlinear Dynamics of ... ‘choice’ to move with the group velocity $v_k$, see (2.4.7)). We have to introduce $A_2$ as a consequence of the non-uniqueness of the solution as provided by the Fredholm alternative and it should be noted that $A_2$ is unimportant for the subsequent analysis. Due to the structure of the expansion we will meet at any level of the analysis this type of ‘solvability’-behaviour.

The final case we consider is $p = 1$, $q = 3$, i.e.

$$\mathcal{L}_{1c} \phi_{13} = f_{13} \equiv f_{13,\text{lin}} + f_{13,\text{nonlin}}$$

(2.4.24)

Here, $f_{13,\text{lin}}$ contains terms which are linear in the amplitude $A$, whereas $f_{13,\text{nonlin}}$ contains all the nonlinear terms. It turns out that the linear terms in $f_{13}$ can be written as:

$$f_{13,\text{lin}} = \mathcal{M} \phi_{12} + \mathcal{N} \phi_{11}$$

(2.4.25)

where

$$\mathcal{M} = i \frac{\partial}{\partial \xi} \left( \frac{\partial \mathcal{L}_1}{\partial k} \right)_c; \quad \mathcal{N} = \left\{ -r \frac{\partial \mathcal{L}_1}{\partial R} + \frac{1}{2} \frac{\partial^2 \mathcal{L}_1}{\partial k^2} \right\}_c + \mathcal{S}$$

(2.4.26)

and $\mathcal{S}$ is a $4 \times 4$-matrix with all elements zeros, except

$$\mathcal{S}(4, 4) = -\frac{\partial}{\partial \tau} + r \left( \frac{\partial \omega}{\partial R} \right)_c - \frac{1}{2} \left( \frac{\partial^2 \omega}{\partial k^2} \right)_c \frac{\partial^2}{\partial \xi^2}$$

(2.4.27)

Substituting (2.4.25)-(2.4.27) in (2.4.24) and using (2.4.20) to evaluate the derivatives of $\mathcal{L}$ with respect to $k$ and $R$ results in

$$\mathcal{L}_{1c} \phi_{13} = f_{13} = \mathcal{L}_{1c} \left[ -\frac{1}{2} \frac{\partial^2 A}{\partial \xi^2} \frac{\partial^2 \phi}{\partial k^2} - i \frac{\partial A_2 \partial \phi}{\partial \xi \partial k} + r \frac{\partial \phi}{\partial R} \right]_c + \mathcal{S} \phi_{11} + f_{13,\text{nonlin}}$$

(2.4.28)

In the derivation of (2.4.28), we have used the same kind of reasoning as in the case of $p = 1, q = 2$, where (2.4.18) was rewritten as (2.4.22). In the present case we have used the differentiation of (2.4.20) with respect to $R$, as well as the second order derivative of (2.4.20) with respect to $k$. We must apply the Fredholm alternative to the right hand side of (2.4.28). This means that we have to determine

$$< f_{13}, \Phi_c > = \int_0^1 f_{13} \Phi_c^A \ dy = 0,$$

where $\Phi_c^A$ are the eigenfunctions of the adjoint linear problem at critical conditions. To avoid calculation of the eigenfunctions of the adjoint problem, we again reduce problem (2.4.28) to one equation for $\phi_{13}^{(4)}$, where $\phi_{13}^{(4)}$ is the fourth component of the vector $\phi_{13}$, in which case the eigenfunctions (i.e. $\sim \cos(p \pi y)$) coincide with their adjoints. Note however that in applying the Fredholm alternative the first term in the right hand side of (2.4.28) vanishes, and thus $\mathcal{S} \phi_{11}$ is the only contribution to the linear terms when we evaluate the inner product. We observe that the reduced problem can be written as $L \phi_{13}^{(4)} = \tilde{g}$, where $L$ follows from (2.3.5) and $\tilde{g}$ reads:

$$\tilde{g} = -k_c^2 (ik_c + CR_c) [f_{13}^z - f_{13}^\xi] + (ik_c + 2CR_c) \frac{\partial^2}{\partial y^2} [f_{13}^z - f_{13}^\xi]$$

$$-ik_c (b - 1) \frac{\partial^2}{\partial y^2} f_{13}^u - (b - 1) k_c^2 \frac{\partial}{\partial y} f_{13}^u + k_c^2 (b - 1)(ik_c + CR_c) f_{13}^\xi.$$

(2.4.29)
where the components \( f^{(i)} \) of the vector \( f_{13} \) can be found in Appendix B. T

edious calculations then show

\[
\tilde{g} = g_1(y) + g_2 \cos(\pi y) + |A|^2 A (g_3 \cos(\pi y) \sin^2(\pi y) + g_4 \cos^3(\pi y))
\]  

(2.4.30)

For \( g_1, \ldots, g_4 \) is again referred to the Appendix B. The Fredholm alternative reads in this case:

\[
\langle \tilde{g}(y), \cos(\pi y) \rangle = \int_0^1 \tilde{g}(y) \cos(\pi y) \, dy = 0
\]  

(2.4.31)

Evaluating expression (2.4.31) finally leads to a nonlinear partial differential equation for the amplitude \( A \):

\[
\frac{\partial A}{\partial \tau} = r(\tau_r + i\nu_r) A - \frac{1}{2} (\nu_k^2 + i\nu_k^2) \frac{\partial^2 A}{\partial \xi^2} + (c_r + ic_l) |A|^2 A
\]  

(2.4.32)

This modulation equation, the so-called Ginzburg-Landau equation, governs all the pattern generating processes. The evolution of a pattern, whether it is periodic or non-periodic (see section 5), is always determined by the time scale \( \tau \) and the spatial scale \( \xi \) (see (2.4.7)). Thus, the time scale on which a solution of (2.4.32) develops into a stable pattern does not depend on the complexity of this pattern. Furthermore, if (2.4.32) has no stable periodic solutions (see section 5), then any initial pattern will immediately (on the \( \tau \)-time-scale) evolve to a non-periodic pattern: an observer will not see a first (time-)step in which the initial pattern becomes periodic and a second step in which this periodic pattern is slowly (in \( \tau \)) modulated into a non-periodic pattern.

Note that the coefficients of the linear terms follow from the expansion for \( \omega \) (see also (2.4.4)). For asymptotic results for the coefficients of (2.4.32) we refer to Appendix B. It should here be remarked that \( c_r \) is smaller than zero (for physically interesting values of \( b, \gamma \) and \( C \)). In terms of the Ginzburg-Landau equation it means that the exponential growth governed by the linear part is counteracted by nonlinear effects, so we may expect finite amplitude solutions.

At this point in the theory the expressions for \( \phi_{11}, \phi_{02}, \phi_{12} \) and \( \phi_{22} \) are known in terms of the model parameters. Thus, we are able to visualize the nonlinear bed profile, using (2.4.11), for an, at this time, undetermined amplitude. A typical result is shown in figure 2.3. Individual waves move with the phase velocity \( |i\omega/k| \), where \( k \) satisfies (2.4.2). On the other hand, the energy of the wave group moves with the group velocity \( |\nu_k| \) (for an asymptotic expression for \( \nu_k \), see Appendix B. It turns out that \( |i\omega/k| > |\nu_k| \) where \( k \) satisfies (2.4.2), i.e. the group velocity is larger than the phase velocity. This phenomenon is called anormal dispersion.

### 2.5 Analysis of the Ginzburg-Landau equation

In this section we shall present some properties of the Ginzburg-Landau equation which are relevant for our morphological model. At first, we consider the possibility of periodic solutions. We shall give conditions on the stability of these solutions and give predictions
Figure 2.3: Various order approximations of the bed profile as described by equation (2.4.11). Solutions up to $O(\varepsilon)$ are known in terms of the parameters $\delta, \varepsilon$ and the amplitude $A$. On the short spatial scale visualized here, the amplitude may be considered as a constant, which is yet unspecified. Here we have taken $b = 6, C = 0.003, \gamma = 1, \varepsilon = 0.33$ and $A = 0.5$. In (a), the $O(\varepsilon)$ approximation of the bed profile is shown, i.e., the linear wave solution at critical conditions. In (b), the nonlinear $O(\varepsilon^2)$ correction on the linear profile is shown. In (c), the $O(\varepsilon^2)$ approximation of nonlinear bed profile is shown. This is the composition of the two profiles shown in (a) and (b). Note the steep wave fronts and the flat wakes caused by the phase difference between the linear profile and the nonlinear $O(\varepsilon^2)$ correction.

We consider periodic solutions of the following form:

$$A(\xi, \tau) = Ge^{i(K\xi+W\tau)} \text{ with } G, K, W \in \mathbb{R}$$  \hfill (2.5.1)

Note that periodic solutions of this type correspond to the bifurcating unstable waves (the alternate bars) of the linearized theory and also appear in the Landau theory (however, then $K$ is fixed, here it is a parameter). The solution with $K = 0$ is called the Stokes wave. Substitution of (2.5.1) in (2.4.32) and splitting the expression in real and imaginary parts
2.5 Analysis of the Ginzburg-Landau equation

yields:

\[ 0 = r\tau + \frac{1}{2} K^2 \tau k^2 + G^2 c_r, \]  
\[ W = r\nu + \frac{1}{2} K^2 \nu k^2 + G^2 c_i. \]

Eq. (2.5.2) gives for every choice of \( K \) and \( r \), the amplitude and the shift in phase (with respect to the critical wave) of the periodic solutions (i.e. the amplitude and the phase of the free bars). We use the periodic solutions (2.5.1) of the Ginzburg-Landau equation where \( K, G \) and \( W \) follow from (2.5.2)-(2.5.3) in order to find an expression for the bottom evolution \( z_b \). Applying (2.4.11) yields:

\[ z_b = \varepsilon A(\xi, \tau) e^{ik_c x + \omega_c t} \cos(\pi y) + \ldots + c.c. \]

\[ = \varepsilon G e^{i(k_c + \varepsilon K)x + (\omega_c + i\varepsilon \nu \varepsilon + \varepsilon^2 W) t} \cos(\pi y) + \ldots + c.c. \]  

(2.5.4)

Thus, in this case, the fundamental wave is modulated, where the modulation is expressed in an \( O(\varepsilon) \) change in the critical wavenumber and an \( O(\varepsilon^2) \) change in the critical frequency. Note that (2.5.2) represents (half) an ellipse in the \((K,G)\)-plane for \( r > 0 \). This is due to the signs of \( \tau_r, \tau_k^2 \) and \( c_r \) (see Appendix B). Thus, periodic solutions only exist for \( r > 0 \), i.e. \( R > R_c \), which agrees with the linear predictions. Figure 2.4 is a contour plot of the maximal amplitude which is reached at \( K = 0 \), i.e. at \( k = k_c \). Note that it can be seen from (2.5.2) that increasing \( r \) leads to increasing amplitude \( G \). To find the maximal amplitude, we have set \( r = R_c \), as is motivated by (2.4.1). It appears that our predicted amplitudes are slightly larger than those obtained from the model of Colombini et al. (1987). A reasonable comparison is expected to be possible if we choose \( b = 3, \gamma = 1 \) and \( \zeta \) between 0.002 and 0.003, which implies that free surface effects in the latter model are negligible. In this case we obtain amplitudes of order 0.4 whereas Colombini et al. (1987) report values of order 0.3. These differences are due to the fact that their model includes the variation of the drag coefficient with the local water depth, the presence of a critical shear stress for erosion and the dependence of the bed slope correction coefficient on the bottom shear stress. On the other hand, slightly larger amplitudes are not unrealistic since the results of Colombini et al. (1987) show more under- than overestimation of observed alternate bar amplitudes.

We remark that (2.5.3) is again a dispersion relation. Since the coefficient \( c_i \) is positive (see Appendix B), it can be seen from (2.5.3) that the frequency \( W \) increases with increasing amplitudes. This means that the total frequency of the alternate bars, which is \(| -i\omega_c + \varepsilon \nu \varepsilon K + \varepsilon^2 W | \), decreases if the amplitude becomes larger or, in other words, the nonlinear bars move slower than is predicted by linear theory. This is a general property of anormal dispersive waves.

The periodic solutions obtained so far are also found by standard Landau theory, in which case they are always stable (see Colombini et al. (1987)). In our model, interactions between various wave components can cause the basic periodic solution (2.5.1) to become unstable. In order to investigate this possibility, we consider a general perturbation of a periodic solution of the type (2.5.1):

\[ A(\xi, \tau) = [G + \rho(\xi, \tau)]e^{i(K\xi + W\tau + \theta(\xi, \tau))} \]  

(2.5.5)
Figure 2.4: Contour plot of the maximal amplitude of the Stokes wave, as determined by (2.5.2), in the $b,C$ parameter space for $\gamma = 1$ and $r = R_c$, where $R_c$ is defined in (2.3.16). The contour line in the middle corresponds to a maximal amplitude of 0.54. The distance between the contour lines is 0.01.

Inserting (2.5.5) into (2.4.32) and using (2.5.2) and (2.5.3) to simplify intermediate results yields after linearizing

$$
\rho_r = r \tau_r, \rho + \frac{1}{2} | \tau_{k^2} | (\rho_{k^2} - 2GK\theta_k - K^2\rho) + \frac{1}{2} \nu_{k^2}(2K\rho_k + G\theta_{k^2}) - 3 | c_r | G^2 \rho
$$

(2.5.6)

$$
G\theta_r = -W\rho + r\nu_r, \rho + \frac{1}{2} | \tau_{k^2} | (2K\rho_k + G\theta_{k^2}) - \frac{1}{2} \nu_{k^2}(\rho_{k^2} - 2GK\theta_k - K^2\rho) + 3c_iG^2 \rho.
$$

(2.5.7)

Due to the structure of these linear equations we may assume

$$
\rho(\xi, \tau) = X(\tau)e^{i\ell \xi},
$$

(2.5.8)

$$
G\theta = Y(\tau)e^{i\ell \xi},
$$

(2.5.9)

with an arbitrary wavenumber $\ell \in \mathbb{R}$. Substitution of (2.5.8)-(2.5.9) into (2.5.6)-(2.5.7), and using (2.5.2)-(2.5.3) to get rid of terms containing $W$ and $K^2$, leads to:

$$
\frac{d}{d\tau} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \ell^2 | \tau_{k^2} | + i\nu_{k^2} \ell K - 2 | c_r | G^2 - \frac{1}{2} \ell^2 \nu_{k^2} - i\ell K | \tau_{k^2} | & \frac{1}{2} \ell^2 \nu_{k^2} + i\ell K | \tau_{k^2} | + 2c_iG^2 \\ \frac{1}{2} \ell^2 \nu_{k^2} + i\ell K | \tau_{k^2} | + 2c_iG^2 & -\frac{1}{2} \ell^2 | \tau_{k^2} | + i\ell K \nu_{k^2} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}
$$

(2.5.10)

Stability of a general periodic solution of type (2.5.1) of the Ginzburg-Landau equation is now reduced to stability of the zero-solution in the $(X,Y)$-system, a linear $2 \times 2$ matrix equation. Hence, we compute the eigenvalues of this matrix and impose the condition that
the real parts of both eigenvalues have to be negative for all $\ell$. This yields, after tedious calculations, a condition on $K^2$:

$$K^2 \leq \frac{r\tau_r \left| c_r \right| \left( \left| \tau_{k^2} \right| + c_i \nu_{k^2} \right)}{2 \left| \tau_{k^2} \right|^2 + \left| \tau_{k^2} \right|^2 c_i^2 + \frac{1}{2} \left| \tau_{k^2} \right|^2 \nu_{k^2} \left| c_r \right| c_i}$$  \hspace{1cm} (2.5.11)

For more details on the stability analysis of periodic solutions of the Ginzburg-Landau equation, we refer to Stuart and DiPrima (1978) or to Matkovsky and Volpert (1993). Expression (2.5.11) gives a band-width in which one can expect finite amplitude solutions. The condition on $K$ defines an ‘inner-parabola’ in the neutral curve in which periodic solutions are stable with respect to general perturbations. Figure 2.5 is a plot of the neutral curve with inner-parabola, for a fixed value of $\delta$ (see (2.3.11)). Note that whenever

$$\chi = \left| c_r \right| \left| \tau_{k^2} \right| + c_i \nu_{k^2} < 0$$  \hspace{1cm} (2.5.12)

(2.5.11) can not be satisfied for any $K$. This means that all periodic solutions of the type (2.5.1) are unstable. In figure 2.5 it means that the inner-parabola vanishes at $K = 0$, thus the Stokes wave is the last periodic solution to become unstable.

Figure 2.5: Neutral curve with inner parabola for the case $b = 5, C = 0.007$ and $\gamma = 1$. The dotted line is the second order approximation of the neutral curve which is represented as a solid line. The dashed line is the boundary of the stable area from nonlinear point of view.

A physical explanation of this stability criterion has been given by Lighthill (1978). Consider a slowly modulated Stokes wave of which the amplitude is described by (2.4.32). The bed forms will have their largest elevations at the top of the envelope wave. Since $c_i > 0$ in our model, the dispersion relation (2.5.3) implies that the phase velocity at the top is smaller than at either sides. Consequently, the bedforms on the downstream side
are enlarged whereas the waves at the upstream sides are shortened. A necessary condition for instabilities to occur is, that there will be an accumulation of energy at the top. Since energy is transported by the group velocity $\nu_k$, this requires in the present model $\partial|\nu_k|/\partial k > 0$ or, in other words, $\nu_k^2 < 0$. This is a necessary element of condition (2.5.12).

The computations give analytical results on the stability, i.e., existence in physical sense, of the alternate bars (periodic solutions (2.5.1)). Stability interval (2.5.11) and, maybe more important, condition (2.5.12) can be derived due to the fact that the Ginzburg-Landau theory admits spatial variation. The Landau theory can only predict the stability of an alternate bar with respect to very special perturbations (those with exactly the same wavenumber as the bar), which for instance yields that all alternate bars are stable. Hence, the non-existence of stable alternate bars (condition (2.5.12)) can not be predicted by the Landau theory. In figure 2.6 we made a plot of the expression $\chi = 0$ for $0.001 \leq C \leq 0.01$ and for $2 \leq b \leq 10$ while $\gamma = 1$. It is readily observed that the physically interesting domain for $C$ and $b$ contains combinations of the parameters where we can expect stable periodic solutions as well as combinations where the instability-criterion (2.5.12) is satisfied i.e. where the periodic solutions are not stable. From figure 2.6 we can conclude that for $b < 3$, the periodic bar pattern as predicted by the Landau theory (Colombini et al. (1987)) is stable, while for $b > 3$, i.e. when the bed is dune-covered, more complicated bed profiles may be expected, depending on the value for $C$. Thus, within the natural parameter region there is a change from existence to non-existence of stable alternate bars.

It should be noted (again), that, due to the character of evolution equation (2.4.32), there is no transient behaviour between a periodic and a non-periodic pattern in the case that

![Figure 2.6: The neutral stability curve $\chi = 0$ ($\chi$ is defined in (2.5.12)), for periodic bar patterns of the type (2.5.1) in the $b,C$-parameter space for $\gamma = 1$, in case of a continuous model. To the left of this curve, standard Landau theory yields valid results while in the other region different bed profiles are to be expected.](image-url)
instability criterion (2.5.12) is satisfied: any initial solution will immediately evolve to a non-periodic pattern, without becoming periodic first.

We will use some numerical techniques in order to study the evolution of solutions of the Ginzburg-Landau equation in the case that all periodic solutions are unstable. First, we bring equation (2.4.32) in standard form (as is done in Doelman (1991), Doering et al. (1989), Keefe (1985)). Substitution of

\[ A = D \cdot A' (\xi', \tau') e^{r \nu \tau'} \quad (2.5.13) \]

in (2.4.32) where

\[ D^2 = \frac{-r \tau_c}{c_r}, \tau' = \frac{1}{r \tau_c} \tau' \quad \text{and} \quad \xi = \sqrt{\frac{-2 r \tau_c}{\tau_k^2}} \xi' \]

yields the rescaled Ginzburg-Landau equation (primes are dropped for convenience)

\[ \frac{\partial A}{\partial \tau} = A + (1 + i \alpha_1) \frac{\partial^2 A}{\partial \xi^2} - (1 + i \alpha_2) |A|^2 A \quad (2.5.14) \]

with \( \alpha_1 = \frac{c_r}{\tau_k^2}, \alpha_2 = \frac{c_t}{c_r} \). Plots of \( \alpha_1 \) and \( \alpha_2 \) are given in figure 2.7. We can not choose \( \alpha_1 \) and \( \alpha_2 \) arbitrary, as is done in some theoretical studies. They are determined by our model parameters and it appears that we end up with pairs \( (\alpha_1, \alpha_2) \) that have not been studied in the context of the Ginzburg-Landau equation before. Because the Stokes wave is the last wave to become unstable, and we are interested in bifurcating solutions from this last stable wave, it is natural to perform a stability analysis around the Stokes wave. Repeating the above general stability analysis for the special case of the Stokes wave, which

![Figure 2.7](image)

Figure 2.7: The coefficients \( \alpha_1 \) (figure (a)) and \( \alpha_2 \) (figure (b)) of the rescaled Ginzburg-Landau equation (2.5.14), as function of \( C \) for several values of \( b \) while \( \gamma = 1 \).
now corresponds to \( \exp(-i\alpha_2\tau) \), for the rescaled equation (2.5.14), we find
\[
\frac{d}{d\tau} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -2 - \ell^2 & \ell^2\alpha_1 \\ -2\alpha_2 - \alpha_1\ell^2 & -\ell^2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}
\] (2.5.15)
the analogon of (2.5.10), and the Stokes wave is stable if perturbations of the type (2.5.5) decay exponentially, i.e., if the eigenvalues of (2.5.15) are smaller than zero. This leads to a condition on \( \ell \): the Stokes wave is stable against perturbations with wave numbers \( \ell \) satisfying
\[
|\ell| \geq \ell_c = \sqrt{-\frac{2(1 + \alpha_1\alpha_2)}{1 + \alpha_1^2}}
\] (2.5.16)
i.e. perturbations with a small wavenumber are the most unstable ones (this has already been observed by Benjamin and Feir (1967), Stuart and DiPrima (1978). Note that the Stokes wave is stable against all perturbations if \( 1 + \alpha_1\alpha_2 > 0; 1 + \alpha_1\alpha_2 < 0 \) is the rescaled version of (2.5.12).

Since we can not apply numerical methods to (2.5.14) on an unbounded domain we restrict \( \xi \) to an interval \([0, 2\pi/q]\) and consider no-flux boundary conditions (i.e. \( \frac{\partial A}{\partial \xi} = 0 \) at \( \xi = 0, 2\pi/q \)). Physically this means that we cut the infinitely straight channel into periodic parts of length \( 2\pi/q \). The boundary conditions are introduced mainly for convenience, they are no of no essential significance to our problem. We can use the spectral method also discussed in Keefe (1985) and Doelman (1991) and introduce:
\[
A(\xi, \tau) = \sum_{n=-\infty}^{\infty} Z_n(\tau) e^{inq\xi}
\] (2.5.17)
Note that \( Z_n = Z_{-n} \), which is a result of the boundary conditions, and that \( q \) corresponds directly to the \( \ell \) in the stability analysis. By decreasing \( q \) we introduce solutions (and thus perturbations) with small wave numbers \( \ell \) to the interval \([0, 2\pi/q]\). In other words, the Stokes wave is a stable solution of the Ginzburg-Landau equation on the interval \([0, 2\pi/q]\) (with no-flux boundary conditions) if \( q > q_c = l_c \) where \( l_c \) is defined in (2.5.16) (for more details, see Doelman (1991)). In figure 2.8 a contour plot is shown of \( l^2_c \) in the \((b, C)\) parameter space for \( \gamma = 1 \). From this is can be seen that for realistic combination of these parameters the critical value \( q = q_c \) ranges between 0 and 0.32. Note that \( 2\pi/q_c \) is the minimum interval length necessary for observing modulations of the Stokes wave as solutions of the rescaled Ginzburg-Landau equation (2.5.14). Using (2.4.6) and (2.5.13) we define the minimum modulation length for the unscaled Ginzburg-Landau equation (2.4.32) as:
\[
L_c = 2\pi
\frac{q_c}{q_c} \sqrt{\frac{\frac{1}{2}r^2}{\varepsilon^2 r^2}}
\] (2.5.18)
in units of channel width. Actual values for \( L_c \) strongly depend on the value of \( \varepsilon \). As we have already discussed we expect the weakly nonlinear theory to yield valid results even if \( \varepsilon \) become of order 1. Consequently, typical values for \( L_c \) range from several tens to a few hundred channel widths, which do not seem unrealistically large. In our experiments
we considered $q \geq 0.05$ such that the channel lengths were less than approximately $300/\varepsilon$ width units.

Note that any initial perturbation of the flat bed evolves on an $O(1/\varepsilon^2)$ time scale to a fully developed periodic or quasi-periodic bed profile. During this transient behaviour, the envelope amplitude propagates with an $O(1)$ group velocity $v_k$, which means that we would actually need a channel with a length of the order $1/\varepsilon^2$ and not of the order of $1/\varepsilon$ as is suggested by $L_c$ (see (2.5.18)). Therefore, a realistic indication of a minimum channel length necessary for actually observing a quasi-periodic modulation behaviour is given by $L_c/\varepsilon$. A consequence of this observation is that, if we want to detect the predicted bed profiles in laboratory or field conditions, we are obliged to choose $\varepsilon$ not too small. Otherwise, the length needed in a laboratory or the length of a straight natural river as predicted by our theory (i.e. $O(1/\varepsilon^2)$) will become unrealistically large.

By substituting (2.5.17) into (2.5.14) one analyses the stability of the Stokes wave ($Z_0(\tau) = \exp(-i\alpha_2\tau), Z_n(\tau) = 0, n \neq 0$), because the terms $\exp(i nq\xi)$ are perturbations to which the Stokes wave eventually (i.e. by sufficiently small $q$) will become unstable. Hence, it is natural to consider $q$ as a bifurcation parameter in the subsequent analysis. Performing the former substitution yields an $\infty$-dimensional system of nonlinear coupled differential
We consider (2.5.19) with $N = 3$ as an approximating system of (2.5.14) (i.e. a 4-D complex or a 8-D real system). This choice is motivated by experiments of Doelman (1991) who investigated more thoroughly system (2.5.19) and found no significant difference in the
Figure 2.9: Long-term behaviour of the bed profile (at a fixed position near the channel bank) as a function of the dimensionless time $t$. In the numerical simulations we have set $b = 6, C = 0.003, \gamma = 1, r = R_c = 26.91$ and $\varepsilon = 0.2$. (a). $q = 0.28$, which is larger than the critical value $q_c = 0.2556$. The solution of the spectral model is a Stokes wave, resulting in a regular oscillatory bed profile. (b). $q = 0.2$ which is between $q_c$ and $q_H = 0.1168$. The spectral model has a periodic solution with all components $Z_n$ have finite, non zero amplitude. As follows from (2.5.20), the behaviour of the bed is quasi periodic with two fundamental frequencies. (c). $q = 0.1 < q_H$. Solutions of the spectral model are quasi periodic with two fundamental frequencies, resulting in quasi periodic bed evolution with three fundamental frequencies.

dynamics in case truncated at $N = 3$, and in case truncated at $N = 31$, which was studied by Keefe (1985). In the numerical simulation of system (2.5.19), we fixed the morphological parameters $b, C, \gamma$ and $\varepsilon^2 r$ and computed the coefficients $\alpha_1$ and $\alpha_2$ in (2.5.14) using the asymptotic expressions presented in Appendix B. The corresponding bed profile, as given by (2.4.11), becomes

$$z_b = \varepsilon A(\xi, \tau) e^{i(k_c x + \omega_c t)} \cdot \cos(\pi y) + \ldots + c.c.$$  

$$= \varepsilon D \sum_{n=-N}^{N} \hat{Z}_n(t) e^{i(C_1_n x + C_2_n t)} \cdot \cos(\pi y) + \ldots + c.c. \quad (2.5.20)$$

where $D$ follows from (2.5.13), $\hat{Z}_n(t) \equiv Z_n(\varepsilon^2 r \tau_t)$ and

$$C_{1n} = \varepsilon n q \sqrt{\frac{\tau k^2}{-2 r \tau_r}} + k_c, \quad (2.5.21)$$

$$C_{2n} = \varepsilon^2 r^2 \tau_r \nu_r + \varepsilon n q \nu_k \sqrt{\frac{\tau k^2}{-2 r \tau_r}} - |\omega_c|. \quad (2.5.22)$$
Here, use has been made of definition (2.4.7), transformation (2.5.13) and representation (2.5.17).

Experiments with \( q > q_c = \ell_c \) (see (2.5.16)) demonstrated that in these cases, solutions of the spectral model converge to a stable Stokes wave, for which \( \hat{Z}_0(t) = e^{i\alpha_2 \varepsilon^2 r \tau t} \) and \( \hat{Z}_n(t) = 0 \) for \( n \neq 0 \). According to (2.5.20), this describes a periodic alternate bar pattern. The corresponding bed profile, at a fixed position of the bank of the channel, as a function of the dimensionless time \( t \) is shown in figure 2.9 (a). At \( q = q_c \), the Stokes wave becomes unstable, due to a pitchfork bifurcation. For slightly smaller \( q \)-values, two new periodic solutions with \( \hat{Z}_0(t) \neq 0 \) for all \( n \) are observed. From (2.5.20) it follows that in this case, the bed profile is composed of a series of travelling waves, each with their own wavenumber and frequency. It appears that all frequencies of this solution follow from integer combinations of two fundamental frequencies which are mutually irrational. Consequently, the temporal and spatial behaviour of the bar pattern is quasi-periodic, even although the solutions of the spectral model are periodic. An example of the bed evolution for such a situation at a fixed position near the bank as a function of time is shown in figure 2.9 (b). If \( q \) is further decreased, the two periodic attractors become unstable due to a Hopf bifurcation at the critical value \( q = q_H = 0.1168 \). Solutions of the spectral model are then itself quasi-periodic with two fundamental frequencies. As a consequence of (2.5.20), the bed profile becomes quasi-periodic with three independent frequencies. An example of the corresponding bed profile is shown in figure 2.9 (c). As can be seen from (2.5.20)-(2.5.22), the bed pattern is not only quasi-periodic in time, but also in space. This behaviour is visualized in figure 2.10, which is a plot of a part of the river bed at a fixed time. The non-periodic behaviour is obvious. The rôle of \( \varepsilon \) in figure 2.9 can also be seen from (2.5.20)-(2.5.22): increasing \( \varepsilon \) causes the time- and spatial scales of the periodic and quasi-periodic behaviour to become smaller, i.e. we need shorter time and smaller distance in space to detect quasi-periodic behaviour. So far, the preceding bifurcation scenario is similar as described by Doelman (1991) and Keefe (1985) who studied solutions of (2.5.19) for \( \alpha_1 = 4, \alpha_2 = -4 \). They demonstrated that for values of \( q < q_H \) chaotic solutions were encountered. Within the context of morphological models, the presence of chaotic solutions would be extremely relevant, since they have limited predictability properties. However, regions with chaotic behaviour could not be located in our simulations. One reason might be that our values for \( q_c \) are an order of magnitude smaller than those found by Doelman (1991) and Keefe (1985). Since all interesting dynamical behaviour occurs in the region \( 0 < q < q_c \), the windows with chaotic behaviour might have become so small that they are missed by our searching procedure. Alternatively, the bifurcation scenario in our case might be different, since our values for \( \alpha_1 \) and \( \alpha_2 \) are quite different, as can be seen from figure 2.7. We have not investigated these possibilities in great detail since deviations from quasi-periodic behaviour occur for small \( q \) values. As will be demonstrated in the next section, these correspond to rather long channels which are not physically realistic.

2.6 Conclusions

In this paper we have studied the nonlinear behaviour of free bars generated by a uni-directional current in an infinitely long straight channel with an erodible bottom and
non-erodible banks. Our main aim was to derive a modified modulation equation describing the amplitude behaviour of the bars and to demonstrate its potential importance for understanding the dynamics of free bars in rivers and laboratory tanks. We have chosen to consider a simple model which only contains the basic mechanisms responsible for bed

| parameter | value | calculated from | determines | value for | value for |
|-----------|-------|----------------|------------| river case | tank case |
| $k_c$     | 0.70  | (2.3.15)       | wave length| 1975 m     | 39.5 m    |
| $R_c$     | 26.97 | (2.3.16)       | channel width| 220 m     | 4.4 m     |
| $|\omega_c|$ | 0.77  | (2.3.17)       | wave period| 9 months   | 9 days    |
| $|c_f|$    | 1.11  | (2.3.15) & (2.3.17) | phase velocity| 7 m day$^{-1}$ | 4.2 m day$^{-1}$ |
| $v_k$     | 1.80  | (App. B)       | group velocity| 11 m day$^{-1}$ | 6.8 m day$^{-1}$ |
| $z_b$     | 0.86  | (2.5.4)        | bar amplitude| 4.3 m     | 8.6 cm    |
| $q_c$     | 0.26  | (2.5.16)       | critical $q$| -         | -         |
| $L_c$     | 65    | (2.5.18)       | min. modulation length| 1.43 · 10$^4$ m | 286 m |

Table 2.1: Model results for the parameter values $b = 6, C = 0.003, \gamma = 1, r = R_c, \varepsilon = 0.2$. For the river case holds: $h_* = 5m, u_* = 1ms^{-1}, T = 3 \cdot 10^6s$. For the tank case holds: $h_* = 0.01m, u_* = 0.1ms^{-1}, T = 10^5s$.

form instabilities. Therefore, we excluded effects like the variation of the drag coefficient with the local water depth (Einstein (1950)) and the dependence of the bed slope correction coefficient in the sediment transport parameterization on the bottom stress (Sekine and Parker (1992)). Finally, we have assumed straight and non-erodible channel banks. Nevertheless, our model reproduces most of the principal aspects of the bar behaviour whereas the amount of mathematical computations is minimized, and we have demonstrated that the agreement with other linear and nonlinear studies is quite satisfactory. The model allows for a basic state, representing a uniform flow over a flat bottom, which, for sufficiently large width-to-depth ratios of the channel turned out to be unstable. A perturbation analysis of the basic state showed in linear theory that alternate bars started to develop with a certain critical wavenumber $k_c$ and a certain critical frequency $|\omega_c|$. A weakly nonlinear theory, which is valid near critical conditions, has been applied in order to derive an amplitude equation for the marginally unstable bed forms. The procedure is similar to that presented by Colombini et al. (1987), who obtained a Landau equation describing the long-term behaviour of a single unstable wave. However, the situation near critical conditions is such, that a narrow spectrum of waves become unstable. Taking that into account, we obtained a Ginzburg-Landau equation describing the evolution of the envelope amplitude of the wave group. This result also includes modulations on a long spatial scale, which are due to the dispersive properties of the wave packet. It has been demonstrated that this approach leads to situations where the periodic alternate bar pattern predicted by Colombini et al. (1987) can become unstable for realistic combinations of the physical parameters. It turns out that a necessary condition for instability is that the bed is dune-covered.
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Figure 2.10: Quasi-periodic behaviour of the bed profile (at a fixed time). We have set $b = 5, C = 0.001, \gamma = 1, r = 5, \varepsilon = 0.2$ and $q < q_H$.

Information on the subsequent dynamical behaviour has been obtained by cutting the channel into periodic intervals with length $2\pi/q$ and using a spectral method. It was found that $q$ should be smaller than a critical value $l_c$, defined in (2.5.16), in order to let the Stokes solution become unstable by the side-band interaction mechanism. Physically, $q_c$ corresponds to a minimum modulation length $L_c$, required to observe an unstable pattern.

We have investigated if such lengths are realized in natural rivers or might be simulated in laboratory tanks. This was done by fixing the morphological parameters at $b = 6, C = 0.003, \gamma = 1, r = R_c$ and $\varepsilon = 0.8$. In table 2.1, results are presented for the corresponding critical width-to-depth ratio $R_c$, critical wavenumber $k_c$, frequency $|\omega_c|$, phase velocity $c_f$, group velocity $\nu_k$, characteristic amplitude $z_b$ of the bars, the bifurcation value $q_c$ and the corresponding minimum modulation length $L_c$. The results have been translated into dimensional predictions of the observed bed forms for both a typical lowland river and a laboratory tank environment. These cases have been characterized by an undisturbed water depth, the intensity of the basic uniform flow and a morphological time scale. Of course the latter is strongly dependent on the sediment properties and has been chosen rather arbitrarily. In particular the chosen value for the laboratory tank requires very fine sediment. As noted at the end of section 5, the evolution of initial perturbations of the flat bed to a fully developed bed profile takes place on a $1/\varepsilon^2$ time scale. Therefore, in order to observe the periodic or quasi-periodic profiles in reality, we need lengths of laboratory tanks or natural straight reaches of the order of $1/\varepsilon^2$ times the width of the tank or the river. If we then want to get realistic values, we must take in our theory $\varepsilon$ of the order unity. Once the bed profile is fully developed, we may detect any quasi-periodic behaviour from the spatial structure of the profile (see figure 2.10).

It appears that the bar amplitudes are typically 15-20 percent of the undisturbed wa-
ter depth. However, this result strongly depends on the selected value for the parameter ε, which measures the difference between the actual- and critical width-to-depth ratio. In Colombini et al. (1987) order 1 values for ε are chosen for which the weakly nonlinear theory still yields useful results, see the discussion in section 4. In this case the bar amplitudes become of the order of the undisturbed water depth which agrees better with observed alternate bars in rivers and laboratory experiments. In order for the rigid lid approximation to be valid, the Froude number should be smaller than approximately 0.1, as can be traced back from eq. (2.2.9)-(2.2.12). This poses rather strong conditions on the intensity of the basic flow which, especially in the case of laboratory tank experiments, are difficult to meet. Furthermore, it is found that the minimum modulation length is approximately 65 channel width units. Both for the river and the tank situation this condition might possibly be satisfied. Our experiments have shown that in these cases the bed profiles behave quasi-periodically, both in space and in time. Although the evolution seems rather complicated, see the time series in figure 2.9, the dynamics are still perfectly predictable. In principle the Ginzburg-Landau equation also allows for chaotic solutions with associated limited predictability properties (Doelman (1991), Keefe (1985)). However, they were not observed in our simulations, resulting in the conclusion that they might only occur for an extremely small region of q-values. Thus the occurrence of chaos seems not very relevant within the context of the present model.

2.6.1 Epilogue

Recently, Komarova and Newell (1995) made an interesting contribution to the analysis presented in this Chapter. In that paper, they show that the analysis presented in the previous sections is not 'complete'. Herewith they mean, that we have overlooked the effect of what they call the mean flow. These are modes with wave number zero, both in longitudinal and in transversal direction. They observe that these modes, which are also called Goldstone modes are neutrally stable for all values of the bifurcation parameter R and should thus be taken into account in the nonlinear stability analysis. The reason that this phenomenon has not been incorporated in this Chapter, is that we became aware of their paper in a rather late stage; besides, the analysis seems to contain some subtle points which have not been completely sorted out yet.

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Appendix A

The asymptotic results for α_u, α_v and α_ζ, defined in (2.3.8), read as follows:

\[
\alpha_u = \delta^{\frac{\bar{\alpha}}{2}} \left\{ 1 - 5\delta^{\frac{\bar{\alpha}}{2}} + O(\delta) \right\} + i \frac{1}{2} \sqrt{2} \delta^{\frac{\bar{\alpha}}{4}} \left\{ 1 + O(\delta) \right\}
\]

\[
\alpha_v = \delta^{\frac{\bar{\alpha}}{2}} \left\{ 1 + O(\delta) \right\} + i \frac{1}{2} \sqrt{2} \delta^{\frac{\bar{\alpha}}{4}} \left\{ 1 - \delta^{\frac{\bar{\alpha}}{2}} + O(\delta) \right\}
\]
Furtheron, we show the asymptotic results for the solutions of the equations $L_{02} \phi_{02} = f_{02}$ and $L_{22} \phi_{22} = f_{22}$ which were considered in section 2.4. In (2.4.15),

$$\phi_{02} = |A|^2 \begin{pmatrix} u_{02s} \sin^2(\pi y) + u_{02c} \cos^2(\pi y) \\ v_{02} \sin(2\pi y) \\ \zeta_{02} \cos(2\pi y) \\ z_{02} \cos(2\pi y) \end{pmatrix}$$

where

$$u_{02s} = \frac{-4 + 3b}{4(b-1)} - (3 + \frac{b(1+b)}{4(b-1)})\delta^\frac{1}{2} + O(\delta)$$

$$u_{02c} = \frac{(-4 + 3b)}{4(b-1)} + (-\frac{5}{2} + \frac{b(1+b)}{4(b-1)})\delta^\frac{1}{2} + O(\delta)$$

$$v_{02} = \delta^\frac{1}{2} + O(\delta^\frac{3}{2})$$

$$z_{02} = \frac{-b}{2(b-1)} - \frac{b(1+b)}{2(b-1)}\delta^\frac{1}{2} + O(\delta)$$

and in (2.4.17),

$$\phi_{22} = A^2 \begin{pmatrix} u_{22s} \sin^2(\pi y) + u_{22c} \cos^2(\pi y) \\ v_{22} \sin^2(\pi y) \\ \zeta_{22s} \sin^2(\pi y) + \zeta_{22c} \cos^2(\pi y) \\ z_{22s} \sin^2(\pi y) + z_{22c} \cos^2(\pi y) \end{pmatrix}$$

where

$$u_{22s} = \frac{-1}{6(b-1)} - \frac{i\sqrt{2(b-6)}}{12(b-1)}\delta^\frac{1}{2} + \frac{(21 - 24b + b^2)}{12(b-1)}\delta^\frac{1}{2} + \frac{\sqrt{2}i(47 + 29b - b^2)}{12(b-1)}\delta^\frac{3}{2} + O(\delta)$$

$$u_{22c} = \frac{1}{6(b-1)} + \frac{\sqrt{2}i(-12 + b)}{12(b-1)}\delta^\frac{1}{2} + \frac{(-45 + 33b + 2b^2)}{12(b-1)}\delta^\frac{1}{2} + \frac{\sqrt{2}i(59 - 41b^2b^2)}{12(b-1)}\delta^\frac{3}{2} + O(\delta)$$

$$v_{22} = \frac{2}{3(b-1)} + \frac{\sqrt{2}i(-8 + 5b)}{6(b-1)}\delta^\frac{1}{2} + \frac{(-29 + 20b + b^2)}{6(b-1)}\delta^\frac{1}{2} + \frac{\sqrt{2}i(-116 - 3b^2)}{24(b-1)}\delta^\frac{3}{2} + O(\delta^\frac{3}{2})$$

$$z_{22s} = \frac{\sqrt{2}i}{3(b-1)}\delta^\frac{1}{2} - \frac{b - 2}{3(b-1)} + \frac{\sqrt{2}i(-20 + 16b^2 + b^2)}{12(b-1)}\delta^\frac{1}{2} + \frac{(-62 + 53b + 5b^2)}{24(b-1)}\delta^\frac{3}{2} + O(\delta^\frac{3}{2})$$

$$z_{22c} = \frac{-i\sqrt{2}}{3(b-1)}\delta^\frac{1}{2} + \frac{b - 2}{3(b-1)} + \frac{\sqrt{2}i(20 - 22b - b^2)}{12(b-1)}\delta^\frac{1}{2} + \frac{38 - 59b + b^2}{24(b-1)}\delta^\frac{1}{2} + O(\delta^\frac{3}{2})$$

Note that the expressions for $\zeta_{02c}$ and $\zeta_{22s}$ and $\zeta_{22c}$ are omitted. It appeared that they where not necessary for the derivation of the nonlinear term of the Ginzburg-Landau equation. This is due to the assumption that the Froude number is small, which physically means that there are no dynamical effects due to variations in the water surface.
Appendix B

In this Appendix we shall specify the expressions that appear in the inhomogeneous term \( \tilde{g} \) (2.4.29) as used in the weakly nonlinear analysis. For convenience we write \( k \) and \( R \) where we mean \( k_c \) and \( R_c \). First, we define the following quantities:

\[
\begin{align*}
 f_{13}^u &= f_{13,1}^u + f_{13,2}^u \cos(\pi y) + |A|^2 A \left[ f_{13,3}^u \sin^2(\pi y) \cos(\pi y) + f_{13,4}^u \cos^3(\pi y) \right] \\
 f_{13}^v &= f_{13,1}^v + f_{13,2}^v \sin(\pi y) + |A|^2 A \left[ f_{13,3}^v \cos^2(\pi y) \sin(\pi y) + f_{13,4}^v \sin^3(\pi y) \right] \\
 f_{13}^\zeta &= f_{13,1}^\zeta + f_{13,2}^\zeta \cos(\pi y) + |A|^2 A \left[ f_{13,3}^\zeta \sin^2(\pi y) \cos(\pi y) + f_{13,4}^\zeta \cos^3(\pi y) \right] \\
 f_{13}^z &= f_{13,1}^z + f_{13,2}^z \cos(\pi y) + |A|^2 A \left[ f_{13,3}^z \sin^2(\pi y) \cos(\pi y) + f_{13,4}^z \cos^3(\pi y) \right]
\end{align*}
\]

where the first two terms in the right hand sides are linear in \( \phi_1 \) and \( \phi_2 \) (they follow from the theory developed in section 2.4 and are thus not concerned here) and where

\[
\begin{align*}
 f_{13,3}^u &= -\left( \frac{1}{2} \alpha_v^2 + \alpha_v \bar{\alpha}_v \right) CR - i k \left( \alpha_u u_{02s} + \bar{\alpha}_u u_{22s} \right) \nonumber \\
 &\quad - 2\pi \left[ \alpha_v (-u_{02c} + u_{02s}) + \bar{\alpha}_v (-u_{22c} + u_{22s}) \right] + 2\pi \left( \alpha_u v_{02} + \bar{\alpha}_u v_{22} \right) \\
 &\quad - 2CR \left[ \alpha_v v_{02} + \bar{\alpha}_v v_{22} + (1 + \alpha_u) (u_{02s} - z_{02}) + (1 + \bar{\alpha}_u) (u_{22s} + z_{22s}) \right] \\
 f_{13,3}^v &= -i k \left( \alpha_u u_{02c} - \bar{\alpha}_u u_{22c} + 4\alpha_v v_{22} \right) \nonumber \\
 &\quad - 2CR \left[ (1 + \alpha_u) v_{02} + (1 + \bar{\alpha}_u) v_{22} \right] - 4\pi \left( \alpha_u v_{02} + \bar{\alpha}_v v_{22} \right) \\
 &\quad - \alpha_v CR \left[ 2 + \alpha_u + \bar{\alpha}_u + u_{02c} + z_{02} \right] - \bar{\alpha}_v CR \left[ 1 + \alpha_u + u_{22c} + z_{22c} \right] \\
 f_{13,3}^\zeta &= -i k \left( \alpha_u u_{02s} - \bar{\alpha}_u u_{22s} + \frac{1}{2} \alpha_v^2 \alpha_v \right) + 2\pi \left( \alpha_v v_{02} + \bar{\alpha}_v v_{22} \right) \nonumber \\
 &\quad - CR \left[ \frac{1}{2} \alpha_v^2 \alpha_v + \alpha_v \left( u_{02s} - z_{02} \right) + \bar{\alpha}_v \left( u_{22s} + z_{22s} \right) \right] \\
 f_{13,3}^z &= -4\pi \left( v_{02} + v_{22} \right) - 5\pi \alpha_v z_{02} + \bar{\alpha}_v \left( -2z_{22c} + 3z_{22s} \right) + \\
 &\quad i k \left( u_{02s} + u_{22s} - \alpha_u z_{02} + \bar{\alpha}_u z_{22s} \right) \nonumber \\
 f_{13,3}^\zeta &= i k \left( u_{02c} + u_{22c} + \alpha_u z_{02} + \bar{\alpha}_u z_{22c} \right) + \pi \left( 2 \left( v_{02} + v_{22} \right) + \alpha_v z_{02} + \bar{\alpha}_v z_{22c} \right)
\end{align*}
\]
\[ f_{13,3}^z = \pi (b - 1) \left[ -\frac{9}{2} \alpha_v^2 \alpha_v + (b - 2) \right] \left( 2 \alpha_u \alpha_u \alpha_v + \alpha_v^2 \alpha_v \right) \]
\[ + \pi (b - 1) \left[ \alpha_v (2u_{02c} - 3u_{02s}) + \alpha_v (2u_{22c} - 3u_{22s}) + 4 (\alpha_u v_{02} + \alpha_u v_{22}) \right] \]
\[ - (b - 1) ik \left[ \frac{1}{2} (b - 2) \left( \alpha_u \alpha_v \alpha_v + 2 \alpha_u \alpha_v \alpha_v \right) + b (\alpha_u u_{02s} + \alpha_u u_{22s}) + 2 (\alpha_u v_{02} + \alpha_u v_{22}) \right] \]
\[ - \frac{\gamma}{R} b k^2 \left( - \frac{1}{2} \alpha_u + \alpha_v \alpha_v + \alpha_v (u_{02s} - u_{22s} + 2 \alpha_v z_{22s}) \right) \]
\[ - 3 \pi^2 b \frac{\gamma}{R} \left( \frac{1}{2} \alpha_v^2 + \alpha_v \alpha_v + u_{02s} + u_{22s} \right) \]
\[ + 2 \pi^2 b \frac{\gamma}{R} \left[ \frac{1}{2} \alpha_u^2 (b - 1) + \alpha_u \alpha_v (b - 1) + u_{02s} + u_{22s} + 4 \alpha_u z_{02} - 2 \alpha_u (-z_{22c} + z_{22s}) \right] \]

\[ f_{13,4}^z = -\frac{1}{2} \pi (b - 1) (b - 2) \left( 2 \alpha_u \alpha_u \alpha_v + \alpha_v^2 \alpha_v \right) \]
\[ - b (b - 1) ik \left[ \frac{1}{2} \alpha_u^2 (b - 2) + \alpha_u u_{02s} + \alpha_u u_{22c} \right] \]
\[ - \pi (b - 1) \left[ \alpha_v u_{02s} + \alpha_v u_{22c} + 2 (\alpha_u v_{02} + \alpha_u v_{22}) \right] \]
\[ - \frac{\gamma}{R} b k^2 \left[ - \frac{1}{2} \alpha_u^2 (b - 1) + \alpha_u \alpha_v (b - 1) + u_{02s} - u_{22c} + 2 \alpha_u z_{22c} \right] \]
\[ - \pi^2 b \frac{\gamma}{R} \left[ \frac{1}{2} \alpha_u^2 (b - 1) + \alpha_u \alpha_v (b - 1) + u_{02s} + u_{22c} + 4 \alpha_u z_{02} - 2 \alpha_u (-z_{22c} + z_{22s}) \right] \]

We shall now specify \( g_i, i = 1...4 \) as they appear in (2.4.29). They read as follows:

\[ g_1 = -k^2 (ik + CR) \left[ f_{13,1}^z - b f_{13,1}^\zeta \right] \]
\[ g_2 = -k^2 (ik + CR) \left[ f_{13,2}^z - b f_{13,2}^\zeta \right] - \pi^2 ((ik + 2CR) \left[ f_{13,2}^z - b f_{13,2}^\zeta \right] \]
\[ + \pi^2 ik (b - 1) f_{13,2}^u - \pi (b - 1) k^2 f_{13,2}^v \]
\[ g_3 = -7 \pi^2 \left( -f_{13,3}^z + f_{13,3}^\zeta \right) (2CR + ik) + 6 \pi^2 \left( -f_{13,4}^z + f_{13,4}^\zeta \right) (2CR + ik) \]
\[ + 7 \pi^2 k (b - 1) f_{13,3}^u - 6 \pi^2 k (b - 1) f_{13,4}^v + 2 \pi (b - 1) f_{13,3}^k - 3 \pi (b - 1) f_{13,4}^k \]
\[ - (CR + ik) k^2 \left( f_{13,3}^z - b f_{13,3}^\zeta \right) \]
\[ g_4 = 2 \pi^2 \left( -f_{13,3}^z + f_{13,3}^\zeta \right) (2CR + ik) - 3 \pi^2 \left( -f_{13,4}^z + f_{13,4}^\zeta \right) (2CR + ik) \]
\[ - 2 \pi^2 (b - 1) f_{13,3}^u + 3 \pi^2 (b - 1) f_{13,4}^u - \pi (b - 1) f_{13,3}^v \]
\[ - (CR + ik) k^2 \left( f_{13,4}^* - b f_{13,4}^* \right) \]

Working out the solvability condition (2.4.30) (thereby using the terms in \( f_{13} \) that are linear in \( \phi_{11} \) and \( \phi_{12} \)) results in the Ginzburg Landau equation:

\[
\frac{\partial A}{\partial \tau} = r \frac{\partial \omega}{\partial R} A - \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2} \frac{\partial^2 A}{\partial k^2} - \frac{1}{4} \left[ \frac{g_3 + 3g_4}{k^2(ik + CR) + \pi^2(ik + 2CR)} \right] |A|^2 A
\]

\[
= r(\tau_r + i\nu_r)A - \frac{1}{2}(\tau_{k^2} + i\nu_{k^2}) \frac{\partial^2 A}{\partial k^2} + (c_r + ic_i)|A|^2 A
\]

where

\[
\tau_r = \frac{2\beta \delta^2}{\gamma} \left( 1 - 4\delta^\frac{1}{2} + O(\delta) \right)
\]

\[
\nu_r = \frac{2\sqrt{2}\beta \delta^\frac{1}{2}}{\gamma} \left( 1 - 3\delta^\frac{1}{2} + O(\delta) \right)
\]

\[
\tau_{k^2} = -\frac{8\beta \delta^\frac{1}{2}}{\pi} \left( 1 + 2\delta^\frac{1}{2} + O(\delta) \right)
\]

\[
\nu_{k^2} = -\frac{5\sqrt{2}\beta \delta^\frac{1}{2}}{\pi} \left( 1 - 6\delta^\frac{1}{2} + O(\delta) \right)
\]

\[
c_i = \frac{(20 - 30b + 9b^2)\sqrt{2}\pi \delta^\frac{3}{2} + (-162 + 294b - 133b^2 + 15b^3)\sqrt{2}\pi \delta^{\frac{3}{2}}}{24(b - 1)} + O(\delta^\frac{5}{2})
\]

\[
c_r = \frac{2\pi}{3(b - 1)} + \frac{(62 - 78b + 19b^2 - 9b^3)\pi \delta^{\frac{3}{2}}}{12(b - 1)} + O(\delta)
\]

The expression for the group velocity reads:

\[
\nu_k = - \left\{ 1 + 7(b - 1)\delta^\frac{1}{2} - 18(b - 1)\delta + O(\delta^\frac{1}{2}) \right\}
\]
Chapter 3

On the Derivation of a Model Problem for Curved Channels

3.1 Introduction

In the previous Chapter we have studied the stability of a flat channel-bed under the influence of a uniform flow, using a simple morphological model which described the most essential features only. The analysis was restricted to straight, rectangular-shaped channels. The result of the analysis was that under certain conditions, the initial flat bed does not remain flat, but becomes unstable with respect to space- and time periodic perturbations. In the subsequent nonlinear analysis we determined an amplitude equation and analysed its solutions. The solutions described the time- and spatial evolution of the amplitude of the bottom configuration, the flow field and the free surface.

A natural next step in the study of this kind of morphological problems would be an analogon of the previous analysis, but then applied to curved channels (where the cross section of the channel remains rectangular). This would be a second step in the direction of gaining understanding of the complex problem of river meandering. With respect to straight channels, there are a few, but important differences.

The first remark has a mathematical character and is related to the change of geometry of the problem. The boundaries of the channel vary in space and this causes complications with respect to the mathematical description of the model and the domain on which the model is defined. A way to handle this problem is to introduce a coordinate transformation in order to map the curved (physical) domain into a straight (computational) domain. The calculations are performed in the computational domain, and the results must be interpreted in the physical domain.

The second remark has a physical character. It is well known that the flow field in curved channels is essentially three dimensional and can therefore not be approximated by means of the shallow water equations (which are two-dimensional equations). In curved channels, there is a very important 3D-effect present which is called secondary flow and which can obviously not be covered by a 2D-model. This phenomenon is caused by the following mechanism. Due to the curvature of the channel the centrifugal forces acting on the flow
cause an initial inclination of the free surface. This results in a pressure gradient, leading to a redistribution of water over the transversal direction. Together with the longitudinal flow this results in a spiral motion, known as secondary flow. This flow is directed ‘inwards’ at the bottom and ‘outwards’ at the free surface. It should be noted that this secondary flow has an effect on the main flow itself and on the direction of the sediment transport. The former is of minor importance (see Kalkwijk and de Vriend (1980)) and will be neglected, while the latter effect is quite important (see de Vriend (1981), Kalkwijk and Booij (1986)) and results in an accumulation of sediment in the inner bend and in aggregation in the outer bend. The resulting (stationary) bed pattern is known as point bars. We emphasize that the pattern of point bars can not be achieved without introducing the secondary flow. The point bars should be distinguished from the free bars, which are a result of an instability mechanism, whereas the point bars are forced by the curvature of the channel. It has been shown by de Vriend (1981) that the main effect of the secondary flow can be taken into account by adding a correction term in the transverse and longitudinal component of the sediment transport terms. Finally, it should be noted that in straight channels, the secondary flow is also present (it is generated by the free bars) but is is so weak that it may be neglected (see Parker and Johannesson (1989)). In the next sections, we will derive a model which is suitable for describing the point bars.

3.2 The model for curved channels

As indicated in the previous section, the model for the flow-bed interactions in curved channels can be derived from the shallow water equations, taking into account some extra terms which take care of the effects caused by the secondary flow. The model itself, together with some considerations about scaling properties, has been discussed in the previous Chapter, and reads in dimensionless and scaled coordinates:

\[
\begin{align*}
(U \cdot \nabla) U + \nabla \zeta &= \mathcal{F} \\
\nabla \cdot (U (1 - z)) &= 0 \\
\frac{\partial z}{\partial t} + \nabla \cdot S &= 0
\end{align*}
\]

where

\[
\mathcal{F} = -CR \left( -1 + \frac{U |U|}{1 - z} \right), \quad R = \frac{y_0}{h_0}, \quad S = (S_x, S_y), \quad U = (u, v)
\]

The variables (which are functions of $t, x$, and $y$) are the same as those used in the previous Chapter. Because we consider a spatially periodic geometry, we take $x \in \left[g_1(x), g_2(x)\right]$, where $g_1$ and $g_2$ define the walls of the channel; they will be specified later on. In the model $U$ denotes the velocity in the $x$ and $y$ direction, $\zeta$ denotes the water level, $z$ the bed level and $S$ the sediment flux. The variables have been made dimensionless with $h_0$ and $y_0$, the unperturbed depth and width of the channel. The boundary conditions are the same as in the case of a straight channel and are given by the fact the the side walls are impermeable for water and sediment, i.e. $U \cdot n = 0$ and $S \cdot n = 0$ where $n$ (which is a function of $x$) is the normal vector of $g_i$ in $(x, g_i(x))$. 
3.2 The model for curved channels

The domain on which the problem is considered is defined as follows. We prescribe a centerline \( c(x) \) of the channel which may in principle be arbitrary. On this centerline, we ‘construct’ straight lines which are locally orthogonal to the centerline (see figure 3.1). The curvature of the centerline must be chosen such that the lines orthogonal to the centerline do not intersect \textit{within the domain}. We assume that the \( c \) is given by \( c(x) = \nu \sin(px) \) where \( \nu \ll 1 \). This assumption is not essential for the forthcoming theory. It keeps however the computations limited. The parameter \( p \) denotes the wave length of the channel. Although it seems reasonable that \( p \) equals in physical situations approximately the wave number of the free bars, we do not assume this a priori but keep \( p \) arbitrary (see also the Remark at the end of section 3.4). Furthermore, we assume that the channel has uniform width. The fact that \( \nu \ll 1 \) means that we study channels with slightly varying geometry.

The aim of this Chapter will be twofold. Firstly, we want to determine a (stationary) solution of permanent form for the curved channel. This will be the analogon of the basic solution representing a uniform flow and a flat bed in case of a straight channel. Secondly, we want to determine the stability properties of this solution. The first goal will be achieved, the second one turns out to be very complicated for this general setting. Therefore, we reformulate the problem such that we arrive at a model problem which contains the essential features of the original problem and for which the calculations, needed for the stability analysis, are easier to perform. The derivation of this model problem will be discussed in section 3.5.

In order to find a basic solution for the model, it is convenient to map the physical, curved domain into a computational, rectangular domain. Equations (3.2.1)-(3.2.3) are given in the physical domain, i.e. in \((x,y)\)-coordinates. In the computational domain, we consider \((s,n)\)-coordinates. They are related by the following nonlinear map \( T : \mathbb{R}^2 \to \mathbb{R}^2 \):

\[
T : \begin{pmatrix} x(s,n) \\ y(s,n) \end{pmatrix} = \begin{pmatrix} \frac{-\nu p \cos(ps)}{\sqrt{1 + \nu^2 p^2 \cos^2(ps)}} n + s \\ \frac{1}{\sqrt{1 + \nu^2 p^2 \cos^2(ps)}} n + \nu \sin(ps) \end{pmatrix} \tag{3.2.5}
\]

This kind of maps are widely used in numerical treatment of problems, defined on ‘non-smooth’ domains. In the subsequent analysis, we make use of the following quantities (which can all be expanded in orders of \( \nu \)):

\[
e_s = \left( \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s} \right) \tag{3.2.6}
\]

\[
e_n = \left( \frac{\partial x}{\partial n}, \frac{\partial y}{\partial n} \right) \tag{3.2.7}
\]

\[
\frac{1}{h_1} = \sqrt{(\frac{\partial x}{\partial s})^2 + (\frac{\partial y}{\partial s})^2} = 1 - \nu np^2 \sin(ps) + O(\nu^2) \tag{3.2.8}
\]

\[
\frac{1}{h_2} = \sqrt{(\frac{\partial x}{\partial n})^2 + (\frac{\partial y}{\partial n})^2} = 1 \tag{3.2.9}
\]

The quantities \( e_s \) and \( e_n \) can be regarded as base vectors of the curved coordinate system and we observe that \( 1/h_1 = |e_s|, 1/h_2 = |e_n| \). Derivatives with respect to \( x \) and \( y \) are
The physical domain in \((x, y)\)-coordinates

The computational domain in \((s, n)\)-coordinates

Figure 3.1: The physical and computational domains. Map \(T\) transforms the computational domain into the physical. On the physical domain, variables are defined in terms of \(x\) and \(y\) coordinates, while on the computational domain, variables are defined in terms of \(s\) and \(n\) coordinates.

Transformed into derivatives with respect to \(s\) and \(n\) by the following expression:

\[
\left( \frac{\partial}{\partial x} \right) = \frac{1}{\mathcal{J}} \left( \frac{\partial y}{\partial s} \frac{\partial y}{\partial n} - \frac{\partial y}{\partial x} \frac{\partial x}{\partial n} \right) \left( \frac{\partial}{\partial s} \right),
\]

where the Jacobian \(\mathcal{J}\) is given by

\[
\mathcal{J} = \frac{\partial x}{\partial s} \frac{\partial y}{\partial n} - \frac{\partial y}{\partial s} \frac{\partial x}{\partial n}.
\]

The curvature of the coordinate lines in the physical space is denoted by \(1/R_s\) and \(1/R_n\), which are defined by

\[
\frac{1}{R_s} = h_1 h_2 \frac{\partial (1/h_1)}{\partial n} = -\nu p^2 \sin(ps) + O(\nu^2)
\]

\[
\frac{1}{R_n} = h_1 h_2 \frac{\partial (1/h_2)}{\partial s} = 0
\]

The fact that \(1/R_n = 0\) corresponds of course with the fact that lines ‘\(s\) is constant’ are mapped (by \(T^{-1}\)) onto straight lines, while lines ‘\(n\) is constant’ are mapped onto curved lines \((n = 0\) is mapped onto \(y(s, 0) = \nu \sin(ps)\), while \(n \neq 0\) is mapped onto \(y(s, n) = \nu \sin(ps) + O(\nu)\)). At this moment, we can also specify \(g_1\) and \(g_2\), the boundaries of the channel: they are parametrized through \(s\) and can be described by \(g_1(s) = (x(s, 1), y(s, 1))\) and \(g_2(s) = (x(s, -1), y(s, -1))\) (see (3.2.5)). We mention again that the assumption concerning the centerline of the channel is not essential. The analysis can be done for an arbitrary centerline \(c(x)\). In that case \(T\) reads

\[
T : \begin{pmatrix} x(s, n) \\ y(s, n) \end{pmatrix} = \begin{pmatrix} \frac{-c'(s)}{\sqrt{1+c'(s)^2}} n + s \\ \frac{\sqrt{1+c'(s)^2}}{n + c(s)} \end{pmatrix}
\]
and all other quantities can be derived in terms of (derivatives of) \( c \). The fact that we use (3.2.5) in the sequel is only because we want to illustrate some ideas which become unnecessarily complicated if we use a general \( c \). Using \( T \) given in (3.2.5), the model becomes (in \((s,n)\)-coordinates):

\[
\begin{align*}
&h_1 u \frac{\partial u}{\partial s} + h_2 v \frac{\partial u}{\partial n} + \frac{uu}{R_s} + h_1 \frac{\partial \zeta}{\partial s} + CRW_1 = 0 \\
&h_1 u \frac{\partial v}{\partial s} + h_2 v \frac{\partial v}{\partial n} - \frac{u^2}{R_s} + h_1 \frac{\partial \zeta}{\partial n} + CRW_2 = 0 \\
&h_1 \frac{\partial}{\partial s} ((1-z)u) + h_2 \frac{\partial}{\partial n} ((1-z)v) + \frac{v(1-z)}{R_s} = 0 \\
&\frac{\partial z}{\partial t} + h_1 \frac{\partial S_s}{\partial s} + h_2 \frac{\partial S_n}{\partial n} + \frac{S_n}{R_s} = 0
\end{align*}
\]

where \( h_1 \) and \( h_2 \) depend on \( \nu \) and can be derived from (3.2.8) and (3.2.9) and where \( W_1, W_2, S_s \) and \( S_n \) are the transformed friction terms and sediment flux. For details about this transformation, see Mosselman (1991) and the references cited there. The variables now depend on \( t, s \) and \( n \) with \( s \in \mathbb{R} \) and \( n \in [-1,1] \). If we recall that the system of equations, describing the dynamics in the straight channel (see (2.2.15)-(2.2.17) can be described as

\[
\frac{\partial}{\partial t} S\Phi = L_R\Phi + N(\Phi)
\]

for \( \Phi = \Phi(x,y) \), with linear operators \( S \) and \( L_R \) and nonlinear operator \( N \), where \( L_R \) depends on a control parameter \( R \), we observe that, using the expansions in \( \nu \) of \( h_1, h_2 \) and \( 1/R_s \), the system (3.2.15)-(3.2.18) can be written as

\[
\frac{\partial}{\partial t} S\Phi = L_R\Phi + N(\Phi) + \nu g(s) (L_1\Phi + N_1(\Phi)) + O(\nu^2)
\]

where \( \nu g(s) \) corresponds with the centerline and where now \( \Phi = \Phi(s,n) \) i.e. it has become a function of \( s \) and \( n \) instead of \( x \) and \( y \). Thus, the slightly varying geometry causes an \( O(\nu) \)-perturbation of the model. We come back on this in section 3.5. Note that \( \frac{\partial}{\partial s} \frac{\partial}{\partial n} + \frac{\partial}{\partial n} \frac{\partial}{\partial s} = 0 \). This is to say that \( T \) is analytical and as a result, the angles between coordinate lines in the \((x,y)\)-plane are preserved in the \((s,n)\)-plane. In other words, if we choose a physical domain in which the coordinate lines are orthogonal, and apply \( T \), we get a computational domain in which the coordinate lines are also orthogonal. In numerical analysis, this kind of maps are often called \textit{conformal}. It is a nice property, because it avoids the appearance of terms in the transformed equations which contain the grid-skewness. Another advantage of this property is, that the boundary conditions are easy to transform, because the normal component of the sediment transport in the \((x,y)\)-plane is transformed to the normal component in the \((s,n)\)-plane. Therefore, the boundary conditions in the \((s,n)\)-plane are \( v = 0 \) and \( S_n = 0 \) for \( n = \pm 1 \); they follow directly from the boundary conditions in the \((x,y)\)-plane.

The sediment transport terms play an important rôle and need to be considered in some more detail. As already remarked, we must take into account the secondary flow, and because our model is still two-dimensional, we can only introduce the \textit{consequences} of this
three dimensional phenomenon and not the phenomenon itself (to do the latter, we must start with the full three dimensional equations). The consequences are introduced on a rather artificial way, which however has been proved to work in practical situations (see Kalkwijk and de Vriend (1980), de Vriend (1981), Struiksma et al. (1985)). We introduce an extra coefficient $A$ in the transport terms, which measures the intensity of the secondary flow, i.e. we model the sediment transport as:

$$S_n = |U|^b \left( \frac{v}{|U|} - \frac{A(1-z)u}{R|U|}(\nu p^2 \sin(ps) - \frac{h_1}{u} \frac{\partial v}{\partial s}) - \frac{\gamma}{R} \frac{\partial z}{\partial n} \right)$$  \hspace{1cm} (3.2.21)

$$S_s = |U|^b \left( \frac{u}{|U|} + \frac{A(1-z)v}{R|U|}(\nu p^2 \sin(ps) - \frac{h_1}{u} \frac{\partial v}{\partial s}) - \frac{\gamma}{R} \frac{\partial z}{\partial s} \right)$$  \hspace{1cm} (3.2.22)

For $A = 0$, we recover the usual sediment transport terms: sediment is moved in the direction of the depth-averaged velocity, corrected for gravitational effects. If $A \neq 0$, the extra terms in $S_n$ and $S_s$ will have as result that the direction of sediment transport changes and will be directed towards the inner bend. For a physical motivation of this correction term, we refer to Struiksma et al. (1985), where also an expression in terms of physical quantities for $A$ is given. Following that approach, it means that $A \approx 10$. Another aspect of the secondary flow is that its intensity develops slowly, starting at the entrance of the bend and reaching its maximum after some time. This effect has been neglected in the present approach. As a last remark of this section, we note that the boundary condition $S_n = 0$ can be rewritten as a boundary condition on $z$: $\partial z/\partial n = -(A/\gamma)(1-z)\nu p^2 \sin(ps)$.

### 3.3 The basic solution

For the derivation of the basic solution of the model presented in the previous section, we take advantage of the small parameter $\nu$, denoting the amplitude of the curved channel. We expand the solution (and all other relevant quantities, such as $h_1, h_2, 1/R_s$ etc.) in a power series in $\nu$ and write for $\Phi = (u, v, \zeta, z)$:

$$\Phi = \phi_0 + \nu \phi_1 + ...$$  \hspace{1cm} (3.3.1)

with $\phi_0 = (u_0, v_0, \zeta_0, z_0)$ etc. and substitute this solution in the model (3.2.15)-(3.2.18). The first order term is easily found to be $(1, 0, 0, 0)$ and represents uniform flow in longitudinal direction over a flat bed with flat water level. Note that this solution satisfies the $O(1)$-part of the boundary conditions which are $v_0 = 0, \partial z_0/\partial n = 0$ for $n = \pm 1$. Considering the $O(\nu)$-terms leads to an equation which can be written as

$$L\phi_1 = J$$  \hspace{1cm} (3.3.2)

where $L$ and $J$ are given by

$$L = \begin{pmatrix}
\frac{\partial}{\partial s} + 2CR & 0 & \frac{\partial}{\partial s} & CR \\
0 & \frac{\partial}{\partial s} + CR & \frac{\partial}{\partial s} & 0 \\
b \frac{\partial}{\partial s} & \frac{\partial}{\partial n} + A \frac{\partial^2}{\partial s^2} & 0 & -\gamma \frac{\partial}{\partial s} \\
\frac{\partial}{\partial s} & \frac{\partial}{\partial n} + A \frac{\partial^2}{\partial s^2} & 0 & -\gamma \frac{\partial}{\partial s} + \frac{\partial^2}{\partial n^2}
\end{pmatrix}, \quad J = \begin{pmatrix} 0 \\
-p^2 \sin(ps) \\
0 \\
0
\end{pmatrix}$$  \hspace{1cm} (3.3.3)

Note that $L$ is almost the same operator as encountered in Chapter 2 for $\omega \equiv 0$ (the only difference is the term containing $A$). The inhomogeneous term $J$ is of course the result...
The basic solution of the fact that the channel is curved. System (3.3.2) can easily be solved as follows. We write \( \phi_1 = \phi_h + \phi_p \) where \( \phi_h \) is the solution of the homogeneous problem and \( \phi_p \) some special solution. It is easy to see that we can write \( \phi_p = (0, f(s), 0, 0) \) and we derive a differential equation for \( f \):

\[
\left( \frac{d}{ds} + CR \right) f(s) = -p^2 \sin(ps)
\]

with solution:

\[
\begin{align*}
 f(s) &= P \sin(ps) + Q \cos(ps) \\
 P &= \frac{-CRp^2}{p^2 + C^2R^2} \\
 Q &= \frac{p^3}{p^2 + C^2R^2}
\end{align*}
\]

What remains is to solve the homogeneous part of (3.3.2). The solution of the homogeneous part has the form \( \phi_h = \phi(n)e^{iks} \) and represents a periodic pattern in the longitudinal direction with wave number \( k \) and with a yet unknown transversal structure. At this point, the wave number \( k \) is arbitrary but we will see later that the boundary conditions will fix \( k \) to \( p \). We substitute \( \phi_h \) in (3.3.2) with \( J \equiv 0 \) and rewrite the set of ordinary differential equations for \( \phi \) by eliminating \( u \) (which depends only algebraically on \( \zeta \) and \( z \)) and introducing \( z_1 = \partial z/\partial n \). Thus, we rewrite the second order system for \( \phi = (u, v, \zeta, z) \) to a first order system for \( \hat{\phi} = (\zeta, v, z, z_1) \) which reads:

\[
\frac{d\hat{\phi}}{dn} = M \hat{\phi}
\]

with

\[
M = \begin{pmatrix}
0 & -(ik + CR) & 0 & 0 \\
\frac{k^2}{\alpha_1} & 0 & ik(1 + \frac{CR}{\alpha_1}) & 0 \\
0 & 0 & 0 & 1 \\
\alpha_2 & 0 & \alpha_3 & 0
\end{pmatrix}
\]

where

\[
\begin{align*}
\alpha_1 &= ik + 2CR \\
\alpha_2 &= \frac{k^2(-iAk + R(b - 1))}{\gamma(ik + 2CR)} \\
\alpha_3 &= \frac{k(k^2(A - \gamma) - ikR(3AC + 2C\gamma + 1) + CR^2(b - 3))}{\gamma(2iCR - k)}
\end{align*}
\]

The solution of this linear problem is then given by

\[
\hat{\phi} = \sum_{i=1}^{4} C_i e^{\lambda_i n} V_i
\]

where \( \lambda_i \) are the eigenvalues of \( M \) and \( V_i \) are the corresponding eigenvectors. The constants \( C_i \) are determined by the \( O(\nu) \)-part of the boundary conditions which read: \( v = 0, z_1 = \)
\[ \frac{dz}{dn} = -(A/\gamma \sin(ps) \) for \( n = \pm 1. \]

Let us now consider symbolically the \( v \) and the \( z_1 \)-component of \( \phi_1 \):

\[ v = v(n)e^{iks} + De^{ips} + c.c. \quad (3.3.14) \]
\[ z_1 = z_1(n)e^{iks} + c.c \] where \( D = \frac{Q + iP}{2} \) for \( P \) and \( Q \), see (3.3.6), (3.3.7) \( (3.3.16) \)

The boundary condition \( v = 0 \) for \( n = \pm 1 \) forces us to choose \( k = p \) (as can be seen from (3.3.14)) and a further application of the boundary conditions results in four equations for the constants \( C_i \) which can easily be solved. Putting things together, we have solved the problem of finding a stationary solution of permanent form for the case of a curved channel. Actual calculations can be performed for any combination of physical relevant parameters. All we have to do is calculate the eigenvalues and eigenvectors of \( M \) (where we must take \( k \) to be equal to \( p \), as a result of the boundary conditions) and solve a linear system of four equations for \( C_i \). It is important to realize however, that the only physical information that we can extract from this solution is insight in the bed pattern. We can draw no conclusions with respect to the flow field, for the obvious reason that the flow field in this model is still two dimensional, where it should be considered as three dimensional for realistic situations. The consequences of the three dimensional flow field on the bottom topography have however been taken into account. We can then continue this process and derive the basic solution up to arbitrary order in \( \nu \). In figure 3.2 we have plotted an example of the resulting bed topography of a curved channel. We note that the point bars are indeed located in the inner bend (slightly shifted) and that there exist small rills near the point bars (at the side walls). Furthermore, we see that the topography in an inner bend differs from the topography in the subsequent (opposite) inner bend.

### 3.4 Stability of the basic solution: Some considerations

Having derived the basic solution (up to some order in \( \nu \)), we want to study the (linear) stability properties of this solution with respect to small perturbations, periodic in time (with (complex) wave number \( \omega \)) and space (with wave number \( k \)). We expect that this will lead, analogous to the theory presented in the previous Chapter, to a neutral stability curve, dividing the \( (k,R) \) space into a part where the basic solution is (linear) stable and a part where it is (linear) unstable. It is to be expected that the ‘new’ stability curve is almost everywhere in an \( O(\nu) \) neighbourhood of the ‘old’ one. However, in the neighbourhood of points of the neutral stability curve where both \( \text{Re}(\omega(k,R)) = 0 \) and \( \text{Im}(\omega(k,R)) = 0 \) there occur degenerations and consequently, one might find an \( O(1) \) correction. A little bit further in this section, we come back on this. To study the non-degenerate situation, we expand the (complex) wave number \( \omega \) in the small parameter \( \nu \) as \( \omega = \omega_0 + \nu\omega_1 + \ldots \). We are as usual interested in the real part of \( \omega \) which denotes whether the basic solution grows or decays in time. By straightforward substitution and collecting equal powers of \( \nu \), we can derive expressions for \( \omega_0, \omega_1 \) etc. To study this situation, we introduce the following
3.4 Stability of the basic solution ...

Figure 3.2: An example of the top- and side view of the bed topography in a curved channel

notation:

\[
\begin{align*}
Re(\omega_0(k, R)) &= H(k, R) & (3.4.1) \\
Re(\omega_1(k, R)) &= G(k, R) & (3.4.2) \\
H(k, R) &= 0 \text{ on } R = h(k) & (3.4.3)
\end{align*}
\]

The perturbation analysis suggests for \( \nu \neq 0 \) the following:

\[
H(k, R) + \nu G(k, R) = 0 \text{ on } R = h(k) + \nu g(k) \quad (3.4.4)
\]

and we want to derive an expression for \( g \) which is the correction on the unperturbed neutral stability curve. Thus, we expand \( H(k, R) + \nu G(k, R) \) around \( R = h(k) \):

\[
\begin{align*}
0 &= H(k, R) + \nu G(k, R) & (3.4.5) \\
&= H(k, h(k) + \nu g(k)) + \nu G(k, h(k) + \nu g(k)) & (3.4.6) \\
&= H(k, h(k)) + \nu \frac{\partial H}{\partial R}(k, h(k))g(k) + \nu G(k, h(k)) & (3.4.7)
\end{align*}
\]

from which we derive an expression for \( g(k) \):

\[
g(k) = -\frac{G(k, h(k))}{\frac{\partial H}{\partial R}(k, h(k))} \quad (3.4.8)
\]
This analysis is of course not correct for $|\frac{\partial H}{\partial R}(k, h(k))| \ll 1$ and for $|G(k, h(k))| \to \infty$. Apart from those cases we are (in principle) able to calculate the perturbed neutral stability curve. The results will depend on $p$, the wave number of the curved channel, which is arbitrary but fixed. There are however a number of problems that we run into if we want to perform the analysis for a certain given setting. We will mention a few of them below.

One of the problems has a computational character. The model that we consider has become more complicated due to the transformation $T$, which also resulted in a more complicated, spatially periodic (instead of constant), basic solution. The computations that we must perform in order to do the stability analysis become quite extensive (they involve for instance the basic solution explicitly). Because it is questionable whether this is a fruitful way of analysis, one might hope that there are other ways which lead to the same kind of information, without actually performing all the calculations.

Another problem is related to the following observation. Let us consider the matrix operator $L$, defined in (3.3.2). As already remarked, $L$ equals (apart from the term involving $\mathcal{A}$) the operator defined in (2.3.3) for $\omega \equiv 0$. If we substitute a solution as suggested in the previous section (see for instance (3.3.14)-(3.3.15) where we already take $k = p$), the matrix operator becomes an algebraic operator with coefficients depending on the model parameters and on $p$ and $R$. This means that there might be a certain combination of wave number and bifurcation parameter $(R_r, k_r)$ such that the matrix becomes singular and thus we will find no finite solution. This should be no surprise, because in that situation, we actually force the system with its natural frequency, which causes resonance (i.e. an infinite response with respect to the amplitudes). This has already been observed by Blondeaux and Seminara (1985), and a number of papers have been written about this problem, see for instance Seminara and Tubino (1989) and Seminara and Tubino (1994). If the point $(R_r, k_r)$ is not close to $(R_c, k_c)$, the minimum of the neutral stability curve, there is no problem in performing a weakly nonlinear analysis (which is a natural extension of a linear analysis), because this is always done around $(R_c, k_c)$.

Another point of attention is the following. When we perform a (linear) stability analysis, the starting point will be to perturb the basic state:

$$\Phi = \phi_b + \varepsilon \phi' \quad (3.4.9)$$

substitute this into the equation and collect the linear terms in $\varepsilon$. In the linear stability theory we can consider $\varepsilon$ to be infinitesimal small with respect to any order of $\nu$ and thus encounter no serious problems. However, when we perform a weakly nonlinear theory, $\varepsilon$ is related to the distance from critical conditions (which are determined in the linear theory) and we must be careful because the results will strongly depend upon the relative magnitude of $\nu$ (which can be considered fixed for a given channel) with respect to $\varepsilon$ (which can be varied at will).

A last problem that will be pointed out here has to do with the wavenumber of the curved channel $p$. For the linear and weakly nonlinear stability analysis it will appear that it is very important whether $p = O(1)$ or $o(1)$. If $p = O(1)$ (the case of a slightly, but not slowly curved channel) there are several situations to consider. It turns out that it is very
important to distinguish between the cases where $p$ is, or is not in the neighbourhood of $2k_c/N$ for some $N \in (k_c$ is the critical wave number, determined by the linear stability analysis). In Chapter 4, we will call this situation resonant. It makes even difference whether $N$ is even or odd. The reason for this will become clear in the next Chapter. The other case which can be considered is $p = o(1)$: slightly, slowly curved channels. This is to say that both the ‘amplitude’ and the wave number of the curvature is small. A subcase of this problem is encountered if both amplitude and wave number are small, but the wave number is much smaller than the amplitude. The analysis for this situation (slightly, very slowly curved channels) differs significantly from the previous case. For the mathematical framework related to this problem, we refer to Eckhaus and Kuske (1995).

Considering the above problems, we feel that it is not very fruitful to study the linear and weakly nonlinear analysis of the physical model of a curved channel in all details. Due to extensive computations, it might very well be possible that fundamental mechanisms that cause important phenomena are eclipsed and thus, it is questionable whether such an approach will provide us with the information that we seek: the influence of the periodic boundaries on the linear and weakly nonlinear stability analysis with respect to the unperturbed, straight case. In a more simple model problem, which contains only the essential features of the ‘real’ problem, we can focus our attention on the stability analysis itself. After having fully understood the model problem we can, guided by the obtained knowledge, tackle the physical problem again.

**Remark**

There is one point related to $p$ which is worthwhile mentioning in this context. If we consider the situation of a straight channel, we have shown in the previous Chapter that free bars with wave number $k_c$ will develop. One can imagine that the flow, which will start to meander with the same wave number $k_c$, causes a initial perturbation of the side walls. This perturbation will also be periodic with wave number $k_c$. Therefore, it is likely that in realistic situations related to river morphology, $p$ equals $k_c$. However, there might be other physical problems where different mechanisms play a rôle and thus, we do not limit ourselves with respect to the meander wavenumber and consider arbitrary (but fixed) $p$.

### 3.5 Derivation of the model problem

The model problem forms the basis for the analysis presented in the next Chapter. In this section, we discuss how one can, starting with a problem defined on a straight domain, derive the equivalent of the problem defined on a curved domain for a general setting. Then, we will make some assumptions that will lead to the model problem which will be analysed in the next Chapter. We use the observations made in the previous section for the specific problem related to the morphodynamics but, as said before, use a more general setting.
We consider an equation of the following type:

$$\frac{\partial \Phi}{\partial t} S = L_R \Phi + N(\Phi)$$  \hspace{1cm} (3.5.1)$$

for a vector of unknowns $\Phi$, depending on two spatial coordinates $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ and on time $t$, a linear operator $L_R$, depending on a bifurcation parameter $R$ and a nonlinear operator $N$. This equation covers the model that we used for the morphological problem, discussed in Chapter 2 of this thesis, but has of course a much wider range of applications. Among them are Rayleigh-Bénard convection, Taylor vortices and Poiseuille flow, to mention but a few. We assume that $\Phi$ is defined on a domain $D$ which is unbounded in $x$ and bounded in $y$ direction and which has a periodic structure in the unbounded direction. Then, as we did in section 3.2, we define a map $T$ which transforms the periodic domain $D$ into a rectangular domain $\tilde{D}$. On $\tilde{D}$, equation (3.5.1) reads:

$$\frac{\partial \Phi}{\partial t} S = \tilde{L}_R \Phi + \tilde{N}(\Phi)$$  \hspace{1cm} (3.5.2)$$

where $\Phi$ now depends on $s \in \mathbb{R}^n$ and $n \in \mathbb{R}^n$ (the relation between $x$ and $y$ on one hand and $s$ and $n$ on the other hand is given by $T$) and where $\tilde{L}_R$ and $\tilde{N}$ have variable coefficients generated by $T$ (they are the equivalents of the coefficients $h_1, h_2, 1/R_s$ etc. in the case of the morphological problem). If the domain $D$ varies only slightly, the variable coefficients (which are periodic because $D$ has a periodic structure) can be expanded in a power series in the small variable $\nu$ which is a measure for the ‘variations’ in the domain.

As suggested by the analysis presented in section 3.3, we may assume that we can calculate a basic solution $\phi(\nu)$, bounded for all $x$, in a power series in $\nu$ where the first order term corresponds with the basic solution, belonging to (3.5.1), defined on the straight domain. Now, let us consider a general perturbation $\Psi$ of this basic solution:

$$\Phi = \phi(\nu) + \Psi$$  \hspace{1cm} (3.5.3)$$

We will derive an equation for $\Psi$, which will depend of course on $\phi(\nu)$. This equation will lead us to the model problem.

By construction, $\Phi = \phi(\nu)$ (i.e. $\Psi = 0$) will be a solution of equation (3.5.2) and stability properties of $\Psi = 0$ are directly related to the stability properties of $\phi(\nu)$. We expand the operators $\tilde{L}_R$ and $\tilde{N}$ in a formal way, using the expansion of the variable coefficients. Thus, we write

$$\tilde{L}_R \Phi \sim \hat{L}_R \Psi + \nu g(s) L_1 \Psi + O(\nu^2)$$  \hspace{1cm} (3.5.4)$$

where $g$ is related to the definition of the domain $D$ and is assumed to be periodic. In (3.5.4), we omitted the terms containing only $\phi(\nu)$; they will eventually drop out because $\phi(\nu)$ is the basic solution of (3.5.1). We used the hat-notation to distinguish $\hat{L}_R$ (which has constant coefficients) from $\tilde{L}_R$ (which has variable coefficients). In the same way, we expand $\tilde{N}$:

$$\tilde{N}(\Phi) \sim \hat{N}(\Psi) + \nu \hat{M}(\Psi) + L_2 \Psi + \nu g(s) L_3 \Psi + O(\nu^2)$$  \hspace{1cm} (3.5.5)$$
3.5 Derivation of the model problem

where $\hat{M}$ still contains variable coefficients. The operator $\hat{N}$ is nonlinear with constant coefficients, the operators $L_2$ and $L_3$ are linear. The operator $L_2$ appears because the basic state $\phi(\nu)$ is allowed to have a nonzero first order part. Thus, putting (3.5.4) and (3.5.5) together, we arrive at the following equation for $\Psi$:

$$\frac{\partial \Psi}{\partial t} S_1 = L_R \Psi + \nu g(s) S_2 \Psi + \mathcal{N}(\Psi) + \nu M(\Psi) + O(\nu^2)$$  \hspace{1cm} (3.5.6)

with

- $S_1 = S$  \hspace{1cm} (3.5.7)
- $S_2 = L_1 + L_3$  \hspace{1cm} (3.5.8)
- $L_R = \hat{L}_R + L_2$  \hspace{1cm} (3.5.9)
- $\mathcal{N} = \hat{N}$  \hspace{1cm} (3.5.10)
- $M = \hat{M}$  \hspace{1cm} (3.5.11)

Now, we make a few more assumptions that facilitates the analysis presented in Chapter 4 but do not affect the fundamental properties of the model.

First of all, we study (3.5.6) in a one dimensional situation, postponing the concerns for the transverse directions for later. We believe that this one-dimensional model exhibits in essence all aspects of pattern formation in systems defined on periodically modulated domains. In other words: in more realistic problems with transversal dimensions the complications will be merely technical, but the modulation equations that we will derive will be the higher dimensional analogon of the equations derived for the one-dimensional problem. Moreover, this reduction is also motivated by recent studies of van Harten (1991) and Eckhaus (1993). There, systems like (3.5.6) are studied in a one dimensional situation and without the periodic perturbation $\sim \nu g(x)$. It follows from these studies that the analysis of the one dimensional situation allows rigorous understanding of the nonlinear evolution of patterns while the methods permit extension to multidimensional ‘real’ problems (see Bollerman (1993), Schneider (1992)).

Secondly, we will neglect the operator $M$ in (3.5.6) and replace $S_{1,2}$ by the identity. These assumptions simplify the computations considerably but have otherwise no significant influence on the analysis. We will also neglect the $O(\nu^2)$ terms.

Finally, we choose $L_R$ to be an operator, containing only even derivatives with respect to $s$. This is to say that we impose some extra structure on the model problem by means of a reflection symmetry $s \rightarrow -s$. The reason for this becomes clear in the next section: there, we will derive a dispersion relation which will have no imaginary part. This simplifies the analysis again considerably.

Taking into account these assumptions, equation (3.5.6) reduces to

$$\frac{\partial \Psi}{\partial t} = L_R \Psi + \nu g(x) \Psi + \mathcal{N}(\Psi)$$  \hspace{1cm} (3.5.12)

with $\Psi = \Psi(x,t), x \in \mathbb{R}, t \in \mathbb{R}^+$, $L_R$ a linear operator depending on a parameter $R$, $\mathcal{N}(\Psi)$ a nonlinear operator, $g(x)$ a $p$-periodic function and $\nu$ a small parameter. This equation will be the starting point for the analysis presented in the next Chapter.
Chapter 4

Nonlinear Stability Analysis for Problems Defined on Periodically Oscillating Domains

4.1 Introduction

In this Chapter, we develop a nonlinear stability theory for equations of the following type:

\[
\frac{\partial \Psi}{\partial t} = \mathcal{L}_R \Psi + \nu g(x) \Psi + \mathcal{N}(\Psi)
\]  

(4.1.1)

where \( \Psi = \Psi(x, t), x \in \mathbb{R}, t \in \mathbb{R}^+, \mathcal{L}_R \) is a linear operator depending on a control parameter \( R \), \( \mathcal{N} \) is a nonlinear operator, \( \nu \) is a small parameter and \( g(x) \) is a \( p \)-periodic function. We write \( g(x) \) as a Fourier series: 

\[ g(x) = \sum_l F_l \exp(ilpx), \]

where \( p = O(1) \), \( F_l = F_{-l} \) and with \( F_0 = 0 \). This latter fact means that the 'perturbation' \( g \) has zero-average, which is of course no limitation: a non-zero average term \( F_0 \) has the same influence on the model problem as an \( O(\nu) \) shift in the control parameter \( R \). This equation is considered to serve as a simple model for the analysis of the evolution of patterns in problems with slightly varying geometry, as is motivated in the previous Chapter. Here, the aim is to gain an understanding on the influence of the varying boundary on the weakly nonlinear stability analysis by studying the model problem (4.1.1).

There are two significantly different cases to consider: \( p = O(1) \) and \( p = o(1) \). The differences between the two situations are already encountered on the linear level. In this Chapter, we concentrate on the case \( p = O(1) \). The other case, \( p = o(1) \), is studied in Eckhaus and Kuske (1995). In section 4.5, we will make some remarks about the main differences between the two analyses (see also Doelman et al. (1995a)).

We summarize the analysis and the main conclusions of this Chapter. First, we consider the unperturbed case, i.e. \( \nu \equiv 0 \). The linear and nonlinear analysis is well known and can very briefly be summarized as follows. The stability of the basic state \( \Psi = 0 \) is considered with respect to perturbations of the type \( \exp(ikx + \omega t) \). It turns out that \( \omega \) can be written as a function of \( k \) and \( R \), the so-called dispersion relation which is denoted by \( \omega = \mu(k, R) \) where \( \mu \) is the symbol of the operator \( \mathcal{L}_R \) (see (4.2.5)). If \( Re(\omega) < 0 \), the...
basic state is stable, while for $Re(\omega) > 0$ the basic state is unstable. Thus, $Re(\omega) = 0$, the neutral stability curve, divides the $(k, R)$-plane into two parts: a stable and an unstable area. We expect bifurcating solutions from the basic state at the minimum $(k_c, R_c)$ of the neutral stability curve. These bifurcating solutions turn out to be slow modulations of the linearly most unstable wave. The amplitude of this modulated wave is determined in the nonlinear theory, using the solutions of a so called amplitude or modulation equation, which is known as the Ginzburg-Landau equation and which, in this particular case, has real coefficients.

Then, we consider the perturbed case $\nu \neq 0$. The linear theory leads to a reinterpretation of the neutral stability curve. The unperturbed curve must now be interpreted on a $p$-periodic cylinder with $p$ the period of the ‘forcing’ $g$ of the model. The ‘new’ neutral stability curve has almost everywhere an $O(\nu^2)$ correction with respect to the unperturbed curve except in the cases where $|\mu(k, R) - \mu(k + Np, R)| = o(1)$, $N$ arbitrary. In that case, the correction is $O(\nu)$. (At the end of this introduction, we recapitulate the definitions of the symbols $O, o, \prec$ and $\succ$, which will be frequently used in this Chapter.) Furthermore, it is shown that the reinterpreted spectrum contains gaps, which are related to the gaps that are present in the spectrum of Hill’s equation (see Magnus and Winkler (1966)). The influence of most wave numbers $p$ on the linear and nonlinear theory is marginal. The critical conditions for the wave number $k$ and the bifurcation parameter $R$ get a $O(\nu^2)$ correction, and the amplitude equation that describes the amplitude of the modulation of the linear most unstable wave remains the Ginzburg-Landau equation, where now the coefficients have a $O(\nu^2)$ correction with respect to the unperturbed case. These corrections depend on the Fourier coefficients $F_l$ of $g(x)$ which are in general complex. It turns out that the dependence is such that the amplitude equation has coefficients with small imaginary parts.

If $p$ is such that that $Np/2$ is ‘close’ to $k_c$, where $k_c$ is the critical wave number belonging to the unperturbed case and $N$ some arbitrary but fixed integer, some unexpected phenomena appear. For those values of $p$, which we will call resonant, one of the results of the linear theory is that the linear most unstable wave has a different structure, compared to the non-resonant case. This wave has now become periodic in $x$, where it was quasi-periodic. Furthermore, some structural modifications with respect to ‘standard’ linear theory have to be taken into account, one of which is that the neutral stability curve ‘splits’ into several curves. For $|R - R_c| \sim O(\nu^4)$ (thus, very close to critical), the modulation equation that we derive is ‘just’ a Ginzburg-Landau equation for a complex amplitude. However, for $R$ sufficiently far away (‘more than $O(\nu^4)$’) from critical, but on the other hand, not too far away (not further than $O(\nu)$) the results for the nonlinear theory are quite drastic. Instead of modulating the critical wave with one, complex, amplitude, we must now consider modulations of the critical wave with two, essentially real amplitudes. The nonlinear theory leads to the derivation of two modulation equations for these amplitudes, which is a coupled system of two (real) Ginzburg-Landau equations. These two equations can be combined to a complex Ginzburg-Landau equation for an amplitude $A$ which contains an additional term proportional to $\overline{A}$. This term only plays a significant role if $\nu^2 \prec |R - R_c| \prec \nu$; we will see that for $\nu^4 \prec |R - R_c| \prec \nu^2$ the dynamical situation is actually one-dimensional. An interesting subcase arises if $N$ is even and $R$ is $O(\nu)$ close to $R_c$: the cubic nonlinearity is extended with quadratic terms. If $R$ gets too far away from critical (‘more than $O(\nu)$’), we
Let us briefly indicate the main implications of this Chapter. The most intriguing one is that the so called *Eckhaus band*, see Eckhaus (1965), which gives a boundary for an area in the parameter space in which periodic solutions are stable, does no longer exist in a relatively large area of the parameter space. This is due to the fact that in that particular area of parameter space, the amplitude equation that we encounter is not a complex equation but a real equation. As a result, the dynamics turn out to be one dimensional, and the (real) amplitude equation does not have periodic solutions with a simple wave structure. In fact, interesting solutions appear to be constant in time and space, which means that there is a very strong pattern selection. The periodic solutions that bifurcate from the basic state have exactly wave number $k_e$, determined in the linear theory, and are not modulated. In another relatively large part of the parameter space, we encounter a Ginzburg-Landau equation for a complex amplitude $A$ with an additional term proportional to $\bar{A}$. The result of this perturbation is that the phase invariance of the solutions, which is present in the complex Ginzburg-Landau equation, vanishes. Finally, we mention that under certain conditions, it is possible that the solutions have different orders of magnitude, i.e. they are either very large or very small, compared with the unperturbed case. Both solutions can be stable, so the order of magnitude of the final pattern (i.e. for $t \to \infty$) depends sensitively on the initial conditions.

The remainder of this Chapter is organized as follows. In section 4.2, the linear and nonlinear theory of the unperturbed case $\nu \equiv 0$ is briefly discussed. In section 4.3, the linear theory of the perturbed model problem is studied, which leads to the discovery of a resonance phenomena. In section 4.4 the nonlinear theory of the non-resonant and the resonant problem is discussed, resulting in various amplitude equations for the different cases. Finally, in section 4.5, we make some brief comments on the difference between $p = O(1)$ and $p = o(1)$.

**Remark**

In this Chapter, we will frequently use the classical Landau order symbols $O$ and $o$. These are defined as follows (see Eckhaus (1979)):

**Definition** Consider any pair of real continuous functions $f(\varepsilon)$ and $g(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$. Then, the symbols $O$ and $o$ are defined as follows:

$f = O(g)$ for $\varepsilon \downarrow 0$ if there exist positive constants $M$ and $C$, such that $|f(\varepsilon)| \leq M|g(\varepsilon)|$ for $0 < \varepsilon < C$.

$f = o(g)$ if $\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)}$ exists and equals zero.

Instead of the order symbols of Landau, we will also use a notation which is due to Hardy.

**Definition** $f \prec g \iff f = o(g)$ (and $f \succ g \iff g = o(f)$)

$f \preceq g \iff f = O(g)$ (and $f \succeq g \iff g = O(f)$)

Finally, we introduce a symbol for sharp order of magnitude estimates:
**Definition** \( f = O_s(g) \Rightarrow f = O(g), \ f \neq o(g) \)

### 4.2 Linear and nonlinear theory of the unperturbed problem

In this section, we consider the unperturbed model problem, i.e. \( \nu \equiv 0 \), and we look at the stability properties of the zero-solution (basic state) of (4.1.1). Thus, we consider:

\[
\frac{\partial \Psi}{\partial t} = \mathcal{L}_R \Psi + \mathcal{N}(\Psi) \tag{4.2.1}
\]

where we take for simplicity

\[
\mathcal{N}(\Psi) = \mathcal{P}_1(\Psi) \mathcal{P}_2(\Psi) \quad \mathcal{P}_1, \mathcal{P}_2 \text{ linear operators} \tag{4.2.2}
\]

The analysis is quite classical and well-developed, and therefore we only go through it briefly, omitting the details. Nevertheless, we consider it useful to present this analysis here since it will act as reference case for the analysis of the perturbed problem.

We consider a perturbation of the zero-solution:

\[
\Psi = 0 + \hat{\varepsilon} \psi_{lin} = \hat{\varepsilon} e^{i k x_0 + \omega t} \psi_{lin} \tag{4.2.3}
\]

substitute this into (4.2.1) and linearise. This yields:

\[
\omega(k, R) = \mu(k, R) \tag{4.2.4}
\]

where \( \mu(k, R) \) is the symbol of the operator \( \mathcal{L}_R \) (see (4.2.1)) and is defined by

\[
\mathcal{L}_R(e^{i k x}) = \mu(k, R)e^{i k x} \tag{4.2.5}
\]

Equation (4.2.4) is a so called dispersion relation, and it is obvious that the sign of \( \text{Re}(\omega) \) determines the linear stability properties of the basic state. Note that, due to the special structure of \( \mathcal{L}_R, \omega \) has no imaginary part. The minimum of the curve \( \omega = 0 \) is denoted by \((k_c, R_c)\). For values of \( R < R_c \), perturbations of the basic state decay exponentially and the basic state is stable, while for \( R > R_c \) the basic state is unstable. Interesting behaviour is expected whenever \( R \) passes through \( R_c \), i.e. when the basic state becomes unstable.

In the nonlinear analysis, we look at the nonlinear evolution of the perturbations of the basic state at near critical conditions. Classically, we model the perturbation by assuming that it is a modulation of the most unstable wave \( E = e^{i k c x} \), i.e. we assume an Ansatz of the following form:

\[
\Psi(x, t, \xi, \tau) = \psi_0 + \varepsilon^2 \psi_{02} + \varepsilon^3 \psi_{03} + \ldots + E[\varepsilon \psi_{11} + \varepsilon^2 \psi_{12} + \varepsilon^3 \psi_{13} + \ldots] + c.c. + E^2[\varepsilon^2 \psi_{22} + \varepsilon^3 \psi_{23} + \ldots] + c.c. + E^3[\varepsilon^3 \psi_{33} + \ldots] + c.c. + \ldots \tag{4.2.6}
\]

where \( \varepsilon = O(\sqrt{R - R_c}) \), and \( \xi \) and \( \tau \) are rescaled spatial and time variables. Recently, Eckhaus (1993) gave a rigorous foundation for this Ansatz by showing that small solutions without an imposed structure do evolve towards the structure described by the Ansatz,
on a faster time-scale then the time-scale at which the nonlinear interactions take place. The classical rescaling of the coordinates is \( \xi = \varepsilon x, \tau = \varepsilon^2 t \). Note that the magnitude of the perturbation, \( \varepsilon \) (see (4.2.3)), is also taken to be of \( O(\varepsilon) \). Expansion (4.2.6) is then substituted into the full, nonlinear model problem (4.2.1) and we compare equal orders of \( \varepsilon^i E^j \). As expected, we find for \( i = j = 0 \) the basic state, i.e. \( \psi_0 = 0 \). For \( i = j = 1 \), we recover the linear problem. Thus, \( \psi_{11} = \hat{\psi}_{\text{lin}} = A(\xi, \tau) \), because the linear operator, formally expanded in powers of \( \varepsilon \), does not act on the slow variables \( \xi \) and \( \tau \) in first order.

Proceeding with the analysis, we find at level \( \varepsilon^1 E \), \( j \neq 1 \) equations which can be solved easily and yield expressions for \( \psi_{ij} \). At level \( \varepsilon^2 E \), the equation for \( \psi_{1i} \) reads:

\[
\mu(k_c, R_c) \psi_{1i} = \chi_{1i} \tag{4.2.7}
\]

where \( \chi_{1i} \) is determined by taking into account the rescaling of the spatial and time variable, the expansion of \( R \) and the nonlinear interactions. For details about the derivation, we refer to Doelman (1990). Because the left hand side of (4.2.7) vanishes at critical conditions \( (\mu(k_c, R_c) = \omega(k_c, R_c) = 0) \), the right hand side must also be equal to zero, i.e. \( \chi_{1i} = 0 \), which gives at \( \varepsilon^3 E \) the first order of the amplitude equation, known as the Ginzburg-Landau equation (the equation arising at \( \varepsilon^2 E \) turns out to be trivial, which is a result of the rescaling of \( \xi \)). Note, that if (4.2.1) depends on other spatial variables, then \( \mu(k, R) \) is an operator, and solving the equivalent of (4.2.7) involves the application of an orthogonality condition (Fredholm Alternative) and leads to the \( n \)-dimensional equivalent of the Ginzburg-Landau equation, where \( n \) is the number of unbounded directions. In this simple case, the Ginzburg-Landau equation reads as follows:

\[
\frac{\partial A}{\partial \tau} = r \omega_R A - \frac{1}{2} \omega_{kk} A_{\xi \xi} + \beta |A|^2 A \tag{4.2.8}
\]

where \( r = (R - R_c)/\varepsilon^2, \omega_R = \partial \omega/\partial R, \omega_{kk} = \partial^2 \omega/\partial k^2 \), both evaluated at critical conditions \( k = k_c, R = R_c; \beta \) is the so-called Landau-coefficient and is the result of nonlinear interactions. The coefficient can be expressed in terms of \( P_1 \) and \( P_2 \), the symbols of the operators \( \mathcal{P}_{1,2} \) (see (4.2.2)) defined by \( \mathcal{P}_{1,2}(e^{ikx}) = P_{1,2}(k)e^{ikx} \):

\[
\beta = \beta_1 + \beta_2 \quad \text{where} \quad \beta_1 = -\frac{1}{\mu(0)} (P_1(k_c)P_2(0) + P_1(0)P_2(k_c)) \quad \beta_2 = -\frac{1}{\mu(2k_c)} P_1(k_c)P_2(k_c) (P_1(2k_c)P_2(-k_c) + P_1(-k_c)P_2(2k_c)) \tag{4.2.9}
\]

The Ginzburg-Landau equation governs the nonlinear evolution of perturbations of type (4.2.3) of the basic state at near critical conditions.

### 4.3 Linear theory of the perturbed problem

In this section, we describe the linear theory of the perturbed model problem. This will lead to a reinterpretation of the neutral stability curve as it is encountered in the previous section.

As before, we perturb the basic state \( \Psi = 0 \) and look at the (linear) stability properties of
the perturbation. Due to structure of the small periodic term \( \nu \sum F_i e^{ilpx} \), the perturbation must now be of the following quasiperiodic structure:

\[
\Psi = 0 + \hat{\varepsilon}_\text{lin} = \hat{\varepsilon} e^{ikx + \omega t} \sum \psi_l(k, R) e^{ilpx} + c.c. \quad (4.3.1)
\]

where \( k \) is an arbitrary wave number, \( \omega \) is the growth rate of the waves and \( p \) is the period of the small periodic effect. As usual, the sign of the real part of \( \omega \) determines the stability properties of the perturbation (4.3.1). With \( \hat{\varepsilon} \), which is again the magnitude of the perturbation, we introduce besides the relative magnitude of \( \hat{\varepsilon} \) with respect to \( \nu \). Due to the structure of \( \psi_{\text{lin}} \) (see (4.3.1)) we have to be careful about the interpretation of the set \( \omega = 0 \) in the \( (k, R) \)-plane. As a result of the periodic function \( g \) in the model problem, the outcome of the linear theory will be periodic in \( k \). To see this, we consider (4.3.1) for \( k = \tilde{k} + np, N \in \mathbb{Z} \) and observe

\[
\psi_{\text{lin}} = e^{i(k + np)x + \omega t} \sum \psi_l(\tilde{k} + np, R) e^{ilpx} + c.c.
\]

Thus, we can study the perturbations given in (4.3.1) for \( k \) limited to an interval with length \( p \); the results we obtain are \( p \)-periodic and therefore \( \omega(k, R) = \omega(k + np, R) \). From now on, we will take \( k \in [k^c_\text{c} - p/2, k^c_\text{c} + p/2] \), where \( k^c_\text{c} \) is the critical wave number belonging to the unperturbed (\( \nu = 0 \)) problem. Geometrically, this means that the graph of \( \omega = \mu(k, R) \) (in the unperturbed case) should now be interpreted as lying on a cylinder. This observation gets a more natural meaning in the sequel of the analysis, when we observe that we get degenerate situations in the neighbourhood of intersections of \( \mu(k, R) \) and \( \mu(k + np, R) \). With the interpretation on the cylinder, these intersections can then be regarded as self-intersections of \( \mu(k, R) \).

Substitution of (4.3.1) (where we have scaled \( \psi_0 = 1 \)) into the model problem (4.1.1) and linearizing the results yields an equation for every coefficient of the Fourier series (4.3.1). For \( l = 0 \), we find a formula for the growth rate \( \omega \), which is a correction of the growth rate as was found in the unperturbed case:

\[
\omega(k, R) = \mu(k, R) + \nu \sum_{s+t=0} F_s \psi_t \quad (4.3.3)
\]

In particular, the equality \( \omega(k, R) = \mu(k, R) \) (see (4.2.4)) no longer holds. For \( l \neq 0 \) we find equations for the coefficients \( \psi_l \), recursively defined:

\[
\psi_l = \nu \frac{\sum_{s+t=l} F_s \psi_t}{\omega(k, R) - \mu(k + lp, R)} \quad (4.3.4)
\]

Thus, for \(|\omega(k, R) - \mu(k + lp, R)| = O(1)| \), we see that \( \psi_l = O(\nu), l \neq 0 \) and since \( \psi_0 = 1 \), setting the index \( t = 0 \) in (4.3.4) yields the leading order term of \( \psi_I, l \neq 0 \):

\[
\psi_l = \nu \frac{F_l}{\mu(k, R) - \mu(k + lp, R)} + h.o.t. \quad (4.3.5)
\]
Substitution of (4.3.5) into (4.3.3) gives an explicit expression for the $O(\nu^2)$ correction on $\omega(k, R)$ in the perturbed case:

$$\omega(k, R) = \mu(k, R) - \nu^2 \sum_{l} \frac{|F_l|^2}{\mu(k + lp, R) - \mu(k, R)} + O(\nu^3)$$

We expect degenerated situations with respect to the magnitude of the correction on $\omega$ whenever $|\omega(k, R) - \mu(k - Np, R)| = o(1)$, for some $N \in \mathbb{Z} \setminus \{0\}$. This turns out to be an important observation and is studied in the next section.

### 4.3.1 Degenerations: $|\mu(k + lp, R) - \mu(k, R)| = o(1)$

We consider the situation in which

$$\mu(k + Np, R) - \mu(k, R) = Q\nu^\lambda$$

for some $Q$ and $\lambda > 0$ and arbitrary but fixed $N$. This means that we look locally around the intersection of the curves $\mu(k, R)$ and $\mu(k + Np, R)$ for general $k$. Before we give the technical details of the analysis, we first describe what happens; the situation is sketched in figures 4.4 and 4.2. In the neighbourhood of the intersection of $\mu(k, R)$ and $\mu(k + Np, R)$ (which, as already been observed, can be interpreted as a self-intersection of $\mu(k, R)$), we observe that there are two corrections on $\mu(k, R)$. One could say that the unperturbed curve 'splits' into two new curves, one of them 'below' (which we will call $\omega^-$ in the sequel) $\mu(k, R)$ and one 'above' (which will be called $\omega^+$), and both are $O(\nu)$ close to the unperturbed curve. If we 'move away' from the intersection, the correction of $\omega^+$ and $\omega^-$ on $\mu$ becomes eventually $O(\nu^2)$. As is indicated in figure 4.2, $\omega^+$ follows the upper branch of $\mu$ and $\omega^-$ follows the lower branch. Thus, although the unperturbed curve shows self-intersections on the cylinder, the perturbed curves $\omega^+$ and $\omega^-$ do not intersect. This 'splitting'-situation occurs at every intersection point: the unperturbed curve splits up into infinitely many branches. Below, we make this reasoning more precise. Let

$$\omega(k, R) = \mu(k, R) + S\nu^\mu + h.o.t.$$  (4.3.8)

where $S$ and $\mu$ have to be determined as function of $\lambda$ and $Q$. Straightforward substitutions of these expressions into (4.3.3) and (4.3.4) lead to:

$$S\nu^\mu = \nu \sum_{s+t=0} F_s \psi_t$$

$$= -\nu^2 \sum_{s+t=0} F_s \frac{\sum_{s'+t'=0} F_{s'} \psi_{t'}}{\mu(k + tp, R) - \omega}$$

$$= -\nu^2 \frac{|F_N|^2}{\mu(k + Np, R) - \omega} - \nu^2 \sum_{s+t=0, s\neq N} F_s \frac{\sum_{s'+t'=0} F_{s'} \psi_{t'}}{\mu(k + tp, R) - \omega}$$

$$= -\nu^2 \frac{|F_N|^2}{Q\nu^\lambda - S\nu^\mu} + O(\nu^2).$$

(4.3.9)

Apparently, we must distinguish between three different cases:
1. \( \lambda > \mu \). In that case, one finds that \( S_{1,2} = \pm |F_N| \) and \( \mu = 1 \). This means that
\[
\omega(k, R) = \mu(k, R) \pm \nu|F_N| + O(\nu^2)
\]
for \( k, R \) such that \( \mu(k, R) - \mu(k + Np, R) = Q\nu^\lambda, \lambda > 1 \)

2. \( \lambda = \mu \). Then (4.3.9) yields at leading order
\[
S_{1,2} = \frac{Q \pm \sqrt{Q^2 + 4|F_N|^2}}{2}
\]
and \( \mu = 1 \) (see figure 4.1 for a plot of \( S_{1,2}/|F_N| = S_{1,2}(Q/|F_N|) \)).

3. \( \lambda < \mu \). Then \( \mu = 2 - \lambda \) and \( S_1 = -|F_N|/Q; S_2 = O(1/\nu^{2-\lambda}) \). Because \( \lambda < \mu \)
and \( \mu = 2 - \lambda \), there holds that \( \lambda < 1 \) and consequently, \( S_2 \gg 1 \).

Figure 4.1: \( S_{1,2}/|F_N| \) as function of \( Q/|F_N| \). The dotted line represents the line \( Q = S \).

From the first case, it follows that in an \( O(\nu) \) neighbourhood of the intersection of \( \mu(k, R) \)
and \( \mu(k + Np, R) \), the correction on \( \omega \) is \( O(\nu) \) instead of \( O(\nu^2) \) as we found in (4.3.6).
The transition of the \( O(\nu^2) \) towards the \( O(\nu) \) correction is described in the third case
and is smooth. Note also, that for all \( \lambda \), there are two solutions for \( S \) (see figure 4.1).
This is especially important for \( \lambda \geq 1 \). It reflects the splitting that we already observed
previously and which is a direct consequence of the periodicity of the domain. The \( Q \)-axis
in figure 4.1, i.e. \( S = 0 \), can be regarded as describing the unperturbed curve \( \mu(k, R) \)
(see (4.3.8)). The other asymptote of the hyperbola, \( S = Q \), is the the unperturbed curve
\( \mu(k + Np, R) \) as can be seen by combining (4.3.7) and (4.3.8). The two solutions \( S_{1,2} \)
can now be interpreted as being the two corrections to \( \mu(k, R) \). Also, the following properties
holds:
\[
\begin{align*}
S_1 &\rightarrow 0 \text{ if } Q \rightarrow -\infty \\
S_1 &\rightarrow Q \text{ if } Q \rightarrow +\infty \\
S_2 &\rightarrow Q \text{ if } Q \rightarrow -\infty \\
S_2 &\rightarrow 0 \text{ if } Q \rightarrow +\infty
\end{align*}
\]

Note that taking the limit for \( |Q| \rightarrow \infty \) corresponds to \( \lambda < 1 \) while \( |Q| \rightarrow 0 \) represents
the case \( \lambda > 1 \). Then, in that view, we can interpret the ‘large’ solution that we found in
the third case for \( \lambda < 1 \): it corresponds with the solution \( S_1 \) for \( Q \) positive and \( S_2 \) for \( Q \)
4.3 Linear theory of the perturbed problem

We observe that for $\lambda < 1$ (alternatively, $\lambda = 1$ and $|Q| \to \infty$, see the second case) a branch of the solutions $|S|_{1,2}$ indeed becomes ‘large’.

Let us consider the transition case $\lambda = 1$ in more detail (one should keep figure 4.1 in mind). We may conclude that in a neighbourhood of $k^* = -Np/2$, ($k^*$ is defined as the $k$-coordinate of the intersection of $\mu(k, R)$ and $\mu(k + Np, R)$), the already announced splitting of $\omega$ in $\omega^+ = \mu(k, R) + S_1 \nu^\mu$ and $\omega^- = \mu(k + Np, R) + S_2 \nu^\mu$ occurs. Note that $\omega^+$ and $\omega^-$ are only $O(\nu)$ close in a neighbourhood of $k^*$. The situation (now for all $\lambda$) has again

![Figure 4.2: The perturbed and unperturbed dispersion relations as function of $k$ for a fixed value of $R \ll R_c$. The unperturbed dispersion relation (interpreted on a cylinder) is represented as a solid line, while the perturbed dispersion relation is represented by the dotted lines. Note that the distance between the perturbed and unperturbed dispersion relation (by a same value of $k$) is either $O(\nu)$ or $O(\nu^2)$.](image)

been sketched and summarized in figure 4.2 in $(k, \omega)$-space. The dotted lines represent $\omega$ as correction of $\mu(k, R)$ and are $O(\nu^2)$ close to $\mu(k, R)$ if $k$ is not $o(1)$ near $k^*$. Note also that the $S_1$ curve is ‘above’ $\mu(k, R)$. Now, this reasoning can be applied for arbitrary $N$, and the reinterpretation of the dispersion relation becomes clear. The perturbed spectrum $\omega$ is not just a correction of the unperturbed dispersion relation $\mu(k, R)$ (which takes arbitrary values in ) but (its graph) ‘splits’ into a (countable) collection of curves.

The fact that the graph of the dispersion relation splits into a collection of curves, implies that there are gaps in the range of $\omega$. More specifically: $\omega(k, R) \notin (\mu(Np/2, R) + \nu |F_N|^2 + O(\nu^2), \mu(Np/2, R) - \nu |F_N|^2 + O(\nu^2))$, for all $N \in - \{0\}$. The creation of the gaps in what can be considered as the spectrum of the linear stability problem is caused by essentially the same mechanism as the one which creates the gaps in the spectrum of
Hill’s equation (see Stoker (1957), Nayfeh (1973)):

\[ \ddot{x} + (\lambda + \nu F(t))x = 0 \quad (4.3.10) \]

for \( x = x(t) \) and where \( F \) is \( T \)-periodic and \( 0 < \nu \ll 1 \) (if \( F(t) = \cos \omega t \), then (4.3.10) is the Mathieu-equation). In the format of Hill’s equation, our model problem is similar to

\[ \frac{\partial^2 \Psi}{\partial x^2} + \left( \omega + \nu \sum F_l e^{ilpx} \right) \Psi = 0 \quad (4.3.11) \]

where \( \partial^2/\partial x^2 \) plays a rôle equivalent to \( L_R \) in (4.1.1). Repeating the analysis for Hill’s equation, applied to (4.3.11) yields that for \( \nu = 0 \), there are bounded solutions for all \( \omega > 0 \). For \( \nu \neq 0 \), there are gaps in the \( \omega \)-spectrum around the resonance values \( \omega = (n\pi/p)^2, n \geq 1 \). The width of the \( n \)-th gap is proportional to the \( n \)-th Fourier-mode \( F_l \). For a value of \( \omega \) in such a gap, (4.3.11) does not have bounded solutions in \( x \). The structure of the computations needed to derive that result is essentially the same as the one applied to the model problem that we are considering now. The analogy also explains why we have chosen for a function in the model problem which is \( O(1) \), the correction is of \( O(\nu) \) instead of \( O(\nu) \), as we found in (4.3.5).

The only two really important parts of \( \omega \) are denoted with \( \omega^+ \) and \( \omega^- \) (see figure 4.2). They determine the new critical conditions for the neutral stability curve at which the basic state becomes linear unstable. (It should be noted however, that this situation is much more subtle if \( p = o(1) \); see section 4.5). From the previous reasoning, it follows that

\[ \max_k \omega^-(k, R) < \max_k \mu(k, R) < \max_k \omega^+(k, R) \quad (4.3.12) \]

for a fixed value of \( R \). Therefore, it can be concluded that the ‘new’ critical \( R \) (which we will call \( R^+ \) in the sequel and is defined by the condition \( \omega^+(k^+, R^+) = \partial \omega/\partial k(k^+, R^+) = 0 \), where \( k^+ \) is the ‘new’ critical wave number) will be smaller than \( R^p_c \), the unperturbed critical value of \( R \). Thus, taking into account the perturbation \( \nu \sum F_l e^{ilpx} \) in the model problem always destabilises the system. Note that this could already have been remarked after computing (4.3.6): near the maximum of \( \mu(k, R) \), it is clear that \( \mu(k + lp, R) - \mu(k, R) < 0 \) for all \( l \neq 0 \) and therefore \( \omega^+(k, R) > \mu(k, R) \). For the sequel, it is remarked that there are also critical conditions to be associated with \( \omega^- \). These conditions will be denoted by \( (k^-, R^-) \).

Let us emphasize the main result of this section (which is summarized in figure 4.2). For \( k \) in the neighbourhood of \( k^* \) (the \( k \)-coordinate of the self-intersection of \( \mu(k, R) \)) and thus \( |\mu(k, R) - \mu(k + Np, R)| = O(\nu^\alpha) \), for some \( \alpha > 0 \), the graph of the unperturbed dispersion relation (which was connected) splits into disjunct curves which are corrections on \( \mu(k, R) \) and where the magnitude of the correction is \( O(\nu^{2-\alpha}) \) for \( \alpha < 1 \) and \( O(\nu) \) for \( \alpha > 1 \). For \( k \) such that \( |\mu(k, R) - \mu(k + Np, R)| = O(1) \), the correction is of \( O(\nu^2) \).

To conclude this section, we make a final remark. As explained, the spectrum of the model
problem splits up into a set of curves, lying on the cylinder. The most important elements of that set are \( \omega^+ \) and \( \omega^- \). Only for \( |k - k^*| = o(1) \), we have that \( |\omega^+(k, R) - \omega^-(k, R)| = o(1) \). We expect interesting behaviour whenever the splitting occurs in the neighbourhood of \( k_c^0 \), the unperturbed critical wave number, i.e. when \( |k_c^0 - k^*| = o(1) \). Recall that \( k^* = -Np/2 \), and thus the splitting occurs in the neighbourhood of \( k_c^0 \) if

\[
p = -\frac{2k_c^0}{N} + o(1)
\]

for some fixed \( N \in -0 \). For such a \( p \), we call the problem resonant and condition (4.3.13) is referred to as the resonance condition. Already on the linear level, being in resonance has a significant influence on the position of \( k^+ \) and \( R^+ \) with respect to \( k_c^0 \) and \( R_c^0 \). Besides, and this is very crucial, in resonance there are two branches of the neutral set, \( \omega^+(k, R) = 0 \) and \( \omega^-(k, R) = 0 \), \( O(\nu) \) near the unperturbed neutral curve \( \mu(k, R) = 0 \). The \( \omega^+ = 0 \) branch is below \( \mu = 0 \), so passing this branch causes the first bifurcation at which the trivial solution of (4.1.1) becomes (linearly) unstable. Both branches correspond to marginally unstable perturbations (of type (4.3.1)). From the point of view of the linearised stability theory it is not important to study the second instability which appears when we cross the \( \omega^- = 0 \) branch, since this branch is always \( O(\nu) \) above \( \omega^+ = 0 \). However, when dealing with the nonlinear problem we will find that the splitting is of crucial importance as long as \( R - R_c = O(\nu) \) and smaller. This is mainly due to the character of the marginally unstable waves associated with these branches. We come back on this character in section 4.4.1.

4.3.2 The starting point for the nonlinear theory: the critical conditions for the non-resonant and resonant problem

The next step in the linear analysis of the perturbed model problem is to find the new critical conditions at which the basic state looses its stability properties. These conditions are an essential starting point for the nonlinear theory because there, we model the perturbation as a modulation of the critical wave. We have to distinguish between the resonant problem and the non-resonant problem. This distinction comes back in the sequel of the analysis: at every point we must consider the resonant and the non-resonant problem. The non-resonant problem however is ‘regular’; we will essentially recover the unperturbed results with \( O(\nu^2) \) corrections. The resonant problem exhibits essentially new phenomena.

The non-resonant problem

First, we analyse the non-resonant problem. In this case, it should be noted that \( |R^- - R^+| = O(1) \). Thus, as a nonlinear analysis is only valid in an \( \varepsilon^2 \)-neighbourhood of \( (k^+, R^+) \), the second critical point \( (k^-, R^-) \) does not play a role. Therefore, we focus our attention on the curve \( \omega^+ \) and the therewith associated critical conditions \( (k^+, R^+) \). It is natural to assume that they are located in a neighbourhood of the old critical conditions, \( k_c^0 \) and \( R_c^0 \). To find them, we need to describe \( \omega^+ \) as function of \( k \) and \( R \) and impose the conditions \( \omega^+(k^+, R^+) = \partial \omega^+/\partial k(k^+, R^+) = 0 \). To calculate the new critical conditions \( k^+ \) and \( R^+ \), we perform an asymptotic analysis in \( \nu \) and we write

\[
k^+ = k_c^0 + K\nu
\]

(4.3.14)
\[ R^+ = R_e^0 + R\nu^2 \]  
(4.3.15)

and expand \( \mu(k + lp, R) \) around \((k_l^0 + lp, R_l^0)\) for all \(l\).

\[
\begin{align*}
  l = 0 & \quad \mu(k, R) = \nu^2 \left[ \frac{1}{2} u_{02} K^2 + R u_{10} + \nu \left( \frac{1}{6} u_{03} K^3 + u_{11} K R \right) + \ldots \right] \\
  l \neq 0 & \quad \mu(k + lp, R) = u_{00} + \nu u_{01} K + \nu^2 \left( \frac{1}{2} u_{02} K^2 + u_{10} R \right) + \ldots
\end{align*}
\]
(4.3.16)

(4.3.17)

where \(u_{02} < 0, u_{10} > 0\). Because the calculations are rather straightforward, we shift them to Appendix A and give here only the results. The full expressions for \(k^+\) and \(R^+\) are found to be:

\[
\begin{align*}
  k^+ &= k_e^0 + \nu^2 \left( V_0 + \frac{u_{11}}{u_{10}} U_0 \right) + O(\nu^3) \\
  R^+ &= R_e^0 + \frac{\nu^2}{u_{10}} (U_0 - \nu \tilde{U}_0) + O(\nu^4)
\end{align*}
\]
(4.3.18)

(4.3.19)

where

\[
\begin{align*}
  U_0 &= \sum_{s+t=0} F_s F_t \left| \frac{0}{u_{00}} \right|^2 < 0 \\
  V_0 &= \sum_{s+t=0} \left| F_s \right|^2 \left( u_{01}^{(t)} \right)^2 \\
  \tilde{U}_0 &= \sum_{s+t=0} F_s U_t
\end{align*}
\]
(4.3.20)

(4.3.21)

(4.3.22)

\(u_{ij}^{(l)}\) are related to the expansion of \(\mu(k + lp, R)\). Thus, it is found that \(k^+ = k_e^0 + O(\nu^2)\) instead of \(k^+ = k_e^0 + O(\nu)\), what one should a priori expect. This is the most striking result of this part of the analysis. Furthermore we see, as was already observed, that \(R^+ < R_e^0\), because \(u_{10} > 0, U_0 < 0\).

**The resonant problem**

Now, we perform an analogous analysis for the resonant problem. As explained in the preceding, the resonant problem is characterized by the fact that \(p\) satisfies

\[ p = -\frac{2k_e^0}{N} + \eta \nu^\beta \]  
(4.3.23)

for some integer \(N\) and \(\eta, \beta \in (\text{see (4.3.13)})\). The factor \(\beta\) measures the detuning from resonance; zero-detuning (i.e. \(\beta \to \infty\)) corresponds to a situation which we will call *exact* resonance. The crucial point in this subsection is that under the resonant conditions, the splitting of \(\omega\) in \(\omega^+\) and \(\omega^-\) occurs in the neighbourhood of \(k_e^0\). This means that we expect a priori two corrections on the critical conditions \((k_e^0, R_e^0)\), one belonging to \(\omega^+\) (and denoted by \((k^+, R^+)\)) and one belonging to \(\omega^-\) (denoted by \((k^-, R^-)\)). However, it turns out that this is only the case for \(\beta > 1/2\). For \(\beta < 1/2\), the situation resembles...
the non-resonant case, as we will show. The results of the forthcoming analysis can be summarized as follows:

\[ k^+ = k_c^0 + O(\nu^{2-3\beta}) \text{ for } 0 < \beta < \frac{1}{2} \]  \hspace{1cm} (4.3.24)

\[ k^\pm = k_c^0 + O(\nu^\beta) \text{ for } \frac{1}{2} < \beta < 2 \]  \hspace{1cm} (4.3.25)

\[ k^\pm = k_c^0 + O(\nu^2) \text{ for } \beta > 2 \]  \hspace{1cm} (4.3.26)

and furthermore, we will see that for \( \beta > \frac{1}{2} \)

\[ p = \frac{2k^\pm}{N} + O_s(\nu^2) \]  \hspace{1cm} (4.3.27)

The implication of (4.3.27) is, that for \( \frac{1}{2} < \beta < 2 \), the system gets ‘better in resonance’. Herewith we mean, that \( k^\pm \) satisfies resonance condition (4.3.13) more accurate than \( k_c^0 \) does. On the other hand, is can be seen that there is never perfect resonance for the new critical conditions, i.e. we will never reach a situation where \( k^\pm \) satisfies resonance condition (4.3.13) exactly, no matter how close we choose \( p \) in a neighbourhood of \( k_c^0 \). It is even so that if \( k_c^0 \) is ‘closer than \( O(\nu^2) \)’ to perfect resonance, the system ‘pushes’ the new critical conditions \( O_s(\nu^2) \) away from resonance. In other words the term \( \sum F_i \exp(ilpx) \) has always \( O_s(\nu^2) \) detuning with respect to the linear most critical wave \( \exp(i k^\pm x) \sum \psi^\pm_l \exp(ilpx) \).

Especially in the nonlinear analysis, we will find that it is more convenient (if not necessary) to base the resonance condition on \( k^\pm \) instead on \( k_c^0 \). Therefore, we will call a problem resonant if \( p \) is such that (4.3.27) is satisfied.

The starting point of the analysis to find the expressions for \( k^\pm \) and \( R^\pm \) will be the assumption:

\[ k^\pm = k_c^0 + K\nu^{\alpha_1} \]  \hspace{1cm} (4.3.28)

\[ R^\pm = R_c^0 + R\nu^{\alpha_2}. \]  \hspace{1cm} (4.3.29)

First of all, in Appendix A, section A.2, we use a significant degeneration argument to show that for \( 2\beta < 1 \), the right choice for \( \alpha_1 \) and \( \alpha_2 \) is: \( \alpha_1 = 1 - \beta, \alpha_2 = 2\alpha_1 \), while for \( 2\beta > 1 \) we must choose \( \alpha_1 = 1/2, \alpha_2 = 2\alpha_1 \). Instead of describing the subcases \( 0 < \beta < 1/2 \) and \( \beta > 1/2 \) in detail, we prefer to give the main results (the higher order corrections on \( k^\pm, R^\pm \)) for each subcase, and refer to Appendix A for detailed calculations. The first subcase we consider is \( 0 < \beta < 1/2 \). In Appendix A, it is shown that we find for \( \psi_N \), the \( N \)-th coefficient of the linear solution, (see (4.3.1)):

\[ \psi_N = \frac{F_N}{u_{01}} \nu^{1-2\beta} + h.o.t. = O(\nu^{1-2\beta}) \]  \hspace{1cm} (4.3.30)

where the coefficients \( u_{ij}^{(l)} \) are again related to the expansion of \( \mu(k + lp) \) (see (4.3.17)). Using (4.3.30), we find:

\[ R^+ = R_c^0 + \frac{-F_{-N}\psi_N}{u_{10}} \nu + h.o.t. = R_c^0 + O(\nu^{2\alpha_1}) \]  \hspace{1cm} (4.3.31)

\[ k^+ = k_c^0 + \frac{u_{10}R^+}{u_{02} \eta} \nu^{2-3\beta} + h.o.t = k_c^0 + O(\nu^{2-3\beta}) \]  \hspace{1cm} (4.3.32)
The coefficients $u_{ij}$ are given in (4.3.16), $\eta$ is given in (4.3.23) Besides, we find as approximations for $\omega^+$ (for $|k - k^+| = o(1), |R - R^+| = o(1))$

$$\omega^+(k, R) = \mu(k, R) + \nu F_N \psi_N + h.o.t. = \mu(k, R) + \frac{|F_N|^2}{u_0^2} \nu^2 - 2\beta + h.o.t. \quad (4.3.33)$$

Using (4.3.30), we observe that

$$\mu(k + Np, R) = -\nu F_N \psi_N + h.o.t. = O(\nu^2) \quad (4.3.34)$$

Being in resonance, $\beta \neq 0$, and thus we see from (4.3.34) that $\mu(k + Np, R)$ becomes small. The terms $\mu(k + lp, R), l \neq 0, N$ remain $O(1)$ (as can be seen from (4.3.17). These two facts become important in the resonant part of the nonlinear theory. In the limit $\beta \to 0$ (which means that we ‘move away’ from resonance), we see that:

$$\psi_N = O(\nu) \quad (4.3.35)$$
$$R^+ = R^o + O(\nu^2) \quad (4.3.36)$$
$$k^+ = k^o + O(\nu^2) \quad (4.3.37)$$
$$\mu(k + Np, R) = O(1) \quad (4.3.38)$$

which is in agreement with the non-resonant theory (see e.g. (4.3.18) and (4.3.19). Furthermore, it is shown in that

$$k^+ + Np = -k^+ + O(\nu^\beta) \quad (4.3.39)$$

and thus, the system remains equally good in resonance.

Then, we look at the case $\beta > 1/2$. The results for $\psi^\pm, R^\pm$ and $k^\pm$ are found to be at leading order (see Appendix A):

$$\psi_N^\pm = \pm \frac{F_N}{|F_N|} \quad (4.3.40)$$
$$R^\pm = R^o + \frac{-F_N \psi_N^\pm}{u_0^2} \nu \quad (4.3.41)$$
$$k^\pm = k^o - \frac{1}{2} \nu^\beta + \frac{1}{2 \mu_2} \left( -\psi_N^\pm (a_2 + b_2) - (a_3 + b_3) \right) \nu^2 + O_s(\nu^{2\beta}) \quad (4.3.42)$$

where $a_2, a_3, b_2, b_3$ depend on $F_j$ and $\mu$ and are defined in Appendix A. Note that we get a two new critical conditions, $(k^\pm, R^\pm)$, belonging to the minimum of $\omega^\pm = 0$. In Appendix A, is is also shown that

$$p = \frac{2k^\pm}{N} + O_s(\nu^2, \nu^{2\beta}) \quad (4.3.43)$$

Thus, we see that for $1/2 < \beta < 2$, the system gets better in resonance. We can give a geometrical meaning to this conclusion, with which we also denote the most important difference between the case $\beta < 1/2$ and $\beta > 1/2$. Therefore, we look at figure 4.3 where we have sketched the curves $\mu(k, R^+)$ and $\mu(k + Np, R^+)$ as function of $k$. For $\beta < 1/2$ (see figure 4.3a), we have that $k^+ = k^o + O(\nu^2)$, and of course, $\omega^+(k^+, R^+) = 0$. Due to
4.4 Nonlinear theory of the perturbed problem

In the nonlinear stability theory we will consider the nonlinear evolution of small solutions of (4.1.1) for $R$ close to critical, i.e. $R^+$. Therefore, we introduce another small parameter $\varepsilon$ and the rescaled control parameter $r$ by $R - R^+ = r\varepsilon^2$: $\varepsilon$ measures the magnitude of
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R \mp R^\pm and r 'controls' R \mp R^\pm within the region R \mp R^\pm = O(\varepsilon^2). Thus, if p = O(1) there are now two small parameters, \varepsilon and \nu; the relative magnitudes of \varepsilon and \nu will cause different types of modulation equations.

Based on the linear theory one models a small solution of (4.1.1) as being a 'modulation' of the 'linearly most critical wave' (see van Harten (1991), Eckhaus (1993)). In other words, one assumes that small solutions of (4.1.1) will look like the unstable solutions found in the linear theory. Being in resonance or not has a significant influence on the structure of these marginally unstable waves. We will analyse this structure in the next section.

4.4.1 The structure of the bifurcating waves

First, we consider the non-resonant case. This is the easiest case, because we do not have to take into account the second neutral stability curve \omega^-. The linear most unstable waves are already given in (4.3.1) Thus, we take as 'basis' for the nonlinear theory:

\Psi = \hat{\varepsilon}A(\xi,\tau) \left( \sum (\psi_l)_c e^{i\xi \pm x} \right) e^{ik^\pm x} + c.c. \tag{4.4.1}

(\psi_l)_c are the coefficients given in (4.3.5) and evaluated at critical conditions. Usually, the magnitude of the perturbation, which we denote by \hat{\varepsilon} is taken to be equal to \varepsilon. We will specify \hat{\varepsilon} whenever necessary.

Then, we consider the resonant case, which is much more subtle. In section 4.3, it is explained that in resonance, we actually determine two new critical conditions (k^\pm, R^\pm), which are both corrections on (k_c^0, R_c^0). As we are looking at bifurcating solutions from the basic state, at first only (k^+, R^+) is of interest, because R^+ < R^-, and the basic state looses its stability at R^+. However, one should realise that |R^+ - R^-| = O(\nu). Therefore, if the bifurcation parameter R is increased O(\nu) above R^+, we must also take into account the second bifurcation point (k^-, R^-) and should consider perturbations of the following form:

\Psi = \hat{\varepsilon} \left( A^+ (\xi,\tau) e^{ik^+ x} \sum \psi_l^+ e^{i\xi \pm x} + c.c. + A^- (\xi,\tau) e^{ik^- x} \sum \psi_l^- e^{i\xi \pm x} + c.c. \right) + h.o.t. \tag{4.4.2}

where the first part of the solution becomes (linear) unstable at (k^+, R^+) and the second part at (k^-, R^-). Note that we introduce two amplitude functions, A^+ and A^-.

Let us consider the expression for the perturbation (4.4.2) more thoroughly. First, we recall that (for \beta > 1, see the Remark at the end of section 4.3.2):

k^\pm = k_c^0 - \frac{1}{2} \eta \nu^\beta + O_s(\nu^2) = \hat{k} + O_s(\nu^2) \tag{4.4.3}

\hat{k} = k_c^0 - \frac{1}{2} \eta \nu^\beta \tag{4.4.4}

|k^+ - k^-| = O_s(\nu^2) \tag{4.4.5}

Thus, up to O(\nu^2) terms, the two solutions bifurcate at the same wave number \hat{k} and for different values of R. It should be noted that \Psi (see (4.4.2)) is the sum of two quasi periodic functions in x, one with frequencies k^+, p and the other one with frequencies k^-, p.
4.4 Nonlinear theory of the perturbed problem

Figure 4.4: Representation of some modes of the solution $\Psi$. The solid lines denote the peaks at $k^+$ and $-k^+$, while the dotted lines denote the 'induced' peaks at $k^+ - Np$ and $-k^+ + Np$. The width of the peaks is $O(\varepsilon)$, while the distance between $k^+$ and $-k^+ + Np$ is $O(\nu^2)$. Thus, for $\varepsilon > \nu^2$, the peaks at $k^+$ and $-k^+ + Np$ (and of course at $-k^+$ and $k^+ - Np$ etc) coincide.

The detuning is always of order $O_s(\nu^2)$. In Fourier space, the first part of $\Psi$ can be visualised by means of peaks located at $\pm k^+$, and, 'induced' by $\sum F_l e^{i lp x}$, peaks at $\pm k^+ + lp$ for all $l$ (see also figure 4.4). Due to the scaling $\xi = \varepsilon x$, the width of all the peaks is $O(\varepsilon)$. However, in resonance we have:

$$|(\pm k^+ + lp) - (\mp k^+ + (l - N)p)| = O_s(\nu^2)$$

Thus, if $\varepsilon > \nu^2$, the peaks at $(\pm k^+ + lp)$ and $(\mp k^+ + (l - N)p)$ can no longer be distinguished (with respect to $\varepsilon$). In other words, the complex conjugated part of $A^+$ in $\Psi$ contains modes that are already contained in the part of $A^+$ itself. The same way of reasoning holds of course for the part of $\Psi$ involving $A^-$. Therefore, for $\varepsilon > \nu^2$, we must rewrite (4.4.2) and it turns out that $\Psi$ becomes periodic in $x$ (instead of quasi-periodic). The detuning that causes the quasi-periodicity of $\Psi$ is 'incorporated' in the $\xi$-length scale. This reasoning is clarified as follows. Suppose $\varepsilon > \nu^2$, and write $k^\pm = \hat{k} + \sigma^\pm \nu^2$, $p = \hat{p} + \sigma \nu^2$ for some $\sigma^\pm, \sigma = O(1)$ and $\hat{p} = -2\hat{k}/N$. Then

$$\Psi = \varepsilon (A^+(\xi, \tau) e^{i k^+ x} \sum \psi_1^+ e^{i px} + c.c. + A^-(\xi, \tau) e^{i k^- x} \sum \psi_1^- e^{i px} + c.c.)$$

$$= \varepsilon \left( e^{i \hat{k} x} e^{i \sigma^+ \nu^2} \xi \sum A^+ \psi_1^ e^{i \frac{p x}{\varepsilon}} e^{i \xi px} + c.c. 
+ e^{i \hat{k} x} e^{i \sigma^- \nu^2} \xi \sum A^- \psi_1^- e^{i \frac{p x}{\varepsilon}} e^{i \xi px} + c.c. \right)$$

We make two observations. First, we note that the evolution on the length scales $\frac{\sigma^\pm \nu^2}{\varepsilon} \xi$ and $\frac{\sigma \nu^2}{\varepsilon} \xi$ is to be considered slow with respect to the evolution on the $\xi$-scale (because
\[ \epsilon \gg \nu^2 \), and therefore \( \psi_{\pm}^{+} \exp(\pm i \nu^2 / \epsilon) \) and \( \psi_{\pm}^{-} \exp(\pm i \nu^2 / \epsilon) \) can be considered as a constant on the \( \xi \)-scale. A second observation is a relation between the Fourier coefficients of \( \psi \) (see (4.4.8)). In deriving it we use the first order of the expression for \( \psi \), given in (4.3.4) and the fact that apart from \( \psi_0 \) (which equals unity) also \( \psi_N \) is of order unity (see (4.3.40)). Furthermore, we use that \( F = F_{-1} \) and \( \mu(k + lp) = \mu(k + (N - l)p) \). The relation reads:

\[
\psi_{\pm}^{+} \psi_{\pm}^{-} = -\nu \left( F_{l} + F_{-N} \psi_{N}^{+} \right) \psi_{N}^{-} \frac{1}{\mu(k + lp)}
\]

and is valid up to \( O(\nu^2) \) terms. Taking these two observations into account, we can rewrite (4.4.7):

\[
\Psi = \hat{\epsilon} \left( e^{ikx} \sum A^+ \psi_{l}^+ e^{il\hat{\psi}^+x} + e^{ikx} \sum A^- \psi_{l}^- e^{il\hat{\psi}^-x} + c.c. \right)
\]

\[
= \hat{\epsilon} \left( e^{ikx} \sum \left[ A^+ \psi_{l}^+ + A^- \psi_{N-l}^+ \right] e^{il\hat{\psi}^+x} + e^{ikx} \sum \left[ A^- \psi_{l}^- + A^- \psi_{N-l}^- \right] e^{il\hat{\psi}^-x} \right)
\]

\[
= \hat{\epsilon} \left( B^+ e^{ikx} \sum \psi_{l}^+ e^{il\hat{\psi}^+x} + B^- e^{ikx} \sum \psi_{l}^- e^{il\hat{\psi}^-x} \right)
\]

(4.4.9)

where \( B^+ = A^+ + \psi_{N}^+ A^+ \) and \( B^- = A^- + \psi_{N}^- A^- \). Note that \( \Psi \) has now become a periodic function in \( x \). The detuning between \( \exp(ik\hat{x}) \) and \( \exp(il\hat{\psi}x) \) is zero (because \( \hat{\psi} = -2k/N \)).

Let us look closer to \( B^+ \) and \( B^- \). From Appendix A, it follows that \( |\psi_{N}^\pm| = 1 + O(\nu^2) \). Thus we find that, although \( A^\pm \) can be chosen anywhere in \( \hat{\epsilon} \), we observe that this is not the case for \( B^\pm \):

\[
B^+ = b^+ e^{-\frac{1}{2}i\gamma + O(\nu)} \text{ for some } b^+ \in \mathbb{C}
\]

(4.4.10)

\[
B^- = ib^- e^{-\frac{1}{2}i\gamma + O(\nu)} \text{ for some } b^- \in \mathbb{C}
\]

(4.4.11)

and \( \gamma = arg(\psi_{N}) \). Hence, we see that in resonance, and for \( \epsilon \gg \nu^2 \), the bifurcating solution (which was arbitrary complex for \( \epsilon \ll \nu^2 \)) is now split into two parts with amplitudes \( B^+ \) and \( B^- \). They live on two perpendicular lines in the complex plane, \( V^+ \) and \( V^- \). The angle of the line with the positive real axes is related to \( \psi_{N}^+ \) which is proportional with the \( N \)-th Fourier mode of the geometrical forcing. It is easy to see that for the case of real Fourier components \( F_l \), we have that \( B^+ = \text{Re}(A) \) and \( B^- = i \text{Im}(A) \).

Summarizing, we see that in resonance, the area of interest is divided into two main parts. The first part is \( \epsilon \ll \nu^2 \). In that case the bifurcating solutions can be written as

\[
\Psi = \hat{\epsilon} A e^{ikx} \sum \psi_{l}^+ e^{il\hat{\psi}^+x} + c.c. + h.o.t.
\]

(4.4.12)

which is quasi-periodic in \( x \) and becomes linearly unstable at \((k^+, R^+)\). The second bifurcation point \((k^-, R^-)\) does not play a role yet. The complex conjugated part of the solution can not be written in the form of the original modes (in other words, the peaks in the Fourier space do not coincide yet), and the splitting of the solution into two parts does not occur. Note that the amplitude \( A \) is arbitrary complex.
For \( \varepsilon \gg \nu^2 \) the solution should be written as (according to the reasoning given in the preceding):

\[
\Psi = \hat{\varepsilon} B^+ e^{ikx} \sum \psi^+_i e^{i\tilde{\rho}x} + \hat{\varepsilon} B^- e^{ikx} \sum \psi^-_i e^{i\tilde{\rho}x} + \text{h.o.t.} \quad (4.4.13)
\]

with \( \tilde{\rho} = 2\hat{k}/N \) and \( \hat{k} \) the first order part of \( k^\pm \). This solution is periodic in \( x \). If \( \varepsilon = O(\nu^2) \), we are in a transition area from the first to the second case (see section 4.4.3, case 3).

For both the resonant case and the non-resonant case, we are then ready to derive amplitude equations for \( A \) (non-resonant case) and \( B^+ \) and \( B^- \) (resonant case). Again, the non-resonant case is ‘regular’, the resonant case is much more interesting.

### 4.4.2 The amplitude equation for the non-resonant problem

As before, this case is characterized by the fact that for every \( N \), we have:

\[
|\tilde{p} - 2\hat{k}/N| = O(1) \quad (4.4.14)
\]

First, we introduce some notation, which is necessary for the substitution of an Ansatz like (4.2.6) into (4.1.1). We expand the linear operators \( L_R, P \) and \( Q \) with respect to \( \varepsilon \).

\[
L_R = L_{Rc + r\varepsilon^2}(\frac{\partial}{\partial x}) + r\varepsilon^2 L_{2c}(\frac{\partial}{\partial x}) + O(\varepsilon^4) \quad (4.4.15)
\]

and decompose further by substituting \( \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \xi} \) for \( \frac{\partial}{\partial x} \)

\[
L_c = \sum_{m=0}^{M} \varepsilon^m L^m (\frac{\partial}{\partial x}) \frac{\partial^m}{\partial \xi^m} \quad (4.4.16)
\]

for some \( M > 0 \). Note that \( L^0(e^{ikx}) = \mu(k, R^c) \). In general, we define

\[
L^m(e^{ikx}) = \mu^{(m)}(k)e^{ikx}, \quad (\text{thus } \mu(k, R^c) = \mu^{(0)}(k)) \quad (4.4.17)
\]

Similar expansions can be made for \( L_{2c} \) (introducing the operators \( L^m_2 \), and the expressions \( \mu^{(m)}(k) \)) and for \( P \) and \( Q \). As Ansatz for the nonlinear theory, we model the perturbation of the basic state by assuming that it is centered around the most unstable wave at the new critical conditions \( R = R^+, k = k^+ \) (For convenience, the upper index + is omitted hereafter). For the non-resonant case, \( E \) is given by (4.4.1), where we should formally put \( \tilde{\varepsilon} \equiv 1, A(\xi, \tau) \equiv 1 \). The magnitude of the perturbation is denoted by \( \tilde{\varepsilon} \), and we scale \( \tilde{\varepsilon} = \varepsilon \).

Thus, \( \Psi \) can be written as:

\[
\Psi(x, t) = \varepsilon^2 \psi^{02} + \varepsilon^3 \psi^{03} + \ldots + E[\varepsilon \psi^{11} + \varepsilon^2 \psi^{12} + \varepsilon^3 \psi^{13} + \ldots] + \text{c.c.} + E^2[\varepsilon^2 \psi^{22} + \varepsilon^3 \psi^{23} + \ldots] + \text{c.c.} + E^3[\varepsilon^3 \psi^{33} + \ldots] + \text{c.c.} + \ldots \quad (4.4.18)
\]

The notation in (4.4.18) is such that the superscript \( ij \) denotes the level \( E^i \varepsilon^j \). As usual, we substitute (4.4.18) into the full nonlinear, perturbed model equation (4.1.1) and compare
equal orders of $\varepsilon^i E^j$. At $i = 1, j = 1$ we find an equation for $\psi^{11}$ which resembles the linear problem, i.e.

$$\left(-L^{(0)} - \nu \sum F_i e^{ilpx}\right) E\psi^{11} = 0$$  \hfill (4.4.19)

Hence, we deduce (since operator $L^{(0)}$ does not depend on the slow variables $\xi$ and $\tau$):

$$\psi^{11} = A(\xi, \tau)$$  \hfill (4.4.20)

As usual, the aim of the nonlinear theory is to derive an amplitude equation for $A(\xi, \tau)$. As we are considering the non-resonant problem, we would expect a priori that $A(\xi, \tau)$ satisfies the Ginzburg-Landau equation (4.2.8) as derived in section 4.2, with an $o(1)$ correction on the coefficients. To be more precise, we expect that the amplitude equation looks like

$$A_\tau = r \omega_R A - \frac{1}{2} \omega_{kk} A_{\xi\xi} + \beta (1 + o(1)) |A|^2 A$$  \hfill (4.4.21)

Note, that in the perturbed case, $\mu(k, R)$ is not equal to $\omega(k, R)$, in contrast to the unperturbed problem. Instead, we see from (4.3.6) that $\omega(k, R) = \mu(k, R) + O(\nu^2)$, so that indeed all the coefficient of (4.4.21) have an $o(1)$ correction, compared with the unperturbed case. Below, we will only give a sketch of the analysis; the detailed calculations are given in Appendix B.

The derivation of the amplitude equation goes always along the same lines. One substitutes (4.4.18) into the model problem and derives equations on the various levels $\varepsilon^i E^j$ of the form

$$\left(L^{(0)} + \nu \sum F_i e^{ilpx}\right) E^j \psi^{ij} = \chi^{ij}$$  \hfill (4.4.22)

which have to be solved. As usual, the $x$-dependence is already incorporated in the structure of the linear solution which, we recall, is quasi-periodic, $\sum \psi_i e^{ilpx}$. Thus, at every level of $\varepsilon^i E^j$, we should consider problem (4.4.22) for every Fourier mode $e^{i(k^+ + lp)x}$; these subproblems can all be solved uniquely. There is one exception: the equation for $l = 0$ on the $\varepsilon^1 E$-level. There, we encounter an algebraic expression where the left hand side vanishes at critical conditions. Thus, the corresponding right hand side must also vanish and as a result, we derive at level $\varepsilon^3 E$ the first order of the amplitude equation. The detailed calculations are given in Appendix B. Here, we only give the results, the first order (in $\varepsilon$) of the amplitude equation with the coefficients given up to $O(\nu^2)$:

$$A_\tau = \omega_R A - \frac{1}{2} A_{\xi\xi} \omega_{kk} + |A|^2 A \left(Q_0^{13} (1 - \nu^2 \sum \frac{|F_i|^2}{\mu(k_c + l p, R_c)^2}) - \nu \sum_{s + t = 0} \frac{F_i Q_i^{13}}{\mu(k_c + l p, R_c)}\right)$$  \hfill (4.4.23)

The coefficients $Q_i^{13}$ are given in Appendix B, eq. (B.25); $Q_0^{13}$ equals in first order the Landau coefficient belonging to the unperturbed problem (see (4.2.9)); the first order correction of $Q_0^{13}$ is of order $\nu^2$. Recall that $Q_i^{(13)} = O(\nu)$ for $t \neq 0$. Therefore, the last term in (4.4.23) is also $O(\nu^2)$. Thus, as mentioned before, all the coefficients of the Ginzburg-Landau equation (4.4.23) have an $O(\nu^2)$ correction with respect to the unperturbed case. Note that the correction of the nonlinear term is complex because the Fourier coefficients $F_i$ are in general complex and the coefficients $Q_i^{(13)}$ are in general not proportional to $F_i$. 


4.4.3 Amplitude equations for the resonant problem

In the resonant problem we must be very careful about the structure of the bifurcating waves. This has already been explained in section 4.4.1. There, it is also explained that there are three subcases to consider, \( \varepsilon < \nu^2 \), \( \varepsilon > \nu^2 \) and \( \varepsilon = O(\nu^2) \). We consider these cases in the sections below.

Case 1: \( \varepsilon < \nu^2 \)

In this case, the analysis goes along the same lines as the analysis of the nonresonant case, described in section 4.4.2. The main difference however, is that for the coefficient \( \psi^+_N \) of the linear solution holds: \( \psi^+_N = F_N/|F_N| = O(1) \), whereas it was \( O(\nu) \) in the nonresonant case (recall again that \( \psi^+_N \) is the \( N \)-th coefficient in the Fourier series of the linear solution, see (4.3.1), and depends (in first order) only on the \( N \)-th Fourier coefficient of the predescribed geometry). As a result, we will see that some of the coefficients that we encounter in the higher order analysis also become \( O(1) \) and even \( O(1/\nu) \).

We follow the line of reasoning of the nonresonant case and assume an Ansatz of the form (4.4.18):

\[
\Psi(x, t) = \varepsilon^2 \psi^{02} + \varepsilon^3 \psi^{03} + \ldots \\
+ E[\varepsilon \psi^{11} + \varepsilon^2 \psi^{12} + \varepsilon^3 \psi^{13} + \ldots] + c.c. \\
+ E^2[\varepsilon^2 \psi^{22} + \varepsilon^3 \psi^{23} + \ldots] + c.c. \\
+ E^3[\varepsilon^3 \psi^{33} + \ldots] + c.c. \\
+ \ldots
\]  

(4.4.24)

Now, we should take \( E \) as given in (4.4.12), because we are in resonance. We substitute (4.4.24) into the full nonlinear model problem (4.1.1), compare like powers of \( \varepsilon^i E^j \), solve at various levels the resulting equations. As we have seen before, this will not be possible at level \( \varepsilon^3 E \), which results in an amplitude equation for \( A \). As the analysis goes along the same lines as described in section 4.4.2, we omit the details and emphasize only the differences.

The solution for the coefficients \( \psi^{02}_0 \) of \( \psi^{02} \) is given in (B.18). Due to the fact that \( \psi^+_N = O(1) \), we have that

\[
\psi^{02}_0 = \frac{-1}{\mu(0)} \left( 2 + 2|\psi^+_N|^2 \right) + h.o.t. \\
\psi^{02}_N = \frac{-2\psi^+_N}{\mu(Np)} + h.o.t. = O(1) \\
\psi^{02}_{-N} = \frac{-2\psi^+_N}{\mu(Np)} + h.o.t. = O(1)
\]  

(4.4.25) (4.4.26) (4.4.27)

Thus, with respect to the nonresonant case, we observe that \( \psi^{02}_0 \) gets an extra term \( 2|\psi^+_N|^2 \) and \( \psi^{02}_N \) and \( \psi^{02}_{-N} \) become \( O(1) \).

A more or less similar result is found for the coefficients \( \psi^{22}_N \) of \( \psi^{22} \) (see Appendix B):

\[
\psi^{22}_N = \frac{-2\psi^+_N}{\mu(2k^+ + Np)} + h.o.t.
\]  

(4.4.28)
\( \psi_{22}^{2N} = -\frac{(\psi_{N}^{+})^2}{\mu(2k^+ + 2Np)} + h.o.t. \)  

(4.4.29)

So, with respect to the nonresonant case, \( \psi_{N}^{22} \) and \( \psi_{22}^{2N} \) become \( O(1) \).

Proceeding with the \( \varepsilon^3 E \)-problem, we find the analogon of equation (4.4.23) and (B.28):

\[-A_{\tau} + r_{\mu} R A - \frac{1}{2} A_{\xi \xi} \mu_{kk} + |A|^2 A \left[ Q_0^{13} + \nu \sum_{s+t=0} F_s \psi_t^{13} \right] = 0 \]  

(4.4.30)

and

\[ \psi_t^{13} = \frac{- (GL)_l - \nu \sum_{s+t=l} F_s \psi_t^{13}}{\mu (k^+ + lp)} = O(\nu) \text{ for } l \neq N \]  

(4.4.31)

For \((GL)_l\) and \(Q_0^{13}\), we refer to Appendix B, where the quantities have been derived for the nonresonant case. The expressions however, are also valid in the resonant case, although it might be possible that the order of magnitude of for instance \((GL)_l\) for the resonant and nonresonant case, differs in magnitude. This is due to the fact that the expressions in (4.4.25)-(4.4.29) differ in magnitude for the resonant and nonresonant case. It is crucial to realize that in resonance, \( \mu(k^+ + Np) = \nu |F_N| \) and \((GL)_N = O(1)\), hence \( \psi_{N}^{13} = O(1/\nu)\). This means that we get an additional \( O(1) \) correction in (4.4.30), namely \( F_{-N} \psi_{N}^{13} \). Tidious calculations (involving elaboration of \( \mu_R \) into \( \omega_R + \mu_{kk} \) into \( \omega_{kk} \)) show that we end up with the following equation for \( A \) up to \( O(\nu) \) terms:

\[ A_{\tau} = r_{\omega_R} A - \frac{1}{2} \omega_{kk} A_{\xi \xi} + 3 \beta |A|^2 A \]  

(4.4.32)

where \( \beta \) is the unperturbed Landau-coefficient given in (4.2.9). One should not get confused about the coefficient in front of the nonlinear term, which is three times the unperturbed coefficient. In the unperturbed case, the structure of the bifurcating wave is just a modulation of \( \exp(ik^c x) \). Now, we modulate a wave which reads in first order \( \exp(ik^+ x) + F_N/|F_N| \exp(i(k^+ + Np)x + c.c.) \) and has obviously a different structure. Therefore, there is no reason to expect that the nonlinear coefficient equals the nonlinear coefficient of the unperturbed case.

**Case 2: \( \varepsilon \gg \nu^2 \)**

In this case, \( \varepsilon \) is so large that the structure of the bifurcating waves has become

\[ \Psi = \varepsilon \left( B^+ e^{ik^c \xi} \sum \psi_t^{+} e^{i\pi x} + B^- e^{ik^c} \sum \psi_t^{-} e^{i\pi x} \right) \]  

(4.4.33)

with \( B^+ \) and \( B^- \) essentially real amplitudes (they ‘live’ on straight lines in the complex plane, see section 4.4.1); they are scaled with \( \varepsilon \). This case should actually be divided into several subcases, for we will see that for increasing \( \varepsilon \) (relative to \( \nu \)), we will pass a number of degenerations. We will discuss the significant degenerations when we encounter them in the analysis. The first part of the solution (4.4.33) becomes linearly unstable at \((k, R^+)\), the second part at \((k, R^-)\). In performing a nonlinear analysis in order to find amplitude equations for \( B^+ \) and \( B^- \), we can no longer assume (4.4.18) as structure of the solutions,
because now, \( \Psi \) is periodic instead of quasi periodic (in \( x \)). Therefore, we assume as as structure of the solutions:

\[
\Psi = \epsilon \Psi^{(1)} + \epsilon^2 \Psi^{(2)} + \epsilon^3 \Psi^{(3)} + \ldots
\]  

(4.4.34)

and expect that the analysis at level \( O(\epsilon^3) \) leads to amplitude equations for \( B^+ \) and \( B^- \). To determine \( \Psi^{(2)} \), we have to take into account the nonlinear interaction of \( \Psi^{(1)} \) and itself. As before, we write

\[
N(\Psi^{(1)}, \Psi^{(1)}) = (B^+)^2 e^{2ikx} \sum_{s+t=l} \psi_s^+ \psi_t^+ P_1(\hat{k} + s \hat{p}) P_2(\hat{k} + t \hat{p}) e^{il \hat{p} x} + \ldots
\]

and

\[
(\Psi^{(1)})^2 = (B^-)^2 e^{2ikx} \sum_{s+t=l} \psi_s^- \psi_t^- P_1(\hat{k} + s \hat{p}) P_2(\hat{k} + t \hat{p}) e^{il \hat{p} x} + \ldots
\]

(4.4.35)

Note that \((B^+)^2\) and \((B^-)^2\) belong to the same one dimensional subspace in (see (4.4.10)-(4.4.11), while \(B^+B^-\) belongs to a one dimensional subspace which is orthogonal to the previous.

At this point in the analysis, we meet the first degeneration: we have to make a distinction between \( N \) is even and \( N \) is odd. Suppose \( N \) is even. Then we can rewrite \( N(\Psi^{(1)}, \Psi^{(1)}) \) in such a way that the structure of \( N(\Psi^{(1)}, \Psi^{(1)}) \) is the same as the structure of the linear solution, i.e.

\[
N(\Psi^{(1)}, \Psi^{(1)}) \sim e^{ikx} \sum \hat{\psi}_l e^{il \hat{p} x}
\]

(4.4.36)

for some \( \hat{\psi}_l \). This can be seen by splitting \( \exp(2ikx) \) into \( \exp(ikx) \exp(ikx) \), bringing \( \exp(ikx) \) into the summation, rewriting \( \exp(il \hat{p} x + \hat{k} x) \) as \( \exp(i(l - N/2) \hat{p} x) \) and resum over the index \( l = l - N/2 \). Then, \( N(\Psi^{(1)}, \Psi^{(1)}) \) can be written as (omitting \( \sim \)):

\[
N(\Psi^{(1)}, \Psi^{(1)}) = (B^+)^2 e^{ikx} \sum_{s+t=l+\frac{N}{2}} \psi_s^+ \psi_t^+ P_1(\hat{k} + s \hat{p}) P_2(\hat{k} + t \hat{p}) e^{il \hat{p} x} + \ldots
\]

\[
(\Psi^{(1)})^2 = (B^-)^2 e^{ikx} \sum_{s+t=l+\frac{N}{2}} \psi_s^- \psi_t^- P_1(\hat{k} + s \hat{p}) P_2(\hat{k} + t \hat{p}) e^{il \hat{p} x} + \ldots
\]

(4.4.37)

for \( \hat{\psi}_l \) which can now be derived from (4.4.37). Thus, already in second order, we find amplitude equations for \( B^+ \) and \( B^- \). These equations contain quadratic terms in \( B^+ \) and
For $N$ odd, the process described above can not be performed (an index $l = -N/2$ is meaningless for $N$ odd) and thus we should proceed the analysis up to order $O(\varepsilon^3)$ as usual. We consider the two subcases $N$ even and $N$ odd in some more detail, starting with the latter case.

- **$N$ odd**

This is the easiest case. As usual, we rescale time and spatial coordinates by $\tau = \varepsilon^2 t$ and $\xi = \varepsilon x$, where $\varepsilon = O(\sqrt{R-R^+})$. The equation arising at $O(\varepsilon^2)$ can be written as follows:

$$
(-L^{(0)} - \nu \sum F_k e^{i\phi_k}) \Psi^{(2)} = L^{(1)} \Psi^{(1)} + N(\Psi^{(1)}, \Psi^{(1)})
$$

This equation can be solved along the lines of section 4.4.2. Naturally, the solution consists out of two parts; one part that solves eq. (4.4.38) for $L^{(1)} \Psi^{(1)}$ and another part that solves the equation for $N(\Psi^{(1)}, \Psi^{(1)})$. These solutions can be found in a straightforward way. Then, we proceed with the analysis at $O(\varepsilon^3)$. At that level, we encounter the following equation:

$$
(-L^{(0)} - \nu \sum F_k e^{i\phi_k}) \Psi^{(3)} = L^{(2)} \Psi^{(1)} + L^{(1)} \Psi^{(2)} + rL^{(0)}\Psi^{(1)} + \frac{\partial}{\partial \tau} \Psi^{(1)} + N(\Psi^{(1)}, \Psi^{(2)})
$$

Now, the right hand side of (4.4.39) contains modes, proportional to the linear solution and we observe again that for $l = 0$ the left hand side of (4.4.39) vanishes and thus, the corresponding right hand side should also vanish for $l = 0$. This yields (after some elaboration) two amplitude equations for $B^+$ and $B^-$ which are found to be at leading order (the coefficients are only given in first order in $\nu$):

$$
B^+_\tau = r^+ \omega^+ R B^+ - \frac{1}{2} \omega^+_{kk} B^+_{\xi\xi} + \beta B^+ \left( (B^+)^2 \psi^+_N + (B^-)^2 \psi^-_N \right)
$$

$$
B^-_\tau = (r^+ - \delta) \omega^- R B^- - \frac{1}{2} \omega^-_{kk} B^-_{\xi\xi} + \beta B^- \left( (B^+)^2 \psi^+_N + (B^-)^2 \psi^-_N \right)
$$

where $R=R^+, \delta = (R^+ - R^-)/\varepsilon^2 = O(\nu/\varepsilon^2)$ and $\omega^+_R = \omega^+_R(\hat{k}, R^+)$, $\omega^-_{kk} = \omega^-_{kk}(\hat{k}, R^+)$ etc. There are two significant degenerations to consider. The first occurs for $\varepsilon = O(\sqrt{\nu})$. Then, the parameter $\delta$ becomes $O(1)$ and the coupled set of equations can be rewritten by re-introducing the complex amplitude $A = B^+ + B^-$ and setting $\hat{r} = r - \delta/2$. Then, we get:

$$
A_\tau = \hat{r} \omega R A - \frac{1}{2} \omega_{kk} A_{\xi\xi} + \beta |A|^2 A + \frac{1}{2} \delta \omega R A
$$

In the sequel of this thesis, we will refer to the equation as non-symmetric Ginzburg-Landau equation. Note that we actually should have distinguished between $\omega^+_R$ and $\omega^-_R$ etc., but the difference between these expressions disappears in the higher order terms. For $\nu^2 \ll \varepsilon \ll \sqrt{\nu}$, the coefficient $\delta$ is much larger than unity and we see from (4.4.41) that
$B^-$ is exponentially damped; we come back on this in the Remark below.
The second degeneration occurs for $\varepsilon = \sqrt{\nu}$. Then, $\delta$ has become of the order of the higher order terms that we have neglected in the derivation of the amplitude equation. So, for consistency, we must then put $\delta \equiv 0$ and what remains is the usual Ginzburg-Landau equation. Thus, the distance between $R$ and the linear stability threshold has become so large that the influence of the $p$-periodic perturbation $\nu g(x)$ (which was due to the periodic geometry) can be neglected.

- **$N$ even**

The crucial point for the case $N$ is even, is to realize that already in second order (i.e. at the $\varepsilon^2 E$-level), amplitude equations for $B^+$ and $B^-$ are derived. As already explained, that is because $N(\Psi(1), \Psi(1))$ can be written in the structure of the linear solution. We consider the usual scalings for $\xi, \tau$ and $R$. Then, the equation follows in a (by now) rather straightforward way: the left hand side of (4.4.38) vanishes for $l = 0$ and thus, the corresponding right hand side must also vanish for $l = 0$. We give the results (omitting the derivation, which is similar to the case $N$ odd), up to $O(\varepsilon)$-terms:

$$B^+_\tau = r^+\omega_R^+ B^+ - (\frac{1}{2} \omega_{kk}^+ + O(\nu)) B^+_{\xi \xi} + \beta B^+ \left((B^+)^2 \psi_N^+ + (B^-)^2 \psi_N^- + O(\nu)\right) + \frac{\nu}{\varepsilon} (\alpha_1 (B^+)^2 + \alpha_2 (B^-)^2)$$  \hspace{1cm} (4.4.43)

$$B^-_\tau = (r^- - \delta)\omega_R^- B^- - (\frac{1}{2} \omega_{kk}^- + O(\nu)) B^-_{\xi \xi} + \beta B^- \left((B^+)^2 \psi_N^+ + (B^-)^2 \psi_N^- + O(\nu)\right) + \frac{1}{2\varepsilon} (\alpha_1 + \alpha_2) B^+ B^-$$  \hspace{1cm} (4.4.44)

where

$$\alpha_1 = 2\psi_N^+ + 2\psi_N^- \psi_N^+, \quad \alpha_2 = 2\psi_N^- + 2\psi_N^+ \psi_N^-$$  \hspace{1cm} (4.4.45)

Note that we did not write all relevant higher order terms (such as $B^+_{\xi \xi \xi}, B^+ B^+_{\xi}$, etc.). This is because there is no combination of the magnitudes of $\varepsilon$ and $\nu$ possible, such that these terms become of $O(1)$-importance. This is not the case for the quadratic and cubic terms in (4.4.43)-(4.4.44).

**Remark**

From (4.4.41) and (4.4.44) one observes that for $\varepsilon \ll \sqrt{\nu}$ (and of course $\varepsilon \gg \nu^2$), the linear coefficient of $B^-$ causes exponential damping of any initial condition. Thus, what remains is only the equation for $B^+$ (in which we must take $B^- = 0$), which replaces the usual Ginzburg-Landau equation. This equation is either a real Ginzburg-Landau equation (if $N$ is odd) or a real Ginzburg-Landau equation with quadratic terms of order $O(\frac{\nu}{\varepsilon})$. It depends on $\varepsilon$ whether the magnitude of these quadratic terms is greater, similar or smaller than the other terms in (4.4.43)-(4.4.44). In either case are the dynamics restricted to $V^+$ or, in other words, the dynamical situation has become one-dimensional, where it was two-dimensional in the unperturbed case.
On the other hand, if $\varepsilon > \sqrt{\nu}$, we can neglect $O(\nu)$-terms and the equations for $B^+$ and $B^-$ (i.e. eq. (4.4.40)-(4.4.41) for $N$ odd and (4.4.43)-(4.4.44) for $N$ even) turn out to be the same (in first order), and can be combined to one equation for $A = B^+ + iB^-$, which is again arbitrary complex! In other words, the splitting of the solution into $B^+ \in V^+$ and $B^- \in V^-$ only occurs for $\nu^2 < \varepsilon < \sqrt{\nu}$, where for $\varepsilon < \sqrt{\nu}$ only $B^+$ plays a rôle.

**Case 3: $\varepsilon = O_\sigma(\nu^2)$**

Now, we will discuss the transition case $\varepsilon = O_\sigma(\nu^2)$. Suppose $\varepsilon = \theta \nu^2$ for some $\theta = O(1)$. In order to be able to describe the transition case correctly, we should rewrite $\Psi$ (see (4.4.2)) in such a way that for $\theta \to 0$, $\Psi$ becomes eq. (4.4.12) (which was ‘valid’ for $\varepsilon < \nu^2$). On the other hand, we must look for a representation of $\Psi$ such that, for $\theta \to \infty$, $\Psi$ becomes eq. (4.4.13), which was ‘valid’ for $\varepsilon > \nu^2$. Therefore, we consider $\Psi_1$ and $\Psi_2$, which are two representations of the same $\Psi$ for $\varepsilon = O(\nu^2)$. In the construction of these functions, we use $k^\pm = \tilde{k} + \sigma^\pm \nu^2, p = \tilde{p} + \sigma \nu^2$ and substitute $
u^2 x = \varepsilon x/\theta = \xi/\theta$.

\[
\begin{align*}
\Psi_1 &= \tilde{\varepsilon} B^+ e^{i\tilde{k}^+ x} \sum \psi_1^+ e^{i\tilde{p}^+ x} e^{i(\tilde{\theta}^2 + 1)\sigma^2 x} e^{i(\tilde{\sigma} \nu^2) \xi} + c.c. \\
\Psi_2 &= \tilde{\varepsilon} B^+ e^{i\tilde{k}^x} e^{i\tilde{p}^x + \tilde{\sigma} x} \sum \psi_1^+ e^{i\tilde{p}^x} e^{i(\tilde{\theta} + 1)\sigma \xi} + c.c.
\end{align*}
\]  

(4.4.47) (4.4.48)

Note that the $B^-$-part is omitted in (4.4.47) and (4.4.48). The crucial point of both representations is that the terms involving $\xi$ can not be considered as constants on the $\xi$-scale, because $\theta = O(1)$. However, for $\theta \to 0$, one observes that $\Psi_1$ becomes similar to 4.4.12, which was the equation derived for $\varepsilon < \nu^2$, and on the other hand, if $\theta \to \infty$, then $\Psi_2$ becomes similar to (4.4.13), the equation derived in the case $\varepsilon > \nu^2$. To show that the transition from $\varepsilon < \nu^2$ to $\varepsilon > \nu^2$ is smooth, we must show that the (solutions of) the amplitude equations that are derived from $\Psi_{1,2}$, tend to the (solutions of) the amplitude equations that are derived for $\varepsilon < \nu^2$ and $\varepsilon > \nu^2$. Although this is intuitively clear, the precise transition is rather subtle. Note for instance, that the phase invariance which exists in the ‘normal’ Ginzburg-Landau equation (and which is consequently present for the case $\varepsilon < \nu^2$) disappears during this transition, since the dynamics become restricted to $V^+$ and $V^-$ (defined in (4.4.10)). This can be interpreted as if the system does no longer admit solutions with arbitrary phase, but has a preference for a specific phase (related to the phase of the $N$-th Fourier mode of the periodic forcing). The precise description of this transition is quite intricate. It turns out that the amplitude of $\Psi_1$ can be written as:

\[
B^+ = A^+ + \tilde{A}^+ \psi^+ e^{iN \frac{\xi}{\sigma} N}
\]  

(4.4.49)

(i.e. the amplitude contains $\xi$-dependent coefficients, compare this with (4.4.10)). One observes that for $\theta \to 0$ the phase invariance is recovered. The amplitudes of $\Psi_2$ can be written as:

\[
B^- = A^+ + \tilde{A}^+ \psi^+ e^{iN \frac{\xi}{\sigma} N}
\]  

(4.4.50)

and in the limit for $\theta \to \infty$, the ‘preferred’ direction $V^+$ appears. The details of this analysis are complex and they are not within the scope of this Chapter.
4.5 Slowly, slightly varying geometry: $p = o(1)$

Throughout this Chapter, we have taken $p$, the wave number of the periodic function which is related to the (periodic) geometry of the boundary of the problem under consideration, to be $O(1)$. The motivation for this choice came from physical observations. Nevertheless, there are physical situations possible in which $p$ is small and the critical wave number for linear stability is $O(1)$. Examples can be found in Eagles (1980) and Walton (1982); both papers consider the Bénard problem. In the terminology used in this Chapter, one could say that the boundaries are not only slowly varying, but also slightly. This problem has, in a more general setting, been analysed in a paper by Eckhaus and Kuske (1995). Although the problems are related, the analysis presented in this Chapter and the analysis presented in Eckhaus and Kuske (1995) is essentially different. For the sake of completeness, we consider the differences very briefly (for a more extensive comparison, we refer to Doelman et al. (1995b)).

First of all, the analysis in Eckhaus and Kuske (1995) is not based on a linear stability analysis, but the outcome of the unperturbed linearised stability problem has been used as the foundation of the nonlinear theory. Thus, sections 4.3 and 4.4 of this Chapter are ‘circumvented’. The $p$-periodic term $\nu g(x)$ is then considered as a non-homogeneous term in the model problem and consequently, appears direct in the modulation equation which takes the form:

$$A_\tau = r \mu_r A - \frac{1}{2} \mu_{kk} + \hat{\nu} \sum F_l e^{ilpx} A + \beta |A|^2 A$$

(4.5.1)

where $\hat{\nu} = \nu/\varepsilon^2$. The linear coefficients of (4.5.1) depend on the unperturbed neutral stability curve, which is obvious because the linear theory of the full, perturbed model problem is omitted. Equation (4.5.1) reflects two of the crucial differences in the analysis presented in this Chapter and in Eckhaus and Kuske (1995). First, because the latter paper is based on the linear stability analysis of the unperturbed model problem, the authors have to take as base-functions of the Ansatz for the nonlinear theory the critical modes, belonging to the unperturbed problem, i.e. $e^{ik_0x}$, whereas the critical modes in the Ansatz, used in this Chapter are either quasiperiodic functions (see (4.4.12)) or periodic functions, but of a more complicated structure than the critical modes of the unperturbed problem (see (4.4.13)). A second difference in the derivation of (4.5.1) is the a priori assumption of $\hat{\nu} = \varepsilon^{-2} \nu$ (or, alternatively, $\varepsilon^2 = O(\nu)$) which is needed because the term $\hat{\nu} \sum F_l e^{ilpx}$ cannot be allowed to be larger than the other terms in (4.5.1).

There is another complication in (4.5.1). Strictly speaking, the Fourier series $\sum F_l e^{ilpx}$ cannot be present completely in the equation, because some modes should actually be taken into account in the analysis of the problem arising at level 03, 23 or 33. This problem can be solved by assuming that there are only a finite number of Fourier-modes present or that $|F_l|$ decays sufficiently fast with $l$.

If one wants to analyse the case of slowly, slightly varying geometry in the way that is presented in this Chapter (and thus, study the case $\varepsilon^2 \preceq \nu$), one runs into trouble, already at the linear level. In the derivation of the coefficients $\psi_l$ of the linear solution (see (4.3.5)),
there occurred degenerations whenever

\[ |\mu(k_c^o - N_p, R_c^o) - \mu(k_c^o, R_c^o)| = o(1) \]  \hspace{1cm} (4.5.2)

If \( p = o(1) \), this is always the case! Moreover, for certain large \( N \) (\( N = O(\frac{1}{p}) \)), condition (4.5.2) is also violated. So, one is always in a degenerate situation (the cylinder on which the neutral stability curve should be interpreted has radius \( o(1) \), the splitting of the neutral curve occurs for wave numbers which are \( o(1) \) close to each-other, and so on). In order to solve this problem, we need probably a totally different approach. This might be an interesting subject for further research.

### Appendix A

In this Appendix, we present the calculations that lead to the asymptotic expressions for the new critical conditions for the non-resonant as well as the resonant problem. We start with the former.

#### A.1 The non-resonant problem

First, we repeat (4.3.3):

\[ \omega(k, R) = \mu(k, R) + \nu \sum_{s+t=0} F_s \psi_t \]  \hspace{1cm} (A.1)

Then, we assume

\[ k = k_c^o + \tilde{K} \nu \]  \hspace{1cm} (A.2)

\[ R = R_c^o + \tilde{R} \nu^2 \]  \hspace{1cm} (A.3)

and repeat the expansions for \( \mu(k + lp, R) \) around \( (k_c^o + lp, R_c^o) \) for all \( l \) (\( \sim \) is omitted):

\[ l = 0 \quad \mu(k, R) = \nu^2 \left[ \frac{1}{2} u_{02} K^2 + Ru_{10} + \nu \left( \frac{1}{6} u_{03} K^3 + u_{11} KR \right) + ... \right] \]  \hspace{1cm} (A.4)

\[ l \neq 0 \quad \mu(k + lp, R) = u_{00}^{(l)} + \nu u_{01}^{(l)} K + \nu^2 \left( \frac{1}{2} u_{02}^{(l)} K^2 + u_{10}^{(l)} R \right) + ... \]  \hspace{1cm} (A.5)

where \( u_{02} < 0, u_{10} > 0 \). These conditions settle the parabolic shape of \( \mu(k, R) \). Also, we have that \( u_{00}^{(l)} < 0 \) for all \( l \). Then, we repeat (4.3.4):

\[ \mu(k + lp, R) \psi_l = -\nu \sum_{s+t=l} F_s \psi_t \text{ or } \psi_l = \frac{-\nu \sum_{s+t=l} F_s \psi_t}{\mu(k + lp, R)} \]  \hspace{1cm} (A.6)

and we use (A.5) to find \( \psi_l \) up to \( O(\nu^2) \). The result is:

\[ \psi_l = \frac{-\nu F_l}{u_{00}^{(l)}} \left[ 1 + \nu K \frac{u_{01}^{(l)}}{u_{00}^{(l)}} \right] + O(\nu^2) \]  \hspace{1cm} (A.7)

The neutral stability curve is determined by the condition \( \omega(k, R) = 0 \). Thus, we find from (A.1) that:

\[ \mu(k, R) = -\nu \sum_{s+t=0} F_s \psi_t \]  \hspace{1cm} (A.8)
and using (A.7) and (A.4), this can be written as

\[
\frac{1}{2} u_{02} K^2 + u_{10} R + \nu \left( \frac{1}{6} u_{03} K^3 + u_{11} K R \right) = U_0 - \nu KV_0 - \nu \tilde{U}_0 + O(\nu^4) \tag{A.9}
\]

with

\[
U_0 = \sum_{s+t=0} F_s F_t = \sum \frac{|F_s|^2}{u_{00}^{s+t}} < 0 \tag{A.10}
\]

\[
V_0 = \sum_{s+t=0} \frac{|F_s|^2}{u_{01}^{(s-t)}} (u_{00}^{(t)})^2 \tag{A.11}
\]

\[
\tilde{U}_0 = \sum_{s+t=0} F_s U_t \tag{A.12}
\]

This expression, which is a relation between \( k \) and \( R \) (i.e. \( \tilde{R} \)), determines the neutral stability curve. As we are interested in the minimum of this curve, we have to take the derivative with respect to \( K \), which yields:

\[
u u_{02} K + \nu \left( \frac{1}{2} u_{03} K^2 + u_{11} R \right) = -\nu V_0 \tag{A.13}
\]

and we can define \( K_c \), the correction on the critical wave number:

\[
K_c = \frac{-\nu}{u_{02}} (V_0 + Ru_{11}) + O(\nu^2) \tag{A.14}
\]

Substitution of (A.14) into (A.9) yields \( R_c \), the correction on the critical value of the bifurcation parameter:

\[
R_c = \frac{1}{u_{10}} (U_0 - \nu \tilde{U}_0) + O(\nu^2) \tag{A.15}
\]

and thus

\[
K_c = \frac{\nu}{u_{02}} (V_0 + \frac{u_{11}}{u_{10}} U_0) + O(\nu^2) \tag{A.16}
\]

Note that \( K_c = O(\nu) \) ! The full expressions for \( k^+ \) and \( R^+ \) are now found to be:

\[
k^+ = k_c^o + \frac{\nu^2}{u_{02}} (V_0 + \frac{u_{11}}{u_{10}} U_0) + O(\nu^3) \tag{A.17}
\]

\[
R^+ = R_c^o + \frac{\nu^2}{u_{10}} (U_0 - \nu \tilde{U}_0) + O(\nu^4) \tag{A.18}
\]

Thus, it is found that \( k^+ = k_c^o + O(\nu^2) \) instead of \( k^+ = k_c^o + O(\nu) \), what one should a priori expect. Furthermore we see that \( R^+ < R_c^o \), because \( u_{10} > 0, U_0 < 0 \).
A.2 The resonant problem

For the resonant problem, there are two subcases to consider with respect to the value of \( \beta \), namely \( 0 < \beta < 1/2 \) and \( \beta > 1/2 \). The starting point for the calculation of the higher order corrections on \( k_c^o \) and \( R_c^o \) is the same for all values of \( \beta \). First, we assume

\[
\begin{align*}
  k &= k_c^o + K_\nu^\alpha_1 \\
  R &= R_c^o + R_\nu^\alpha_2
\end{align*}
\]  

(A.19)  

(A.20)

and expand \( \mu(k + lp, R) \) for all \( l \):

\[
\begin{align*}
  \mu(k, R) &= \sum_{i,j} u_{ij} R^i K_{i} J_{j}^{\alpha_1 + \alpha_2} \\
  \mu(k + lp, R) &= \sum_{i,j} u_{ij}^{(l)} R^i K_{i} J_{j}^{\alpha_1 + \alpha_2} \\
  \mu(k + Np, R) &= \sum_{i,j} u_{ij} R^i (\eta^\beta + K_\nu^\alpha_1) J_{j}^{\alpha_2}
\end{align*}
\]  

(A.21)  

(A.22)  

(A.23)

where we used the symmetry of \( \mu \) and the resonance condition. We use the expansion of \( \mu(k + Np, R) \) (see (A.23)) to get the equivalents of equation (A.6) for \( l = N \) and \( l = 0 \):

\[
\begin{align*}
  \psi_N \left[ \frac{1}{2} u_{02}(\eta^\beta + K_\nu^\alpha_1)^2 + u_{10} R_\nu^\alpha_2 + \ldots \right] &= -\nu \sum_{s+t=N} F_s \psi_t \\
  &= -\nu F_N + O(\nu^2) \quad \text{(A.24)} \\
  \frac{1}{2} u_{02} K_\nu^2 \nu^{2\alpha_1} + u_{10} R_\nu^\alpha_2 + \ldots &= -\nu \sum_{s+t=0} F_s \psi_t \\
  &= -\nu F_N \psi_N + O(\nu^2) \quad \text{(A.25)}
\end{align*}
\]

Equations (A.24) and (A.25) are the two equations that are needed to calculate the higher order corrections on \( k_c^o, R_c^o \) and \( \psi_N \). Depending on the value of \( \beta \), it might be necessary to take into account the \( O(\nu^2) \) terms in the right hand sides of (A.24) and (A.25). These can be calculated, using (A.6) for \( l \neq 0, N \). By a significant degeneration argument, it follows directly from (A.24) and (A.25) that:

- if \( 2\beta < 1 \) then \( \alpha_1 = 1 - \beta, \alpha_2 = 2\alpha_1 \)
- if \( 2\beta > 1 \) then \( \alpha_1 = \frac{1}{2}, \alpha_2 = 2\alpha_1 \)

Now, we recall that in resonance, \( \mu(k + Np, R) \) becomes \( o(1) \) and consequently, \( \psi_N \) becomes \( O(\nu^\gamma) \), for some \( \gamma > 0 \). Taking this into account, we consider (4.3.4), where we already put \( \omega(k, R) = 0 \) and where we use (A.22) to expand \( \psi_l \):

\[
\begin{align*}
  \psi_l &= -\nu \frac{\mu(k + lp, R)}{\mu(k + lp, R)} \sum_{s+t=l} F_s \psi_t = -\nu \frac{\mu(k + lp, R)}{\mu(k + lp, R)} (F_l + F_{l-N} \psi_N + \text{h.o.t.}) \\
  &= -\nu (F_l + F_{l-N} \psi_N) \left( \frac{1}{u_{00}^{(l)}} - \frac{u_{01}^{(l)} K_\nu^\alpha_1}{(u_{00}^{(l)})^2} + \text{h.o.t.} \right) \quad \text{(A.26)}
\end{align*}
\]
Using these formulas, we can write down the expansions for $\mu(k, R)$ and $\mu(k + Np, R)$, which we are going to use to calculate the higher order corrections on $k_c^o$ and $R_c^o$:

$$
\mu(k, R) = \frac{1}{2} u_{02} K^2 \nu^{2\alpha_1} + R u_{10} \nu^{\alpha_2} + \frac{1}{6} \nu^{3\alpha_1} u_{03} K^3 + u_{11} K \nu^{\alpha_1+\alpha_2} + \text{h.o.t.} \\
= -\nu \left[ F_{-N} \psi_N + \nu a_0 + \nu \psi_N a_1 + \nu^{1+\alpha_1} K \psi_N a_2 + \nu^{1+\alpha_1} K a_3 + \text{h.o.t.} \right] 
$$  (A.27)

$$
\mu(k + Np, R) = \frac{1}{2} u_{02}(\eta \nu^\beta + K \nu^{\alpha_1})^2 + R u_{10} \nu^{\alpha_2} - \frac{1}{6} u_{03}(\eta \nu^\beta + K \nu^{\alpha_1})^3 - \nu u_{11}(\eta \nu^\beta + K \nu^{\alpha_1})^2 \nu^{\alpha_2} \text{h.o.t.} \\
= -\nu \left[ F_N + \nu b_0 + \nu \psi_N b_1 + \nu^{1+\alpha_1} K \psi_N b_2 + \nu^{1+\alpha_1} K b_3 + \text{h.o.t.} \right] \frac{1}{\psi_N} 
$$  (A.28)

where $a_i, b_i$ are defined as follows:

$$
a_0 = -\sum_{l \neq N} \frac{|F_l|^2}{u_{l0}^2} \quad b_0 = -\sum_{l \neq N} \frac{F_l F_{N,-l}^*}{u_{l0}^2} \\
a_1 = -\sum_{l \neq N} \frac{F_l F_{l,-N}^*}{u_{l0}^2} \quad b_1 = -\sum_{l \neq N} |F_l|^2 \\
a_2 = \sum_{l \neq N} \frac{F_l F_{l,-N}^*}{u_{l0}^2} \quad b_2 = \sum_{l \neq N} \frac{F_l^2}{u_{l0}^4} \\
a_3 = \sum_{l \neq N} \frac{F_l^2}{u_{l0}^4} 
$$  (A.29)

So, note that $a_0 = b_1, a_3 = b_2, \overline{a_1} = b_0$ and $\overline{b_2} = b_3$. Using (A.27) and (A.28), we are able to calculate the higher order corrections on $k_c^o$ and $R_c^o$. We distinguish between $\beta < 1/2$ and $\beta > 1/2$.

- **The case** $\beta < 1/2$

From (A.28), it follows immediately that:

$$
\psi_N = \frac{-\nu F_N}{G(\nu^\beta)} = O(\nu^{-2\beta}) 
$$  (A.30)

where

$$
G(\nu^\beta) = \sum_{\beta \geq 2} \frac{u_{\alpha i}}{i!} \nu^{i\beta} 
$$  (A.31)

Comparing terms proportional to $\nu^{2\alpha_1}$, we find from (A.27)

$$
\frac{1}{2} u_{02} K^2 + u_{10} R = F_{-N} \psi_N 
$$  (A.32)

As we are mainly interested in the minimum $(K_c, R_c)$ of (A.32), we set the derivative with respect to $K$ of (A.32) equal to zero, which yields $R_c = -\frac{F_{-N} \psi_N}{u_{10}}$ and $K_c = 0$. So, apparently, the leading order correction on $k_c^o$ is smaller than what we assumed in (A.19) (i.e. $\nu^{\alpha_1}$). In order to find the right magnitude of the correction, we assume that

$$
k = k_c^o + K \nu^\gamma 
$$  (A.33)
for some \( \gamma > 0 \) and look for a significant degeneration in (A.27) and (A.28). Substitution in (A.28) leads to a balance between

\[
u_{02}\eta K^{\nu+\beta} \sim u_{10} R_c \nu^{2\alpha_1}\]

(A.34)

so that \( \gamma = 2 - 3\beta \) (recall that \( \alpha_1 = 1 - \beta \)) and

\[
K_c = \frac{u_{10} R_c}{u_{02}\eta}
\]

(A.35)

Higher order corrections can be calculated straightforward.

**The case \( \beta > 1/2 \)**

Essentially, the analysis is analogous to the previous case. However, as explained before, we have for \( \beta > 1/2 \), that \( \alpha_1 = 1/2 \) and \( \alpha_2 = 1 \). Therefore, in order to find \( \psi_N \), we subtract (A.27) and (A.28) which yields up to \( O(\nu^3) \):

\[
\psi^\pm_N = \pm \frac{F_N}{|F_N|} - \frac{1}{F_{-N}} a_0 + a_1 \psi^\pm_N - b_0 - b_1 \psi^{(0)}_N
\]

(A.36)

Then, comparing terms proportional to \( O(\nu) \), we find again equation (A.32):

\[
\frac{1}{2} u_{02}K^2 + u_{10} R = F_{-N} \psi^{(0)}_N
\]

(A.37)

which gives as before \( R_c = -\frac{F_{-N} \psi^+_N}{u_{10}} \), and \( K_c = 0 \). In order to find the leading order of the correction on the critical wave number, we pose

\[
k = k^0_{c0} + K \nu^\gamma
\]

(A.38)

and we look as before for a significant degeneration, where we now have to balance between

\[
u_{02}\eta \nu^{2\beta}, \nu^{2+\gamma} K \psi_N b_2 \sim \frac{1}{2} u_{02}K^2 \nu^{2\gamma}
\]

(A.39)

There are two possibilities for \( \gamma \). For \( 1/2 < \beta < 2 \) we find a significant degeneration if \( \gamma = \beta \) and calculations similar to the case \( \beta < 1/2 \) show that

\[
k^\pm = -\frac{1}{2} \eta \nu^\beta - \frac{\nu^2}{2u_{02}} \left( \psi^\pm_N a_2 + a_3 + \psi^\pm_N b_2 + b_3 \right) + O_s(\nu^{2\beta}) + h.o.t.
\]

(A.40)

The \( O_s(\nu^\beta) \) terms are only important for \( 1/2 < \beta < 1 \), but are otherwise not essential for the theory and are therefore not calculated explicitly.

For \( \beta > 2 \) we must take the other possibility, i.e. \( \gamma = 2 \). Then, it turns out that

\[
k^\pm = -\frac{1}{2} \eta \nu^\beta - \frac{\nu^2}{2u_{02}} \left( \psi^\pm_N a_2 + a_3 + \psi^\pm_N b_2 + b_3 \right) + h.o.t.
\]

(A.41)

Finally, we note that if \( \beta > 1/2 \) then \( |\psi^+_N| = 1 + h.o.t. \), as can be seen from (A.36). One can even show by straightforward calculations and using the expressions for \( a_i \) and \( b_i \) that if \( \beta > 2 \), then \( |\psi^+_N| = 1 + O_s(\nu^2) \).
Appendix B

In this Appendix, we present the calculations that are necessary to derive the amplitude equations for the non-resonant problem. First, we make some preparations in the form of two remarks which will turn out to be useful for the forthcoming analysis.

Remark

We consider the linear problem again:

$$\left( \frac{\partial}{\partial t} - L - \nu \sum F_l e^{i\mu_l x} \right) \psi_{\text{lin}} = 0$$  \hspace{1cm} (B.1)

Substitution of

$$\psi_{\text{lin}} = e^{ikx + \omega x} \sum \psi_l e^{i\mu_l x}$$  \hspace{1cm} (B.2)

yields

$$\sum \left( \omega \psi_l - \mu(k + lp) \psi_l - \nu \sum_{s+t=l} F_s \psi_t \right) e^{i\mu_l x} = 0$$  \hspace{1cm} (B.3)

Then, we take the derivative of (B.3) with respect to $k$, and evaluate at critical conditions (denoted by a subscript 'c'). This gives after some elaboration, where we should take into account that $\omega_c = \left( \frac{\partial \omega}{\partial k} \right)_c = 0$:

$$\sum \left( -\mu_c(k + lp) \left( \frac{\partial \psi_l}{\partial k} \right)_c - \nu \sum_{s+t=l} F_s \left( \frac{\partial \psi_l}{\partial k} \right)_c \right) e^{i\mu_l x} = \sum \left( \frac{\partial \mu(k + lp)}{\partial k} \right)_c \psi_l e^{i\mu_l x}$$  \hspace{1cm} (B.4)

This equation will be useful in the forthcoming analysis. Furthermore, it turns out that we need a relation between $\mu_R$ and $\omega_R$, and between $\mu_{kk}$ and $\omega_{kk}$. This can be found as follows.

Remark

We repeat (4.3.6):

$$\omega(k, R) = \mu(k, R) + \nu \sum_{s+t=0} F_s \psi_t^{11}$$

$$= \mu(k, R) - \nu^2 \sum \frac{|F_l|^2}{\mu(k + lp, R) - \omega(k, R)} + O(\nu^3)$$  \hspace{1cm} (B.5)

Differentiation with respect to $R$ and $k$ gives, after some straightforward manipulation,

$$\mu_R = \omega_R \left( 1 + \sum \frac{\nu^2 |F_l|^2}{\mu(k + lp)^2} \right) - \nu^2 \sum \frac{|F_l|^2 \mu_R(k + lp)}{\mu(k + lp)^2} + O(\nu^3)$$  \hspace{1cm} (B.6)

$$\mu_{kk} = \omega_{kk} \left( 1 + \sum \frac{\nu^2 |F_l|^2}{\mu(k + lp)^2} \right) - \nu^2 \sum \frac{|F_l|^2 (\mu_{kk}(k + lp) - 2\mu_k^2(k + lp))}{\mu(k + lp)^3} + O(\nu^3)$$  \hspace{1cm} (B.7)
It can be shown that the coefficient of $\omega_R$ and $\omega_{kk}$ is the same up to arbitrary order. Relations (B.6) and (B.7) are important tools in the derivation of the amplitude equation.

Let us now proceed with the analysis. After substitution of (4.4.18), we compare the terms proportional to $\varepsilon^2 E$, and we find the following equation:

$$(-L - \nu \sum F_l e^{ilpx}) E \psi^{12} = \frac{1}{i} A_\xi L^{(1)} \psi^{11} 

\text{(B.8)}$$

As solution, we pose $\psi^{12} = A_\xi \frac{\xi}{i} \sum \psi^{12}_l e^{ilpx}$, and find, using the first Remark:

$$A_\xi \left[ - \sum \psi^{12}_l \mu(k_c + lp)e^{ilpx} - \nu \sum \sum F_s \psi^{12}_l e^{ilpx} \right]$$

$$= \frac{1}{i} A_\xi \sum \psi^{11}_l \mu(k_c + lp)e^{ilpx} 

\text{(B.9)}$$

$$= \frac{-1}{i} A_\xi \left[ \sum \psi^{11}_l \mu(k_c + lp)e^{ilpx} + \nu \sum \sum F_s \psi^{11}_l e^{ilpx} \right] 

\text{(B.10)}$$

where we have used the notation $\psi^{11}_{l,k} = \partial/\partial k (\psi^{11}_l)$ etc. Hence, we immediately see a solution for $\psi^{12}_l$:

$$\psi^{12}_l = \psi^{11}_{l,k} 

\text{(B.11)}$$

Thus the full solution at the $\varepsilon^2 E$-level becomes:

$$\varepsilon^2 E \psi^{12} = \varepsilon^2 e^{ik_c x} \left[ -i A_\xi \sum \psi^{11}_l e^{ilpx} + B(\xi, \tau) \sum \psi^{11}_l e^{ilpx} \right] 

\text{(B.12)}$$

where $B(\xi, \tau)$ is the next order (with respect to its expansion in $\varepsilon$) of the amplitude function $A$. The analysis then continues by solving the equations arising at order $\varepsilon^2$ and at $\varepsilon^2 E^2$, precisely as was done in the unperturbed situation where the functions $\psi^{pq}$ are Fourier series with components proportional to $exp(ilpx)$ instead of constants, as was the case in the unperturbed problem. This makes things a bit more complicated, but essentially, the analysis does not change. Let us look at the equation arising at order $\varepsilon^2 E^2$. We have to take into account nonlinear interactions between the linear solution and itself and we find the following equation:

$$(-L - \nu \sum F_l e^{ilpx}) E^2 \psi^{22} = N(E \psi^{11}, E \psi^{11}) 

\text{(B.13)}$$

Note that the right hand side of (B.13) is proportional to $A^2$, thus we may assume that $\psi^{22}$ can be written as:

$$\psi^{22} = A^2 \sum \psi^{22}_l e^{ilpx} 

\text{(B.14)}$$

and substitution of $E^2 \psi^{22}$ into (B.13) yields:

$$= \sum \psi^{22}_l \mu(2k_c + lp)e^{ilpx} - \nu \sum F_s \psi^{22}_l e^{ilpx} = N(E \psi^{11}, E \psi^{11})$$

$$= A^2 e^{2ik_c x} \sum \left( \sum \psi^{11}_s \tilde{P}_1(k_c + sp)\psi^{11}_l P_2(k_c + tp) \right) e^{ilpx} 

= A^2 e^{2ik_c x} \sum Q^{22}_l e^{ilpx} 

\text{(B.15)}$$
where we have abbreviated
\[ Q_l^{22} = \sum_{s+t=l} \psi_s^{11} P_1(k_c + sp) \psi_t^{11} P_2(k_c + tp) \] (B.16)

where \( P_{1,2} \) are defined below (4.2.8). Note that it follows from (B.15) that \( Q_l^{(22)} = O(\nu) \) for \( l \neq 0 \) and \( Q_0^{(22)} = O(1) \). The reason for this comes from the fact that \( \psi_l = O(\nu) \) for \( l \neq 0 \) and \( \psi_0 = 1 \). (Recall that \( \psi_l \) are the coefficients of the linear solution, see (4.3.1)). This equation can be solved in a straightforward way by
\[ \psi_l^{22} = -Q_l^{22} - \nu \sum_{s+t=l} F_s \psi_t^{22} / \mu(2k + tp) \] (B.17)

We observe that \( \psi_0^{22} = -Q_0^{22} / \mu(2k) + h.o.t. = O(1) \) and \( \psi_l^{22} = O(\nu) \) for \( l \neq 0 \).

The next problem that we encounter appears at order \( \varepsilon^2 \). The analysis of this problem goes along the same lines as the one at order \( \varepsilon^2 E^2 \). The problem at \( \varepsilon^2 \) reads:
\[ (-L - \nu \sum F_l e^{ilpx}) \psi_0^{02} = N(E \psi_1^{11}, \overline{E \psi_1^{11}}) \] (B.18)

and again, we note that the right hand side of (B.18) is proportional to \( |A|^2 \). Hence, we assume that \( \psi_0^{02} \) can be written as
\[ \psi_0^{02} = |A|^2 \sum \psi_t^{02} e^{ilpx} \] (B.19)

which yields after substitution:
\[ - \sum \psi_t^{02} \mu(lp) e^{ilpx} - \nu \sum_{s+t=l} F_s \psi_t^{02} e^{ilpx} = N(\psi_1^{11}, \overline{\psi_1^{11}}) \]
\[ = \sum \left( \sum_{s+t=l} \psi_s^{11} P_1(k_c + sp) \overline{\psi_t^{11}} P_2(-k_c - tp) + \sum_{s+t=l} \psi_t^{11} P_2(k_c + tp) \overline{\psi_s^{11}} P_1(-k_c - sp) \right) e^{ilpx} \]
\[ = \sum Q_l^{02} e^{ilpx} \] (B.20)

where we have again abbreviated
\[ Q_l^{02} = \sum_{s+t=l} \psi_s^{11} P_1(k_c + sp) \overline{\psi_t^{11}} P_2(-k_c - tp) + \sum_{s+t=l} \psi_t^{11} P_2(k_c + tp) \overline{\psi_s^{11}} P_1(-k_c - sp) \] (B.21)

We find
\[ \psi_l^{02} = -Q_l^{02} - \nu \sum_{s+t=l} F_s \psi_t^{02} / \mu(lp) \] (B.22)

and we observe that \( \psi_0^{02} = O(1) \) while for \( l \neq 0 \), \( \psi_l^{02} = O(\nu) \).

Then, we proceed with the problem arising at order \( \varepsilon^3 E \). At this level we expect, guided by the knowledge obtained from the unperturbed case, that the amplitude equation for \( A \)
will arise. As usual, we now have to take into account nonlinear interactions as well as the rescaled variables and the expansion of the bifurcation parameter. Symbolically, the equation can easily be written down:

\[ (-L - \nu \sum F_i e^{ilpx}) \psi_{13}^{13} = \]

\[ \frac{\partial}{\partial \tau} \psi_{11}^{11} + rL^{(0)} \psi_{11}^{11} + L^{(1)} \psi_{11}^{11} + L^{(2)} \psi_{11}^{11} + N(E \psi_{11}^{11}, \psi_{02}^{11}) + N(\overline{E \psi_{11}^{11}}, E^2 \psi_{22}^{11}) \]  
(B.23)

The term \( L^{(1)} \psi_{11}^{12} \) gives rise to terms involving \(-iB_\xi \sum \psi_{11}^{11} \exp(ilpx)\). In order to get rid of the terms containing \( B_\xi \), we substitute:

\[ \psi_{13}^{13} = \sum \psi_{13}^{13}(\xi, \tau) e^{ilpx} - iB_\xi \sum \psi_{11}^{11} e^{ilpx} \]  
(B.24)

Note that now the coefficients \( \psi_{13}^{13} \) themselves are functions of \( \xi \) and \( \tau \) which is a result of the fact that the r.h.s. of eq. (B.23) has now become a differential operator instead of an algebraic operator. Substitution of (B.24) into (B.23) yields:

\[ - \sum \psi_{13}^{13} \mu(k_c + lp)e^{ilpx} - \nu \sum \sum F_s \psi_{13}^{13} e^{ilpx} = \]

\[ - A_r \sum \psi_{11}^{11} e^{ilpx} + A \sum \psi_{11}^{11} \mu_R(k_c + lp)e^{ilpx} \]

\[ - A\xi \sum \left( \frac{1}{2} \psi_{11}^{11} \mu (k_c + lp) + \psi_{11}^{11} \mu_{kk}(k_c + lp) \right) e^{ilpx} + |A|^2 A \sum Q_{13}^{13} e^{ilpx} \]  
(B.25)

The definition for \( Q_{13}^{13} \) is similar to the definitions of \( Q_0^{02} \) and \( Q_1^{22} \) and should be clear. We abbreviate the right hand side of (B.25) by \((GL)_l\), i.e. we write

\[ - \sum \psi_{13}^{13} \mu(k_c + lp)e^{ilpx} - \nu \sum \sum F_s \psi_{13}^{13} e^{ilpx} = \sum (GL)_l e^{ilpx} \]  
(B.26)

At the \( l = 0 \)-level of the \( O(\varepsilon^3 E) \) problem, the left hand side of (B.26) vanishes and consequently, the right hand side must also vanish for \( l = 0 \). Thus we find:

\[ - A_r + \mu_R A - \frac{1}{2} A\xi \mu_{kk} + |A|^2 A Q_0^{13} + \nu \sum_{s+t=0} F_s \psi_{13}^{13} = 0 \]  
(B.27)

while comparing terms proportional to \( \exp(ilpx) \) yields for every \( l \neq 0 \):

\[ \psi_{13}^{13} = \frac{-(GL)_l - \nu \sum_{s+t=l} F_s \psi_{13}^{13}}{\mu(k_c + lp)} = O(\nu) \]  
(B.28)

Essentially, (B.27) is the ‘perturbed’ Ginzburg-Landau equation, where the perturbation is expressed by a correction of the coefficients. Note that the last term of (B.27) is \( O(\nu^2) \). The first order of \( Q_0^{13} \) coincides with the Landau coefficient, defined in (4.2.9). We note that the linear coefficients of equation (B.27) are \( \mu_R \) and \( \mu_{kk} \) instead of \( \omega_R \) and \( \omega_{kk} \) what we would expect (see (4.4.21)). However, we can elaborate (B.27) in such a way that the linear coefficients become \( \omega_R \) and \( \omega_{kk} \). This is done by substituting (the leading order of) \( \psi_{13}^{13} = -(GL)_l / \mu(k_c + lp) \) (see (B.28)) into (B.27) and use the (leading order of) \( \psi_{11}^{11} \) to elaborate \((GL)_l\) (recall that \((GL)_l\) depends in a complicated way on \( \psi_{11}^{11} \), see (B.26)).
this tedious process of substitutions, we must also take into account the relations between 
\( \mu_R \) and \( \omega_R \) and between \( \mu_{kk} \) and \( \omega_{kk} \) (see (B.6) and (B.7)), to find up to \( O(\nu^2) \):

\[
A_\tau = r\omega_RA - \frac{1}{2}A\xi\omega_{kk} + |A|^2A \left( Q_0^{13}(1 - \nu^2 \sum \frac{|F_l|^2}{\mu(k_c + lp, R_c)^2} - \nu \sum_{s+t=0} \frac{F_sQ_t^{13}}{\mu(k_c + tp, R_c)} \right) \tag{B.29}
\]

As mentioned before, all the coefficients of the Ginzburg-Landau equation (B.29) have an
\( O(\nu^2) \) correction with respect to the unperturbed case. Note that the correction of the nonlinear term is complex!
Chapter 4. Nonlinear Stability Analysis for ...
Chapter 5

On the Solutions of the Non-Symmetric Ginzburg-Landau Equation

5.1 Introduction

In this Chapter we will discuss some of the solutions of the amplitude equations, derived in the previous section. We recall that we derived not one, but various equations, depending on the relative magnitude of $\nu$ with respect to $\varepsilon$. The important parameter in the forthcoming analysis will be the ratio between $\nu$ and $\varepsilon^2$ which we defined as $\delta = \nu/\varepsilon^2$. We assume that we are in resonance, i.e. $|p - 2k_c/N| = o(1)$, (see (4.3.13) for the definition of resonance; $p$ and $k_c$ are introduced in section 4.1).

Usually, one fixes $\varepsilon$ and denotes the distance above criticality by $r\varepsilon^2$, where $r$ is the bifurcation parameter which can be varied. Changing $r$ means that we change the distance to $R_c$. Now, the situation is more subtle. First of all, we have two neutral stability curves $\omega^+$ and $\omega^-$. The curve $\omega^+$ is associated with amplitude $B^+$ and $\omega^-$ is associated with $B^-$. Thus, if we mention ‘above criticality’, we must specify whether we mean critical conditions belonging to $\omega^+$ or $\omega^-$. As we already know, the distance between the two critical conditions is $O(\nu)$ (and thus fixed). It can lead to ambiguity to use $r$ as bifurcation parameter, because then we can only measure the distance with respect to the minimum of $\omega^+$ and we may overlook the influence of the curve $\omega^-$. To overcome this, we use $\delta$ (which measures the relative magnitude of $\varepsilon^2$ with respect to $\nu$) as bifurcation parameter and fix $r$ to 1. Introducing $\delta$ still allows us to consider variations in $\varepsilon$. Suppose for instance, that $\varepsilon = O_s(\sqrt{\nu})$. Then, within the regime $\varepsilon = O_s(\sqrt{\nu})$ (which means $\delta = O_s(1)$) we can consider $\delta$ ‘large’. This indicates the transition to the regime $\varepsilon \ll \sqrt{\nu}$. Similarly, $\delta$ small corresponds to the transition to the regime $\varepsilon \gg \sqrt{\nu}$. In the sequel, we use the notation $\delta \ll 1$ and ‘$\delta$ small’, as a kind of abbreviation to indicate that within the regime $\varepsilon = O_s(\sqrt{\nu})$, $\varepsilon$ is large with respect to $\sqrt{\nu}$. Thus, for instance, $\varepsilon = M\sqrt{\nu}$ for some $M, 1 < M < M_0$. It is emphasized that in the context of this Chapter, we use $\delta \ll 1$ and ‘$\delta$ small’ merely as a notation. A similar line of thoughts holds for $\delta \gg 1$ and ‘$\delta$ large’.

Let us repeat the equations which hold for this case (see (4.4.42); we applied some cosmetic
rescalings, in particular $A \to A \exp(i\gamma/2)$ which means that the subspaces $V^+$ and $V^-$ (see (4.4.10)-(4.4.11)) can be replaced by the real and imaginary axis):

$$A_\tau = \hat{r}A + A\xi - |A|^2A + \frac{\delta A}{2}$$

(5.1.1)

where $\hat{r} = 1 - \delta / 2$ (recall that we fixed $r$ on 1). As already noted in the previous Chapter (see the Remark, made in section 4.4.3), only for $\varepsilon = O_{\delta}(\sqrt{\nu})$ the non-symmetric term $\sim \delta A$ is present. We recall that $\delta \ll 1$ means that we are 'relatively far' above both minima ('far' in this context means of course still $O(\varepsilon^2)$ above the minimum of $\omega^+$). Somewhere in the region $\delta = O(1)$, starting with $\delta$ small and studying the dynamics for increasing $\delta$, we expect that the influence of $\omega^-$ becomes substantial. Then, increasing $\delta$ even more means that we go further and further down (because $\delta$ is proportional to the inverse of $\varepsilon^2$) and for $\delta$ large, we approach the minimum of $\omega^+$. Using $\delta$ as bifurcation parameter implies that in the $(k,R)$-space, we can never pass through the minimum of $\omega^+$. Because the trivial solution is linearly stable below $\omega^+$, this means that we do not miss interesting behaviour.

### 5.2 Space and time independent solutions

First of all, we pay some attention to the most simple, i.e. space and time independent, solutions for the parameter regime $\varepsilon > \nu^2$. Recall that for $\varepsilon < \nu^2$, we should consider a slightly different equation, see Chapter 4, sections 4.4.1 and 4.4.3. For $\delta = 0$, there are (independent of the value of $\varepsilon$) solutions $A = G\exp(i\theta), \theta$ arbitrary and thus, these solutions are phase invariant. Due to the perturbation $\sim \delta A$, these solutions do no longer exist for $\delta \neq 0$. It is not difficult to see that the the only possible constant solutions are:

$$A = Ge^{i\theta} \text{ with } \theta = 0, \pi, \text{ and } G = 1$$

(5.2.1)

$$A = Ge^{i\theta} \text{ with } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ and } G = \sqrt{1 - \delta} \ (\delta < 1)$$

(5.2.2)

A simple stability analysis (putting $A = G\exp(i\theta) + \hat{a}\exp(il\xi)$ and linearizing) shows that (5.2.1) is always stable and (5.2.2) is always unstable, as solution of equation (5.1.1). Now, we will briefly describe how the phase invariant character of the solutions is destroyed for $\delta \neq 0$.

Therefore, we consider the situation in the $(Re(A), Im(A))$-phase plane. By direct linear analysis, we see that solutions (5.2.1) (which are real) are nodes and (5.2.2) (which are purely imaginary) are saddle points. The trajectories are as indicated in the two left subfigures of figure 5.1. The outer circle corresponds with points for which $Re(A) = 0$, the inner circle with $Im(A) = 0$. The gap between them is $\delta$ and vanishes for $\delta \to 0$. This means that in the limit $\delta \to 0$ both circles coincide and hence, consist of (a continuum) of fixed points: the phase invariant character of the solution is restored.

Another way to show the restoration of the phase invariance is to transform equation (5.1.1), using polar coordinates: $Re(A) = \gamma \cos(\phi), Im(A) = \gamma \sin(\phi)$. This yields:

$$\gamma_\tau = \gamma(1 - \gamma^2) + \frac{\delta}{2}(\cos^2 \phi - \sin^2 \phi)$$

(5.2.3)

$$\phi_\tau = -\delta \gamma \sin \phi \cos \phi$$

(5.2.4)
and it is immediately clear that $\phi_\tau \to 0$ for $\delta \to 0$: phase invariance is restored.

Then, we consider $\varepsilon \ll \sqrt{\nu}$, which means that $\delta$ has become so large that the imaginary part of $A$ is exponentially damped, so that only the (real) equation for the real part of the amplitude remains. After some cosmetic rescalings and writing $B$ for $B^+$, the equation reads:

$$B_\tau = B + B_{\xi \xi} + \frac{\nu}{\varepsilon} \lambda B^2 - B^3$$

(5.2.5)

A priori, we scaled $B$ with $\varepsilon$. Depending on the situation we will see that it is necessary to adapt this scaling in some cases. Recall that $\lambda \equiv 0$ if $N$ is odd: the stationary, space independent solutions of (5.2.5) are then just $\pm 1$ which is in agreement with the case $\varepsilon = O_s(\sqrt{\nu})$, and within that regime, $\varepsilon$ small with respect to $\sqrt{\nu}$ (in other words, $\delta$ large). For $\lambda \neq 0$, there are two significant different situations possible, depending on the relative magnitude of the quadratic and cubic term. Comparing them, we see that for $\varepsilon \ll \nu$, the cubic term is $o(1)$ with respect to the quadratic term and for $\varepsilon \gg \nu$, it is the other way around. Now, we consider three cases with respect to the solutions of (5.2.5):

- $\nu \ll \varepsilon \ll \nu^{1/2}$

There are two solutions, both of order $\varepsilon$. These solutions should be written down as:

$$B_{1,2} = \varepsilon \hat{B}_{1,2}$$

with

$$\hat{B}_{1,2} = \pm 1 - \frac{\lambda \nu}{\varepsilon} + \cdots$$

(5.2.6)

In the sequel, the hat is always omitted. Both solutions are stable which can be seen immediately by a simple stability analysis and which is what one should expect because these solutions tend to $\pm 1$ for increasing $\varepsilon$; this corresponds to the situation, described in the beginning of this section, i.e. $\varepsilon$ small with respect to $\sqrt{\nu}$ but still $\varepsilon = O_s(\sqrt{\nu})$.

- $\varepsilon = O_s(\nu)$

There are two solutions, both of order $\varepsilon$.

$$B_{1,2} = \frac{1}{2} \lambda \pm \frac{1}{2} \sqrt{\lambda^2 + 4}$$

(5.2.7)

It turns out that, depending on the sign of $\lambda$, either $B_1$ or $B_2$ is stable. Apparently, one of the stable solutions of the previous case becomes unstable for decreasing $\varepsilon$ (meanwhile keeping $\varepsilon = O_s(\nu)$). This can easily be shown if we take into account the $O(\frac{\nu}{\varepsilon})$ terms (which are $O(1)$ is $\varepsilon = O_s(\sqrt{\nu}))$ in (5.2.6)

- $\nu^2 \ll \varepsilon \ll \nu$

In this case, there are two solutions of different magnitude: $\hat{\varepsilon} = O(\varepsilon^2/\nu)$ and $\hat{\varepsilon} = O(\nu)$. The former corresponds with $B = -1 + O(\varepsilon^2/\nu^2)$ and is stable, the latter with $B = -\lambda + O(\varepsilon^2/\nu)$ and is also stable, but becomes unstable for $\varepsilon \to \nu$; in that case, both solutions become of the same order of magnitude $\hat{\varepsilon} = O(\varepsilon)$. This is in agreement with the previous two observations. The situation has been summarized in figure 5.1.
\[ \varepsilon = \mathcal{O}_s(\sqrt{\nu}) \]

\[ \nu \ll \varepsilon < \sqrt{\nu} \]

\[ \varepsilon = \mathcal{O}_s(\nu) \]

\[ \nu^2 \ll \varepsilon < \nu \]

\[ N \text{ even} \]

\[ N \text{ odd} \]

\[ N \text{ arbitrary} \]

Figure 5.1: Visualisation of time- and space independent solutions for \( \varepsilon \gg \sqrt{\nu} \) and smaller, up to \( \varepsilon \ll \nu^2 \). Going from left to right in the figure, \( \delta \) increases. The circles denotes where \( \text{Re}(A) \) and \( \text{Im}(A) \) vanishes. The x-axis corresponds with \( \text{Re}(A) \) and the y-axis corresponds with \( \text{Im}(A) \). For \( \varepsilon \ll \mathcal{O}_s(\sqrt{\nu}) \), the circles have actually no meaning, because the solution has become real. They are still drawn to function as reference with respect to the order of magnitude of the solutions. The circles have radius \( \mathcal{O}(\varepsilon) \).

### 5.3 Spectral solutions

Now, we look at space periodic solutions with a more complicated structure. Again, we consider (5.1.1). In this, and in the next section, we assume that \( \delta = o(1) \) so that we can perform an asymptotic analysis in the small parameter \( \delta \). In physical terms, this means that \( R \) is relatively large with respect to \( R_c \) (as measured in terms of \( \sqrt{\nu} \)). Before starting the analysis, we consider briefly the unperturbed case \( \delta \equiv 0 \). In that case, the equations reduce to the usual Ginzburg-Landau equation and it is easy to see that there are simple stationary, space periodic solutions with one Fourier mode only: \( A(\xi) = G \exp(i k \xi) \), where \( G = g \exp(i \theta) \), \( g \in \mathbb{R} \), \( \theta \) arbitrary and \( k^2 + g^2 = 1 \). Thus, the existence criterion is given by \( k^2 \leq 1 \). We recall a classical result:

- Solutions of the type described above are stable for \( |k| < \frac{1}{\sqrt{3}} \) (see Eckhaus (1965); this is known as the Eckhaus stability criterion).

Thus, the solutions \( G \exp(i(k \xi + \theta)) \) form a two parameter family in the parameters \( k \) and \( \theta \).

It is obvious that (5.1.1) no longer admits these solutions. Hence, a natural question that arises whenever \( \delta \neq 0 \), \( \delta = o(1) \) is:

- What happens to the stable solutions in the Eckhaus band?

This question is the starting point of the subsequent sections.
5.3 Spectral solutions

5.3.1 Preliminaries

Before giving all the details of the analysis, we emphasize that we are interested in the influence that the non-symmetric perturbation $\sim \delta A$, $(\delta = o(1))$ has on the stable solutions within the Eckhaus band. A priori, it is unclear if solutions within the Eckhaus band survive the perturbation, apart from the solutions with wavenumber close to zero. Therefore, we first study the existence of solutions of the form

$$A(\xi, \tau) = Ge^{ik\xi} + \delta B(\xi, \tau)$$  \hspace{1cm} (5.3.1)

where $G = gexp(i\theta)$, $\theta$ arbitrary. This means that we consider perturbations of the (stable and unstable) ‘Eckhaus’-solutions; in section 5.4 we study the stability properties of these solutions.

We derive an equation for the perturbation $B$, which reads up to $O(\delta)$:

$$B_\tau = \hat{r}B + B_{\xi\xi} - (2|G|^2B + G^2e^{2ik\xi}B) + \frac{Ge^{-ik\xi}}{2}$$  \hspace{1cm} (5.3.2)

It is clear that the $O(1)$-part of (5.3.1) (which consists of a single mode with wave number $k$) generates at $O(\delta)$ higher harmonics (with mode $-k$ and mode $3k$) and a correction on the $k$-mode itself. This will happen at any level in $\delta$, in other words, at $O(\delta^2)$ the modes $-3k$ and $5k$ are generated, and modes $-k, k$ and $3k$ will get an $O(\delta^2)$ correction and so on. A priori, one would expect that the spectral solution has time dependent Fourier modes, i.e. modes $exp(in(k\xi + W\tau))$ with $W = O(\delta)$. However, it turns out that these modes do not occur, which means that spatially periodic solutions of (5.1.1) are stationary. This can be seen as follows. We observe that

$$A(\xi, \tau) = Ge^{i(k\xi + W\tau)}$$  \hspace{1cm} (5.3.3)

with $G \in$ (i.e. $G = gexp(i\theta)$) is a solution of (5.1.1) for $\delta = 0$ and $W = 0$. Time periodic solutions (if they do exist) will generally be created by a Hopf-bifurcation from a stable fixed point. A necessary condition for a Hopf-bifurcation at $\delta = 0$ is that it least two of the eigenvalues belonging to the fixed point are pure imaginary, while the other eigenvalues have negative real part. We will show that this is not possible, i.e. there occurs no Hopf-bifurcation for $\delta = 0$ which means that for small $\delta$, there do not exist time-periodic solutions which merge with a solution of the type (5.3.3) as $\delta \downarrow 0$. In other words: $W$ in (5.3.3) should be zero. To show this, we linearize around (5.3.3). Thus, we put:

$$A(\xi, \tau) = Ge^{ik\xi} + \rho(\xi, \tau)e^{ik\xi}$$  \hspace{1cm} (5.3.4)

$$\rho(\xi, \tau) = \int_{-\infty}^{\infty} \rho_l(\tau)e^{il\xi}dl$$  \hspace{1cm} (5.3.5)

with $\rho_l$ the Fourier coefficients of $\rho$. Substitution and collecting equal modes results in a decoupling (which is due to the cubic character of the nonlinearity and to the fact that $\delta = 0$) where $\rho_l$ is coupled to $\overline{\rho}_l$. Symbolically, the perturbation equations can be written as:

$$\begin{pmatrix} \dot{\rho}_l \\ \overline{\rho}_l \end{pmatrix} = \mathcal{M} \begin{pmatrix} \rho_l \\ \overline{\rho}_l \end{pmatrix}$$  \hspace{1cm} (5.3.6)
where
\[ M = \begin{pmatrix} 1 - k^2 - 2kl + l^2 - 2(1 - k^2) & -(1 - k^2) \\ -(1 - k^2) & 1 - k^2 + 2kl - l^2 - 2(1 - k^2) \end{pmatrix} \] (5.3.7)

The eigenvalues of \( M \) (which depend on \( l \) and \( k \)) determine the stability properties of the solution \( G \exp(ik\xi) \). A simple calculation shows that the eigenvalues can be written as:
\[ \lambda_{1,2} = -\left( (1 - k^2) - l^2 \pm \sqrt{(1 - k^2)^2 + 4k^2l^2} \right) \] (5.3.8)

It is easy to check that the eigenvalues can never be purely imaginary. This completes the argument. Hence, \( W \) in (5.3.3) should be zero. It should be noted however, that bifurcations to closed orbits can occur by other mechanisms than the Hopf-bifurcation, see for instance Marsden and McCracken (1976) for an example. It is not likely that these mechanisms play a role here, because the condition that at least some of the eigenvalues should be in \( i \) is necessary for a local bifurcation to take place (see again Marsden and McCracken (1976)).

With this analysis, we have also re-derived the well known Eckhaus stability criterion, which states that solution (5.3.3) is stable for \( |k| < \sqrt{1/3} \). This can easily be observed from (5.3.8): for stability, \( \lambda \) should be negative for all \( l \) which immediately gives the Eckhaus stability boundary. Finally, note that this observation does not imply that time-dependent solutions do not exist for all \( \delta \). For large \( \delta \), it is still possible that there occur bifurcations in which time-dependent solutions are created.

### 5.3.2 Remark

In the sequel, we encounter for \( \delta \neq 0 \) a solution of (5.1.1) with a full Fourier series, instead of only one mode, as in (5.3.3). The stability analysis of that kind of solutions is a little bit more subtle. We find it useful to pay already some attention to this analysis. We do that by re-considering the stability analysis of the solution (5.3.3) (with \( W \equiv 0 \)). Instead of (5.3.4), we propose
\[ A(\xi, \tau) = Ge^{ik\xi} + \rho(\xi, \tau) = Ge^{ik\xi} + \sum_{j=-\infty}^{\infty} \rho_j e^{ijk\xi} \] (5.3.9)
with \( \rho_{j,l} \) the Fourier coefficients of \( \rho_j \). This is of course unnecessarily complicated in this context, but on the other hand, it is exactly what one should do if the solution has a full Fourier series. Substitution and collecting equal modes results again in a decoupling where now \( \rho_{j,l} \) is coupled to \( \overline{\rho}_{-j+2,l} \). The perturbation equations can then symbolically be written as
\[ \begin{pmatrix} \dot{\rho}_{j,l} \\ \dot{\overline{\rho}}_{-j+2,l} \end{pmatrix} = M_j \begin{pmatrix} \rho_{j,l} \\ \overline{\rho}_{-j+2,l} \end{pmatrix} \] (5.3.11)
5.3 Spectral solutions

with $\mathcal{M}_j$ now depending on $l, j$ and $k$:

$$
\mathcal{M}_j = \begin{pmatrix}
\mathcal{E}_1 & \mathcal{E}_2 \\
\mathcal{E}_2 & \mathcal{E}_3
\end{pmatrix} \quad (5.3.12)
$$

\begin{align*}
\mathcal{E}_1 &= 1 - j^2k^2 - 2jkl - l^2 - 2(1 - k^2) \quad (5.3.13) \\
\mathcal{E}_2 &= -k^2 - 1 \quad (5.3.14) \\
\mathcal{E}_3 &= 1 - (2 - j)^2k^2 + 2(2 - j)kl - l^2 - 2(1 - k^2) \quad (5.3.15)
\end{align*}

It is readily observed that $\mathcal{M}_1 = \mathcal{M}$ (see (5.3.7)). It is also easy to see that the graph of the eigenvalues of $\mathcal{M}_j$ is the same for all $k, j$, up to a translation $j \to (1 - j)k$. This means that the eigenvalues of $\mathcal{M}_j$ are centered around $l = (1 - j)k$; the results for $j \neq 1$ are just the translated results for $j = 1$.

### 5.3.3 The structure of the spectral solution

Let us now continue the analysis. Because we know that we can expect (at least for small $\delta$) stationary solutions, we assume that a general spectral solution of (5.1.1) can be written as:

$$
A(\xi) = \sum_{n=-\infty}^{\infty} A_n(\tau)e^{ink\xi} \quad (5.3.16)
$$

where

$$
A_1 = G = ge^{i\theta} + h.o.t. \quad (5.3.17)
$$

$$
A_{1\pm n} = O(\delta^n), \ n \neq 0 \quad (5.3.18)
$$

This assumption is motivated by the fact that we want to study ‘analogons’ of the solutions $A(\xi) = G e^{ik\xi}$ which satisfies the symmetric equation. Besides, in section 5.5, we use numerical simulations; the structure of possible solutions is then precisely as given in (5.3.16) (truncated at some $N$). A direct consequence of the assumption is that we study equation (5.1.1) on a finite interval with length $2\pi/k$ and periodic boundary conditions. An alternative view is that one divides the interval into a number of subintervals with equal lengths $2\pi/q$, solve the equation on the subintervals (with periodic boundary conditions) and glue the intervals back together.

It is easy to check that assumption (5.3.16) is invariant under the transformation $A_n \to A_n e^{in\theta}$. This means that without loss of generality, we can take $A_1 \in \mathbb{C}$ (this is to say that $\theta \equiv 0$ and $g = G \in \mathbb{R}$). Substitution of (5.3.16) into (5.1.1) leads to an infinite system of coupled ordinary differential equations for $A_n(\tau)$ which can be written as

$$
\dot{A}_n = (\delta - n^2k^2)A_n - \sum_{k+l+m=n} A_k A_l \overline{A}_m + \frac{\delta}{2} \overline{A}_n \quad (5.3.19)
$$

In the sequel, we will write for notational convenience $B, C, D, \ldots$ instead of $A_2, A_3, A_4, \ldots$. To find the functions $B, C$ etc., we study the higher order parts of system (5.3.19). From
the preceding we know that \( B \), the \( O(\delta) \) part of the solution, contains three modes, \( k \), \(-k\) and 3\( k \); i.e. we write
\[
B(\xi) = \sum_{l=-1,1,3} B_l e^{ik\xi} \quad (5.3.20)
\]
Substitution leads to an equation for \( B_1 \) (recall that \( G \in \mathbb{C} \)):
\[
0 = (1 - k^2 - 2G^2)B_1 - G^2 \overline{B_1} - \frac{1}{2}G
\]
\[
= -G^2 (B_1 + \overline{B_1}) - \frac{1}{2}G \quad (5.3.21)
\]
The solutions of \( (5.3.21) \) can be written as
\[
B_1 = -\frac{1}{4G} + ib_1, \ b_1 \text{ arbitrary} \quad (5.3.22)
\]
For the coefficients \( B_{-1} \) and \( B_3 \) we get a coupled system of algebraic equations:
\[
B_{-1} = (1 - k^2 - 2G^2)B_{-1} - G^2 B_3 + \frac{1}{2}G \quad (5.3.23)
\]
\[
B_3 = (1 - 9k^2 - 2G^2)B_3 - G^2 B_{-1} \quad (5.3.24)
\]
which can easily be solved. We find
\[
B_{-1} = \frac{G(1 - 9k^2 - 2G^2)}{2(G^4 - (1 - k^2 - 2G^2)(1 - 9k^2 - 2G^2))} \quad (5.3.25)
\]
\[
B_3 = \frac{G^3}{2(G^4 - (1 - k^2 - 2G^2)(1 - 9k^2 - 2G^2))} \quad (5.3.26)
\]
At this point we see that we have to exclude \( k = 0 \) from the analysis, because it gives rise to an singular system of equations for \( B_{-1} \) and \( B_3 \). This indicates singular behaviour for \( k \to 0 \), i.e. for the Stokes wave. However, this is no surprise because we already noted that for \( k = 0 \), we only found four solutions, two (stable) real and to (unstable) pure imaginary.

The analysis can then be continued up to arbitrary order in \( \delta \). At every next order of \( \delta \), two new modes are generated, while all the existing modes get a correction in their amplitudes. Note that the coefficient of the mode \( \exp(ik\xi) \) gets an arbitrary imaginary part at every level of \( \delta \). This is consistent with the observation that there are no time-periodic solutions in the form of a Fourier series possible. The results can now be summarized as follows. Considering \( (5.1.1) \) with \( (5.3.16) \), the solution reads:
\[
A(\xi) = (G + \delta B_1 + \delta^2 C_1 + \cdots)e^{ik\xi + i\theta} + \text{higher harmonics of } O(\delta) \quad (5.3.27)
\]
\[
B_1 = -\frac{1}{4G} + ib_1, \ b_1 \text{ arbitrary}
\]
\[
C_1 = c_{\text{real}} + ic_{1}, \ c_1 \text{ arbitrary, } c_{\text{real}} = c_{\text{real}}(G)
\]
Note that these solutions form a three-parameter family (in the parameters \( k, \theta \) and \( b_1; c_1 \) is just the higher order correction on \( b_1 \)) whereas in the unperturbed case the solutions formed a two-parameter family (phase independent solutions).

To answer the question posed in the Introduction of this Chapter, we must consider the stability of the stationary, space periodic solutions \( (5.3.27) \).
5.4 Stability properties of space periodic solutions

Let us describe the procedure that we are going to follow. Starting with the results of the previous section, we take advantage of the small parameter $\delta$ and analyse the modifications of the unperturbed eigenvalues and eigenvectors when a perturbation is applied.

The idea remains the same as in the previous section. We study the stability of the solution given in (5.3.27), which we denote by $A_p$, against a general perturbation $\rho$. Thus, we substitute

$$A(\xi) = A_p(\xi) + \rho(\xi, \tau)$$

and derive an equation for $\rho$:

$$\rho_\tau = \hat{r}_\rho + \rho_{\xi \xi} - \left(2|A_p|^2 \rho + A_p^2 \rho \right) + \frac{1}{2} \delta \bar{\rho}$$

Then, taking Remark 5.3.2 into account, we write $\rho$ as:

$$\rho(\xi, \tau) = \sum_{j=-\infty}^{\infty} \rho_j e^{ij\xi} \text{ where}$$

$$\rho_j(\xi, \tau) = \int_{-\infty}^{\infty} \rho_{j,l}(\tau) e^{il\xi} dl$$

After some calculations, we find equations for the coefficients $\rho_{j,l}$ which reads:

$$\dot{\rho}_{j,l} = H_j \rho_{j,l} - \left( J_2 \bar{\rho}_{-j+2,l} + J_0 \bar{\rho}_{-j,l} + J_4 \bar{\rho}_{-j+4,l} \right)$$
$$- (L_0 \rho_{j,l} + L_2 \rho_{-j-2,l} + L_{-2} \rho_{-j+2,l}) + \frac{1}{2} \delta \bar{\rho}_{-j,l}$$

where the coefficients $H_j, J_i$ and $L_i$ can be calculated explicitly. They depend on $\delta$ and are given in Appendix A. It is important to note that (apart from $H_j$) only $L_0, J_2 = O(1)$; the other coefficients are $O(\delta)$. Thus, for $\delta \to 0$, equation (5.4.4) reduces to the equation encountered in the unperturbed case (see (5.3.11)). The task is now to calculate the eigenvalues of the matrix $\mathcal{M}$ which determines the stability of the zero-solution of (5.4.4), and thus of $A_p$. This matrix $\mathcal{M}$ is infinite dimensional and, unlike the unperturbed case, does not decouple in $2 \times 2$-matrices. Nevertheless, a lot of structure remains, and symbolically, the $\mathcal{M}$ can be represented as $\mathcal{M} = \mathcal{N}_0 + \delta \mathcal{N}_1 + \delta^2 \mathcal{N}_2 + \ldots$ where

$$\mathcal{N}_0 = diag(\mathcal{A}_i^{(0)})$$
$$\mathcal{N}_1 = diag(\mathcal{B}_i^{(1)}, \mathcal{A}_i^{(1)}, \mathcal{B}^{(1)})$$
$$\mathcal{N}_2 = diag(\mathcal{C}_i^{(1)}, \mathcal{B}_i^{(2)}, \mathcal{A}_i^{(2)}, \mathcal{B}^{(2)}, \mathcal{C}^{(1)})$$

The subscript `*' means adjoint and the superscript denotes the order in $\delta$. All the matrices in (5.4.5)-(5.4.7) are $2 \times 2$ and can be calculated using the expansions in $\delta$ of $H_i, J_i$ and $L_i$; the relevant parts are given in Appendix A. Note that the matrices $\mathcal{A}_i^{(0)}$ correspond to $\mathcal{M}_i$ (see (5.3.11)). So, $\mathcal{N}_0$ is an infinite dimensional matrix with on the diagonal $2 \times 2$ matrices. It is important for the subsequent analysis to note that the (infinite dimensional) matrices $\mathcal{N}_j$ are self-adjoint.
The aim is to solve the eigenvalue equation $\mathcal{M}\mathbf{v} = \lambda \mathbf{v}$ where both $\lambda$ and $\mathbf{v}$ are expanded in $\delta$. The first order part of the eigenvalue equation leads to the unperturbed eigenvalue $\lambda_i^{(0)}$ (which is a function of $k$ and $l$) and the eigenvector $\mathbf{v}^0 = (\ldots, 0, v_{i_0}^{(0)}, 0 \ldots)$ where $v_{i_0}^{(0)}$ is the eigenvector belonging to the $2 \times 2$ matrix $\mathcal{A}_i^{(0)}$. (For notational convenience, we do not write the subscript ‘$i$’ any longer: the subscripts of $\mathcal{A}^{(j)}$ must be taken from now on relative to ‘$i$’ i.e. $\mathcal{A}^{(j)}_i$ should be read as $\mathcal{A}^{(j)}_i$, $\mathcal{A}^{(j)}_i$ should be read as $\mathcal{A}^{(j)}_{i+1}$ etc.) The $O(\delta)$ problem can symbolically be written as:

$$\mathcal{N}_0 \mathbf{v}^1 + \mathcal{N}_1 \mathbf{v}^0 = \lambda^{(0)} \mathbf{v}^1 + \lambda^{(1)} \mathbf{v}^0$$

(5.4.8)

Then, we look at the $i$-th and $i \pm 1$-th components of the system (recall that we only write subscripts relative to $i$):

- $i$-th component: $A_i^{(1)} v_i^{(0)} + A_i^{(0)} v_i^{(1)} = \lambda^{(0)} v_i^{(1)} + \lambda^{(1)} v_i^{(0)}$ (5.4.9)
- $i + 1$-th component: $A_i^{(1)} v_{i+1}^{(0)} + B_i^{(1)} v_{i+1}^{(0)} = \lambda^{(0)} v_{i+1}^{(1)}$ (5.4.10)
- $i - 1$-th component: $A_{i-1}^{(0)} v_{i-1}^{(0)} + B_i^{(1)} v_{i-1}^{(0)} = \lambda^{(0)} v_{i-1}^{(1)}$ (5.4.11)
- $i + j$-th component: $A_j^{(0)} v_j^{(1)} = \lambda^{(0)} v_j^{(1)}$ (5.4.12)

From (5.4.10- 5.4.12) we immediately find

$$v_i^{(1)} = -(A_i^{(0)} - \lambda^{(0)})^{-1} B_i^{(1)} v_i^{(0)}$$

(5.4.13)

$$v_{i-1}^{(1)} = -(A_{i-1}^{(0)} - \lambda^{(0)})^{-1} B_i^{(1)} v_{i-1}^{(0)}$$

(5.4.14)

$$v_j^{(1)} = 0 \text{ for } j \neq 0, \pm 1$$

(5.4.15)

In principle, $v_j^{(0)}$ can also be calculated (see (5.4.16)), but this is not needed for the analysis and is thus omitted. In order to find $\lambda^{(1)}$, the first order correction to the eigenvalue, we make use of the fact that we are dealing with self-adjoint matrices. This means that the eigenvectors are orthogonal (with respect to the Hermitian inner product) and that the image of $(\mathcal{A}^{(0)} - \lambda^{(0)} I)$ is spanned by $w^{(0)}$ where $w^{(0)}$ is the second eigenvector of $\mathcal{A}^{(0)}$. Thus, by (5.4.9):

$$(\mathcal{A}^{(1)} - \lambda^{(1)}) v^{(0)} = -(\mathcal{A}^{(0)} - \lambda^{(0)}) v^{(1)} = \mu w^{(0)}$$

(5.4.16)

for a certain $\mu$. and taking the inner product on both sides with $v^{(0)}$ gives $< (\mathcal{A}^{(1)} - \lambda^{(1)}) v^{(0)}, v^{(0)} > = 0$ and we can calculate $\lambda^{(1)}$.

Putting things together, we have found an expression for the eigenvalue and the $O(\delta)$ correction:

$$\lambda(k, l) = \lambda^{(0)}(k, l) + \delta \lambda^{(1)}(k, l) \text{ where}$$

$$\lambda^{(0)} = k^2 - l^2 - 1 + \sqrt{(1 - k^2)^2 + 4kl^2}$$

(5.4.17)

$$\lambda^{(1)} = \frac{\sqrt{(1 - k^2) + 4kl^2(k^2 - 1 - 2kl)}}{\sqrt{(1 - k^2)^2 + 4k^2l^2} (-2\sqrt{(1 - k^2) + 4k^2l^2} + 2kl)} - \frac{-(1 - k^2)^2 + 4k^2l^2) + 2kl(k^2 - 1)}{\sqrt{(1 - k^2)^2 + 4k^2l^2} (-2\sqrt{(1 - k^2) + 4k^2l^2} + 2kl)}$$

(5.4.18)
We know a priori that $\lambda(k,0) = 0$, because we can always construct a perturbation in the direction of the periodic solution which is marginally stable. Furthermore, calculations show that $\lambda^{(1)} > 0$ for all $k,l \neq 0$ and $\lambda^{(1)}(k,0) = 0$. Thus, the perturbation has a destabilizing effect and we can immediately conclude that the Eckhaus band cannot become wider. In order to see whether it becomes more narrow or not, we expand $l$ around zero, setting $l = \delta l_1$. This yields for $\lambda(k,l)$:

$$
\lambda(k,l) = \delta^2 \left( l_1^2 \left( -1 + \frac{2k^2}{1 - k^2} \right) \right) - \delta^3 \frac{k^2 l_1^2}{(1 - k^2)^2} + h.o.t.
$$

(5.4.20)

where the $O(\delta^3)$ term in (5.4.20) comes from the expansion of $\delta \lambda^{(1)}$. Thus, we observe that for $k$ sufficiently far inside the Eckhaus band (that is, $|k| < \sqrt{1/3} + O(\delta)$) the $O(\delta^3)$ destabilizing correction can never be bigger than the $O(\delta^2)$ terms, and all solutions with $k$ satisfying $|k| < \sqrt{1/3} + O(\delta)$ are stable. What remains to be analysed are those solutions which are $O(\delta)$ close to the boundary of the Eckhaus band. So, we put

$$
k = \sqrt{\frac{1}{3}} - k_1 \delta, \ k_1 > 0
$$

(5.4.21)

and expand (5.4.20). This yields:

$$
\lambda(k_1,l_1) = l_1^2 \left( -3\sqrt{3}k_1 + \frac{3}{4} \right)
$$

(5.4.22)

and for stability, we must have $\lambda(k_1,l_1) < 0$ for all $l_1$. Thus, a necessary condition is $k_1 > \frac{1}{4\sqrt{3}}$ and apparently, for

$$
|k| < \sqrt{1/3} - \frac{1}{4} \delta \sqrt{1/3} + O(\delta^2)
$$

(5.4.23)

we have stable spectral solutions of the type (5.3.27). We conclude that the Eckhaus band becomes $O(\delta)$ narrower. This answers the question posed in the Introduction. Note that the arbitrary coefficient $b_1$ does not show up in the correction.

The above analysis is restricted to small $\delta$. It seems however interesting to look at the dynamics of (5.1.1) for $\delta = O(1)$ and large, within the regime $\varepsilon = O_\varepsilon(\sqrt{\nu})$. Obviously, this can no longer be done with an asymptotic analysis and therefore, we turn to numerical simulations. Because we cannot simulate an infinite dimensional model as given in (5.3.19), we have to truncate solution (5.3.16) at some mode, say $N$. One needs to be careful about the choice of $N$. One one hand, $N$ must be large enough from convergence point of view, on the other hand $N$ should be small to keep the computations into reasonable bounds. We come back on this in section 5.5.3.

One must immediately realise that truncation has an important consequence for the Eckhaus stability criterion, which states that waves with wave numbers $q > \sqrt{1/3}$ are unstable, and they become unstable by the side-band mechanism. This mechanism indicates that perturbations with a very small wave number (i.e. very long waves) are the most unstable ones, see Stuart and DiPrima (1978). This observation however is a crucial point, because
in the case that we are dealing with a finite interval, the ‘most destabilizing’ waves do not fit in the interval.

If we want to understand the stability properties of solutions of the truncated system for large $\delta$, we must first understand the consequence of truncation for the case $\delta \equiv 0$. After this, we can study the influence on that situation for $\delta = O(1)$. We expect to see ‘somewhere’ in the dynamics the presence of $\omega^-$ and $\omega^+$, the neutral stability curves.

## 5.5 A finite-dimensional model of the non-symmetric Ginzburg-Landau equation

In this section we fix a wave number and consider the spectral solution (5.3.16), truncated at mode $N$. In this finite dimensional analysis, we denote the wave number with $q$; we do this to distinguish this case from the previous section where we denoted the wave number with the symbol $k$. Note that $q$ determines the length of the finite interval (which is $2\pi/q$) that we consider. First, we consider $\delta = 0$.

### 5.5.1 The symmetric problem

We will look at the spectral solution

$$A(\xi) = \sum_{n=-N}^{N} Z_n e^{inq\xi}$$

and we determine the stability properties of what we denote as a pure $p$-mode. These solutions can be considered as the finite dimensional equivalents of the solutions $G\exp(ikx)$ that satisfy the Ginzburg-Landau equation (i.e. (5.1.1) with $\delta \equiv 0$) and read as follows:

$$Z_p = \sqrt{1 - p^2q^2}$$

$$Z_n = 0, \ n \neq p$$

We immediately observe that the existence criterion is given by $q^2 \leq 1/p^2$ (which can be considered as the equivalence of $k^2 < 1$). A stability analysis of this solution (putting $Z_p = \sqrt{1 - p^2q^2} + \varepsilon z_p, Z_n = \varepsilon z_n$, and linearizing) leads to:

$$\dot{z}_p = -2(1 - p^2q^2)z_p$$

$$\begin{pmatrix}
\dot{z}_n \\
\dot{z}_{2p-n}
\end{pmatrix} =
\begin{pmatrix}
\mathcal{E}_1 & \mathcal{E}_2 \\
\mathcal{E}_2 & \mathcal{E}_3
\end{pmatrix}
\begin{pmatrix}
z_p \\
z_{2p-n}
\end{pmatrix}$$

$$\mathcal{E}_1 = 1 - n^2q^2 - 2(1 - p^2q^2)$$

$$\mathcal{E}_2 = p^2q^2 - 1$$

$$\mathcal{E}_3 = 1 - (2p - n)^2q^2 - 2(1 - p^2q^2)$$

Obviously, the pure-$p$-mode is stable whenever $q^2 < 1/p^2$ (see (5.5.4); this is exactly the existence criterion) and if the eigenvalues of systems (5.5.5) have negative real parts for all $n$. It is now readily seen that the results for a pure $p$-mode and a pure $-p$-mode are
Table 5.1: The ‘expected’ and the calculated stability bounds for the pure 0,1,2 and 3-mode identical. However, we must be careful. The spectral model is truncated at mode $N$. Thus, if we consider the stability of the $p$-mode, the coupling between $z_n$ and $z_{2p-n}$ only exists if $n$ and $2p - n$ are both in the interval $[-N,N]$. If $2p - n \notin [-N,N]$, equation (5.5.5) reduces to

$$\dot{z}_n = \left(1 - n^2q^2 - 2(1 - p^2q^2)\right)z_n$$

(5.5.6)

and we find stability if $(1 - n^2q^2) - 2(1 - p^2q^2) < 0$. For the cases where both $p$ and $2p - n$ are in $[-N,N]$ the eigenvalues of the matrix given in (5.5.5) must have negative real parts. These conditions determine a number of critical $q_p$’s, below which the pure $Z_p$ mode is stable. The minimum of these values is the ‘$p$-discrete analogon’ of the Eckhaus criterion. Because this is an important line of thoughts, we emphasize the results using $p = 3$.

With $q_a$, we denote the ‘analytical’ stability boundary, obtained by Eckhaus, see Eckhaus (1965): $q_a = \sqrt{1/3}$. Naïvely, we might expect that the pure 3-mode $Z_3 exp(3i\xi \xi)$ becomes unstable if $3q > q_a$, thus $q > \sqrt{1/27}$. However, if we perform the calculations that are mentioned above, it turns out that the 3-mode becomes unstable at $q_3 = \sqrt{1/18}$ (see table 5.1) and we observe that $q_a < q_3$. This latter fact should be no surprise because we already noted that the solutions become unstable by the side-band mechanism. Due to the finite interval (which was a consequence of the truncation) the ‘most unstable’ perturbations, which have a wave number arbitrarily close to zero, do not fit in the interval. Instead, the pure 3-mode only becomes unstable if there is a perturbation possible which fits in the interval. A similar story can be told for the pure 2 and 1 mode. For these cases, we also find that $q_{1,2}$, the discrete stability boundaries, are larger than $q_a$. The results (for $N = 3$) are summarized in table 5.1. In the Remark at the end of this section, we will give a motivation for this choice of $N$. It will appear that $N = 3$ is in some sense a minimum, which enables us to study two different types of solutions. Increasing $N$ does not add essentially new elements to the analysis. Besides, $N = 3$ will turn out to be large enough from convergence point of view.
5.5.2 The non-symmetric problem

Let us now look at the influence of \( \delta \) with respect to these pure \( p \)-modes. To give an idea of things that may happen, we analyse the perturbed case on the basis of \( N = 3 \). At this point we do not, unlike the numerical computations that we presented in Chapter 2, section 2.5, assume that \( Z_n = Z_{-n} \). The reason is that in that particular case, the Stokes wave (i.e. the pure zero-mode), is a global attractor of the resulting dynamical system. Thus, assuming that \( Z_n = Z_{-n} \) disables us to consider the stability properties of pure modes (apart from the pure zero-mode). Dropping this assumption means that we consider a 7-dimensional complex or 14-dimensional real system. Now, due to the perturbation, a pure \( p \)-mode does no longer exist for \( \delta \neq 0 \). From the previous analysis we know that if we consider a pure \( p \)-mode for the unperturbed case, the results for the perturbed case are that the \(-p, 3p, -3p, 5p\) etc. modes become excited. Thus, for \( p = 3 \) (and \( N = 3 \)), this means that only the 3 and the -3 mode exist, while for a pure 2-mode, the 2 and -2 mode exist. In case of a pure 1-mode, the \(-1, 1\) and \(-3\) modes will be excited for \( \delta \neq 0 \). The latter case is the most complicated one. The 0-mode is the only pure mode which exists for \( \delta \neq 0 \). When we talk in the sequel about ‘pure modes’, we actually mean these modes; the perturbed analogon of the pure modes for \( \delta = 0 \).

Let us first consider the pure 3-mode. To find the amplitudes of the 3 and \(-3\)-mode, we have to solve the following system of equations, (we have already taken into account the phase-invariance, which means that we only have to look for real solutions in \( z_3 \) and \( z_{-3} \); see (5.3.16) and the text below that)):

\[
(1 - \frac{1}{2} \delta - 9q^2)z_{-3} - (2z_{-3}^3 + z_{-3}z_3) + \frac{\delta}{2}z_3 = 0 \tag{5.5.7}
\]

\[
(1 - \frac{1}{2} \delta - 9q^2)z_3 - (2z_3^3 + z_3^2z_{-3}) + \frac{\delta}{2}z_{-3} = 0 \tag{5.5.8}
\]

These equations have 8 solutions for \( z_3 \) and \( z_{-3} \):

\[
z_3 = -z_{-3} = \pm \sqrt[3]{\frac{1 - \delta - 9q^2}{3}} \tag{5.5.9}
\]

\[
z_3 = z_{-3} = \pm \sqrt[3]{\frac{1 - 9q^2}{3}} \tag{5.5.10}
\]

\[
\begin{align*}
\begin{cases}
  z_{-3} &= \pm \frac{1}{4} \left( \sqrt{C_1 + \sqrt{C_2}} \right) \left( \sqrt{C_2} - C_1 \right) \\
  z_3 &= \mp \frac{1}{2} \sqrt{C_1 + \sqrt{C_2}} \\
  z_{-3} &= \pm \frac{1}{4} \left( \sqrt{C_1 - \sqrt{C_2}} \right) \left( \sqrt{C_2} + C_1 \right) \\
  z_3 &= \pm \frac{1}{2} \sqrt{C_1 - \sqrt{C_2}}
\end{cases}
\end{align*} \tag{5.5.11}
\]

\[
\begin{align*}
  C_1 &= 2(1 - 9q^2) - \delta \tag{5.5.13} \\
  C_2 &= \left( 2(1 - 9q^2) - \delta \right)^2 - 4\delta^2 = C_1^2 - 4\delta^2 \tag{5.5.14}
\end{align*}
\]

We are actually only interested in the solutions which can be considered as pure 3-modes for \( \delta \to 0 \). In that view, one might argue that we can neglect (5.5.9) and (5.5.10): for
δ → 0, these solutions (which are both real !) tend to solutions in which the 3-mode as well as the -3-mode are present. However, solution (5.5.10) turns out to be very important, because, despite that it does not tend to a pure 3-mode for δ → 0, it can bifurcate into one. The solution (5.5.9) is indeed neglected. It turns out not to be important for our analysis; besides it is always unstable. The solutions (5.5.11) and (5.5.12) show the beformentioned behaviour: they tend to pure modes for δ → 0. In figure 5.2, we have plotted all the solutions as function of δ for two specific values of q. This shows that the remaining 6 solutions are of interest: either they are pure modes or they can bifurcate into pure modes. Figure 5.2 also shows that the the rôle of a pure 3 and -3 mode can be interchanged. Results, obtained for a pure 3-mode can be directly applied to the pure -3-mode.

The stability properties of these solutions can easily be studied by direct linear analysis, considering them as solutions of the 7-dimensional complex system. The results are shown in figures 5.2 and 5.3. These figures should be interpreted as follows. In figure 5.3, we have plotted curves in the q,δ-plane, which bound the areas of stability of the different types of solutions. In figure 5.2, we consider the situation for two fixed values of q, q = 0.1 and q = 0.2. The heavy lines denote the amplitude of the 3-mode and the thin lines denote the amplitude of the -3 mode, both as function of δ. Stable solutions are represented by solid lines while dashed lines indicate unstable solutions. We can distinguish between various areas.

First of all, we note that we only consider wave numbers q < √1/18. This is because we study bifurcating solutions from a stable, pure 3-mode. As indicated in table 5.1, the 3-mode is unstable for q > √1/18. We discuss the behaviour of the relevant solutions for q < q* and q > q*, where q* is defined as the intersection of the three curves, plotted in figure 5.3. A simple analysis shows that q* = 1/6.
Consider a fixed $q < q^*$, $\delta = 0$ and the pure 3-modes given in (5.5.11)-(5.5.12). As indicated in figure 5.3, solution (5.5.10) is unstable for these conditions. Then, we increase $\delta$. As a result, the amplitude of the -3-mode increases, while the amplitude of the 3-mode decreases. For $\delta = 2/3 - 6q^2$, both amplitudes have become equal and the solutions (5.5.10)-(5.5.12) are similar. Increasing $\delta$ further, (5.5.11) and (5.5.12) ‘vanish’ (that is, they become complex but, as already explained, we should only consider real solutions for $z_3$ and $z_{-3}$) and (5.5.10) (which is independent of $\delta$!) becomes stable. This latter solution exists and remains stable for $\delta \to \infty$. The situation for $q = 0.1$ is plotted in figure 5.2 (a).

For $q > q^*$ the situation is a little bit different. Solutions (5.5.11) and (5.5.12) are stable for $\delta$ small, but become unstable for $\delta = 1 - 18q^2$. For $\delta$ larger than $1/3 + 6q^2$, the (real) solution (5.5.10) is stable, but becomes unstable at $1/3 + 6q^2$. The question is then whether the solutions that bifurcate from (5.5.11) and (5.5.12) at $\delta = 1 - 18q^2$ ‘match’ the solutions that bifurcate from (5.5.10) at $\delta = 1/3 + 6q^2$. From an eigenvalue analysis, we see that at $\delta = 1 - 18q^2$ (that is, coming from below in figure 5.3) the real part of the 0-mode becomes unstable. We can then numerically follow the bifurcating solution and consider the bifurcation scenario further. This has not been studied in detail. On the other hand, if we come from above in figure 5.3, an eigenvalue analysis shows that at $\delta = 1/3 + 6q^2$, the imaginary part of the 0-mode becomes unstable and a subsequent bifurcation, where the imaginary part of the -3,3 and the real part of the 0-mode becomes unstable, happens at $\delta = 2(1 + 35q^2)/9$. This indicates that, at least for $q$ close to $q^*$, both bifurcation scenarios (that is, coming from below and coming from above, see figure 5.4) do not match.
Figure 5.4: A blow up of figure 5.3 in the neighbourhood of $q_* = 1/6, \delta = 1/2$. At the curves $\delta = 1 - 18q^2$, solution (5.5.11) bifurcates, at $\delta = 1/3 + 6q^2$, solution (5.5.10) bifurcates, and a subsequent bifurcation of this solution occurs at $\delta = 2(1 + 35q^2)/9$.

Summarizing, we conclude that in the light shaded area, we have a stable, complex solution and an unstable, real solution, while in the dark shaded area, we only have a stable, real solution. In the non-shaded areas, bounded by $\delta = 1 - 18q^2, \delta = 1/3 + 6q^2$ and $q^2 = 1/18$, there is no stable solution with only a 3-mode and a -3-mode. We conclude from these observations, and this is emphasised, that for all $q < \sqrt{1/18}$ and $\delta > \max(1/3 + 6q^2, 1 - 18q^2)$ there are pure 3-modes which are stable and real.

A similar story can be told for a pure 2-mode. The results are not shown; they are qualitatively the same as the results for the 3-mode. The stability properties however, are harder to analyse which is the result of the structure of the linearised system (recall that this is a $14 \times 14$ matrix). In figure 5.5, we have shown the equivalence of figure 5.3 for the 2-mode.

Then, we study the situation for a pure 1-mode. The equations that we have to solve in order to find the amplitudes for the $-3, -1, 1$ and 3 mode for $\delta \neq 0$ read:

\[
(1 - \frac{1}{2}\delta - 9q^2)z_3 - \sum_{k+l+m = 3 \atop |k|, |l|, |m| \leq 3} z_k z_l z_{-m} + \frac{\delta}{2} z_3 = 0 \quad (5.5.18)
\]
In the light of the previous cases (concerning a pure 3 and 2 mode) we expect that for a fixed $q$ (where $q$ is of course smaller than $\sqrt{2/5}$, see table 5.1), there exists a $\delta$ for which the total solution becomes real again. Using this information (which means that we can assume $z_1 = z_{-1}, z_3 = z_{-3}$) it is possible to solve the remaining system of equations analytically and study the stability properties. These results (which are again independent of $\delta$ !) are shown in figure 5.6. Again, stable solutions are represented by solid lines, unstable solutions by dashed lines. Then, we can determine for every $q$ the critical $\delta$ for which this solution becomes unstable and construct the equivalents of figures 5.3 and 5.5 (see figure 5.7). The bifurcating stable solutions can then numerically be followed for decreasing $\delta$. One of these solutions must of course tend to the pure 1-mode for $\delta \to 0$.
Finally, we consider the situation for a pure 0-mode, which is a little bit different. As already mentioned, this is the only pure mode which exists for the perturbed case and it is easy to see that the amplitude of this (complex!) mode equals 1 or $\sqrt{1 - \delta/2}$. A stability analysis of this mode is relatively easy because there occurs a coupling between the $n$ and $-n$ mode only (and the 0-mode is coupled to itself). The analysis shows that $z_0 = \pm 1$ is stable for all $\delta$ and $z_0 = \pm i\sqrt{1 - \delta/2}$ is always unstable.

There is one other point which is worthwhile mentioning in this context. From Chapter 4, section 4.4.3 we know that for $(k, R)$ such that we are below $\omega^-$, there exist only real solutions. In the spectral theory, applied to the non-symmetric problem, we observed that, looking at the pure $p$-modes ($p = 1, 2, 3$) there exist a curve $\delta = h_p(q)$, such that for $\delta > h_p(q)$ the solution has become real. Thus, in some sense, we can interpret the curves $h_p(q)$ as discrete analogons of $\omega^-$. Furthermore, the results so far show that for all $q$ smaller than the values indicated in table 5.1 and for $\delta$ sufficiently large, there exist stable, real, periodic solutions (with period $2\pi/q$) of the spectral model, truncated at $N = 3$. These solutions are independent of $\delta$ and can be considered as bifurcating solutions from the pure 0, 1, 2, 3-mode. We observe that the period goes to infinity for $q \to 0$. Now, we recall the equation that is ‘valid’ for $\delta \gg 1$, which is given in Chapter 4, section 4.4.3 and which is just a real Ginzburg-Landau equation. After some rescaling, the equation looks like:

$$A_\tau = A + A_{\xi\xi} - A^3 \quad (5.5.19)$$

It is known that this equation does not have stable, stationary space-periodic solutions. We will show this in Appendix B. This fact however, does not contradict our results concerning the stable spectral solutions, which are also stationary and periodic. These solutions satisfy the spectral model (which is an approximation of the stationary problem associated with
Therefore, the solutions of the spectral model can be regarded as approximations of the stationary solutions satisfying (5.5.19). The fact that they are found to be stable may very well be due to the low dimensional truncation with respect to the number of modes. Indeed, it has been shown in Doelman and Titi (1993) that the estimates for the necessary number of truncation modes \( N \) depend quite sensitive on \( q \): for \( q \to 0 \), \( N \) becomes very large. This might be an explanation for the stability of the low-dimensional, real spectral solution. On the other hand, it is known that (5.5.19) allows front solutions, which can be represented as heteroclinic connections in the \((A, A_\xi)\)-phase plane. In Appendix B, we will show that these fronts are stable solutions. The stable, \((2\pi/q)\)-periodic solutions that we found in the spectral theory can, for \( q \to \infty \), be considered as approximations of these fronts. Thus, the stability properties of these solutions fit well within the frame.

### 5.5.3 Considerations on the truncation number

Let us reflect upon the truncation number \( N \). Throughout this section, we took \( N = 3 \). This choice is not arbitrary. We based this choice on the thought to keep \( N \) as small as possible, but on the other hand we do not want to take \( N \) so small that we exclude interesting behaviour. Therefore, we took \( N = 3 \): it allows us to study the (relatively simple) analogon of pure 2,3-modes for the case \( \delta \neq 0 \) (which contain only \( \pm 2, \pm 3 \)-modes) as well as the more complicated analogon of the pure 1-mode, which contains the \( \pm 1 \) as well as \( \pm 3 \)-modes. Besides, for the spectral models of the Ginzburg-Landau equation (i.e. the problem under consideration, with \( \delta = 0 \)), it is well known that the norm of the Fourier-modes of solutions exhibits exponential decay. This indicates that the Ginzburg-Landau equation can be approximated by very low dimensional Galerkin projections. These facts have been shown in Doelman and Titi (1993). Although the analysis has not been carried out for the non-symmetric Ginzburg-Landau equation, it is clear that the arguments of Doelman and Titi (1993) can also be applied here, and it is expected that the analysis gives similar results: (5.1.1) can be approximated by low dimensional Galerkin projections. Furthermore, we considered the solutions (5.5.10)-(5.5.12) for \( N = 3 \), which are of course also solutions for \( N = 4 \) and \( 5 \). Then we determined their stability properties for \( N = 3, 4, 5 \). The results are shown in figure 5.8. We observe that there is a good agreement between the results which indicates once more that a low dimensional truncation is justified.

### 5.5.4 A remark about travelling wave solutions for \( \delta \) large

The situation \( \delta \gg 1 \) corresponds with \( \varepsilon \ll \sqrt{\nu} \) or, in other words, we are so far below \( \omega^- \) that we should consider the amplitude equation for \( B^+ \) only, (the real part of the amplitude). This equation is real, and reads (see (4.4.43)-(4.4.44); we applied again some cosmetic scalings and write \( B \) for \( B^+ \)):

\[
B_\tau = B + B_\xi \xi + \gamma B^2 - B^3 \tag{5.5.20}
\]

The parameter \( \gamma \) can be determined from (5.2.5); here, we only recall that \( \gamma \equiv 0 \) corresponds to \( N \) is odd and that \( \gamma \to 0 \) for \( \varepsilon \to \sqrt{\nu} \). This is to emphasize that the quadratic term only plays a rôle for \( \varepsilon \) sufficiently small. If we would replace the right hand side of
5.5 A finite-dimensional model of ...  

(5.5.20) with $B_{\xi\xi} + F(B)$ for some general cubic function $F$ with two or three zeroes, the equation is sometimes called Fisher's equation (see Fife (1979)); it can also be considered as a special case of Nagumo's equation (McKean (1970)). Apart from the stationary, space independent solutions which have already been studied in section 5.2, it is well known that there exist also travelling waves and travelling fronts as solutions of this equation. The theory about existence, uniqueness and stability of such fronts has been the subject of many studies, see for instance Henry (1981), Fife (1979), Grindrod (1991). We repeat some of the results that are interesting in this context.

A travelling wave solution of (5.5.20) is defined as

$$B(\xi, \tau) = W(\xi - V\tau) = W(z) \quad (5.5.21)$$

with $z = \xi - V\tau$. This solution can be visualized as a function which, viewed in an appropriately moving coordinate system, appears constant. Substitution of (5.5.21) into (5.5.20) leads to an ordinary differential equation for $W$:

$$W_{zz} = -W + VW_z - \gamma W^2 + W^3 \quad (5.5.22)$$

This equation can easily be studied in the $W, W_z$ phase-space. Depending on the values of $V$ and $\gamma$, there are several possibilities, three of which have been sketched in figure 5.9. They are of particular interest, because of the occurrence of homoclinic or heteroclinic connections between two equilibria. They correspond with (traveling) pulses and fronts, respectively. In the case of the fronts, the solution can be written down explicitly, as is done in Henry (1981). For this case, the solution reads

$$W(z) = \left( \frac{1}{a + e^{-bz}} \right) + d, \quad (5.5.23)$$
$$a = \frac{-1}{\sqrt{\gamma^2 + 4}}$$

$$b = -\sqrt{\frac{4 + \gamma^2}{2}}$$

$$d = \frac{-\gamma + \sqrt{\gamma^2 + 4}}{2}$$

and the corresponding speed $V$ is given by

$$V = -\frac{\gamma}{\sqrt{2}}$$

Thus, one observes that for $\gamma = 0$ (the case where the quadratic term in (5.5.20) vanishes), the front is stationary. In Henry (1981), it is shown that the fronts are stable solutions of (5.5.20).

We emphasize that, speaking in general terms, it is a far from trivial problem to say something about the stability properties of solutions of an ordinary differential equation, considered as solutions of the partial differential equation. In special cases however, some results are known (see for instance Appendix B). Nevertheless, as a first step to gain insight in a general structure of solutions of the PDE, it seems interesting enough to study the stationary ODE. This approach is followed in the next Chapter, where we consider the non-symmetric Ginzburg-Landau equation, and make use of the fact that it can be viewed as a Hamiltonian system.

### Appendix A

In this appendix, we specify the terms $H_j$, $J_i$ and $L_i$. They read:

$$H_j = 1 - \frac{\delta}{2} - l^2 - j^2k^2$$  \hspace{1cm} (A.1)

$$J_0 = 2\delta GB_{-1}$$  \hspace{1cm} (A.2)

$$J_2 = G^2 + 2\delta B_1 G$$  \hspace{1cm} (A.3)

$$J_4 = 2\delta GB_3$$  \hspace{1cm} (A.4)
Appendix B On the stability of space periodic solutions

In this appendix, we will prove the following theorem.

Theorem B.1 Consider the equation

\[ A_t = A_{zz} + F(A) \] (B.1)

for a real-valued function \( A(z,t), z \in \) and \( F \in C^1 \). Let \( a(z) \) be a stationary, \( p \)-periodic solution of

\[ a_{zz} + F(a) = 0 \] (B.2)

with arbitrary period \( p \). Then, \( a(z) \) is an unstable solution of (B.1).

The idea for the proof has already been given by Doelman et al. (1995b). In that paper, it is proven that quasi periodic solutions in space of the complex Ginzburg-Landau equation are unstable. Here, we restrict ourselves to the ‘usual’ and non-symmetric real equation.

Proof

Consider a \( p \)-periodic solution \( a(z) \) of (B.1). We linearize around this solution, i.e. we put \( A = a + U \) and derive an equation for \( U \):

\[ U_t = F'(a)U + U_{zz} \] (B.3)

Then, we look for solutions \( U(z,t) = e^{\lambda t}V \), and substitution leads to the following equation for \( V \):

\[ V_{zz} + V(F'(a) - \lambda) = 0 \] (B.4)

We are going to show that there exist solutions \( V \), satisfying (B.4) with \( \lambda > 0 \) and which are bounded for all \( z \). Hence, the perturbations \( U \) of the solution \( a(z) \) grow exponentially and thus, the solution \( a(z) \) is unstable. Equation (B.4) is known as Hill’s equation and has some well known properties which turn out to be useful in this context. Most of them can be found in Magnus and Winkler (1966). The basis of the proof is the following property:
Theorem B.2 (Oscillation Theorem, see Magnus and Winkler (1966)) To every differential equation (B.4), there belong two monotonically decreasing infinite sequences of real numbers $\lambda_0, \lambda_1, \lambda_2, \ldots$ and $\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4, \ldots$ such that (B.4) has a solution of the same period as $F'(a)$ if and only if $\lambda = \lambda_n, n = 0, 1, 2, \ldots$ and a solution of twice the period of $F'(a)$ if and only if $\lambda = \lambda'_n, n = 1, 2, 3$. The $\lambda_n$ and $\lambda'_n$ satisfy the inequalities

$$\ldots > \lambda_4 \geq \lambda_3 > \lambda'_4 \geq \lambda'_3 > \lambda_2 \geq \lambda_1 > \lambda'_2 \geq \lambda'_1 > \lambda_0$$

(B.5)

The solutions of (B.4) are stable in the intervals

$$(\lambda_3, \lambda'_4), (\lambda'_3, \lambda_2), (\lambda_1, \lambda'_2), (\lambda'_1, \lambda_0), \ldots$$

(B.6)

We emphasize that the stability intervals are bounded from above by $\lambda_0$. The position of $\lambda = 0$ with respect to the intervals is the crucial point in the proof.

Now, we note that $V = a_z$ is a solution of (B.4) for $\lambda = 0$. This can easily be verified, taking into account that $a(z)$ is a solution of (B.2). Then, the main point is to decide whether it is possible that $\lambda_0 = 0$. If that is the case, all perturbations $\exp(\lambda t)V, V$ bounded, decay (because $\lambda < \lambda_0 = 0$) and we have stability. However, we recall from Magnus and Winkler (1966) the following:

**Theorem B.3** The periodic solution, belonging to $\lambda_0$ with the same period as $F'(a)$ has no zeroes.

Obviously, $a_z$ (which is a solution for $\lambda = 0$) does have zeroes, and consequently, $\lambda_0 \neq 0$. But this means exactly that, according to Theorem B.2, there exist solutions of (B.4) with $\lambda > 0$. Thus, $A(z)$ is unstable as solution of (B.1). This completes the proof of theorem B.1.
Chapter 6

A Hamiltonian Approach to Analyse the Non-Symmetric Ginzburg-Landau Equation

6.1 Introduction

In this Chapter, we will follow a different approach to find stationary solutions of the non-symmetric Ginzburg-Landau equation. The motivation for this Chapter comes from the fact that in the previous Chapter, we studied approximating solutions in the form of a Fourier series which, for actual calculations, was truncated at a fairly low mode-number. In order to get more insight in the structure of solutions of the non-symmetric equation, we study the equation once more, but from a different viewpoint.

We consider equation (5.1.1) which is used in Chapter 5 and derived in Chapter 4. We refer to the latter Chapter for the background of the equation, which reads:

\[ A_{\tau} = (1 - \delta)A + A_{\xi \xi} - |A|^2 A + \frac{\delta A}{2} \]  

(6.1.1)

In the sequel of this Chapter, we will consider the stationary equivalent of (6.1.1), i.e. we set \( A_{\tau} \equiv 0 \). Below, we make some observations, formulate our objectives and summarize the main results. In the next sections, we give the details of the analysis.

An important observation is that (6.1.1) (with \( A_{\tau} \equiv 0 \)) can be associated with an integrable Hamiltonian system for arbitrary \( \delta \). This can be seen as follows. First, we split the amplitude \( A \) into its real and imaginary parts. In order order to keep track with the usual notation used in Hamilton theory, we set \( Re(A) = q_1, Im(A) = q_2 \) and consider the spatial variable \( \xi \) as ‘time’. Then, we define a potential function \( V \) as \( V(q_1, q_2) = \frac{q_1^2}{2} + \frac{(1-\delta)q_2^2}{2} + \frac{(q_1^2 + q_2^2)^2}{4} \) and use \( V \) to rewrite (6.1.1) (with \( A_{\tau} \equiv 0 \)) as:

\[
\ddot{q}_1 = -\frac{\partial V}{\partial q_1} = -q_1 - q_1(q_1^2 + q_2^2)  
\]  

(6.1.2)
\[\ddot{q}_2 = -\frac{\partial V}{\partial q_2} = -(1 - \delta)q_2 - q_2(q_1^2 + q_2^2) \quad (6.1.3)\]

Further, we rewrite this second order system as a first order system, setting \(\dot{q} = p\), where \(q = (q_1, q_2)\), \(p = (p_1, p_2)\), and derive the Hamilton equations:

\[
\begin{align*}
\dot{q}_1 &= p_1 \quad (6.1.4) \\
\dot{p}_1 &= -q_1 + q_1(q_1^2 + q_2^2) \quad (6.1.5) \\
\dot{q}_2 &= p_2 \quad (6.1.6) \\
\dot{p}_2 &= -(1 - \delta)q_2 + q_2(q_1^2 + q_2^2) \quad (6.1.7)
\end{align*}
\]

with Hamiltonian

\[H = \frac{1}{2} \left( (p_1^2 + p_2^2) + w_1 q_1^2 + w_2 q_2^2 - \frac{1}{2}(q_1^2 + q_2^2)^2 \right) \quad (6.1.8)\]

where

\[
\begin{align*}
w_1 &= 1 \quad (6.1.9) \\
w_2 &= 1 - \delta \quad (6.1.10)
\end{align*}
\]

The fact that (6.1.1) can be seen as a Hamiltonian system has been well known for \(\delta \equiv 0\): equation (6.1.1) reduces in that case to a complex Duffing equation. Surprisingly enough, introducing the additional term \(\sim \delta A\) does not destroy the Hamiltonian character of the equation. In the subsequent sections, we show another surprisingly fact: system (6.1.4)-(6.1.7) is integrable for all values of \(\delta\). Again, for \(\delta \equiv 0\), this was already known, for \(\delta \neq 0\), it is more than we would expect (see section 6.3 for more details). It implies that the integral manifolds on which the solutions live in the four dimensional phase-space, are two dimensional. Let us denote the Hamiltonian and the second integral for this problem by \(H(q, p, \delta)\) and \(J(q, p, \delta)\) respectively (in the sequel, we only write the arguments of \(H\) and \(J\) in the cases that omitting them may lead to ambiguity. If we write only one argument, it is to emphasize the \(\delta\)-dependency). The main aim of this Chapter will be to determine for which value of \(H\) and \(J\) there exist bounded solutions for \(\delta \geq 0\). Due to the fact that we are dealing with an integrable Hamiltonian system, these bounded solutions (if they exist) ‘live’ on compact manifolds. Thus, we can reformulate the main aim by the following objectives:

- Can we denote in the \((H(\delta), J(\delta))\)-plane a subspace which corresponds with integral manifolds which are compact, or which have compact parts ?
- What is the nature of the solutions on the bounded parts of the integral manifolds ?
- Can we describe the (if any) bifurcation scenario in the \((H(\delta), J(\delta))\)-plane ?

It turns out that it is very illustrative to study first the symmetric case \(\delta \equiv 0\). The information obtained from this case can be used to study the more complicated case \(\delta \neq 0\).
6.1 Introduction

6.1.1 A few words on Morse-theory

For \( \delta < 1 \) (and in particular, \( \delta = 0 \)) we can use Morse-theory to obtain easily a part of the answer on the first two questions posed above. Therefore, we recall the following facts, which can be found in Verhulst (1990). First, we observe that the origin is a non-degenerate critical point of the \( C^\infty \) function \( H \). Functions with such a property are called Morse-functions. It is clear that \( H \) can be expanded in the neighbourhood of the origin as:

\[
H(p, q) = c_1 q_1^2 + c_2 p_1^2 + c_3 q_2^2 + c_4 p_2^2 + h.o.t \tag{6.1.11}
\]

with positive coefficients \( c_1, \ldots, c_4 \). Because there are no \( - \) signs in the quadratic part of \( H \), the index of the origin is zero. Now we can apply the Morse-lemma which states the remarkable fact that in the neighbourhood of the origin, there exist a coordinate transformation such that \( H(p, q) \to G(\tilde{p}, \tilde{q}) \) where \( G(\tilde{p}, \tilde{q}) \) is a Morse-function with the origin as critical point, with index zero and which contains only quadratic terms. Thus, this implies that in the neighbourhood of the origin, the level sets are locally diffeomorphic with a 3-sphere, on which the flow is quasi-periodic (for this latter fact, see for instance Sanders and Verhulst (1979)). Finally, we note that the origin corresponds with \( H = 0 \), and the family of (non-degenerate) critical points \( p = 0, q_1^2 + q_2^2 = 1 \) corresponds with \( H = 1/4 \). It is obvious that the topology of the integral manifolds can only change in the critical points. Thus, we observe that for \( 0 \leq H \leq 1/4 \), the integral manifolds have bounded parts. However, this estimate does not hold for all values of \( J \). In order to get the complete answer on the first question, we need another method.

6.1.2 Summary of the results

At this point, we find it useful to summarize the results of this Chapter. For \( \delta = 0 \), there exist an area \( C \subset \mathbb{R}^2 \) with boundary \( \partial C \) such that for \( (h, j) \in C \) (where \( h \) and \( j \) are values of \( H \) and \( J \) for some fixed initial conditions) the integral manifolds consist of four components (see figure 6.1 for a sketch of \( C \)). Two of the components are 2-tori on which (quasi)-periodic solutions live; the third and fourth component is unbounded and contains unbounded solutions. On \( \partial C \), there occur bifurcations. Either each of the tori has shrunk to a circle or the tori degenerate into tori which cross-section contains a saddle point of the associated vector field. A third possibility is that the two tori merge and form one ‘envelope’ torus. Which bifurcation occurs depends on the position on \( \partial C \). If \( (h, j) \notin C \cup \partial C \), the integral manifolds are unbounded and contain only unbounded solutions.

Then, we study the situation for \( \delta \neq 0 \). A way to do this is to consider the evolution of \( C \) as function of \( \delta \). This is motivated by the fact that (6.1.1) depends continuously upon \( \delta \). It turns out that for \( \delta \neq 0, C \) is divided into two areas, \( C_1(\delta) \) and \( C_2(\delta) \) which evolve as a function of \( \delta \) and which are separated by a straight line with gradient \( 2\delta \). For \( 0 < \delta < 2/3 \), both \( C_1 \) and \( C_2 \) can be ‘associated’ with compact parts of integral manifolds and hence, with bounded solutions. That is to say, we can choose initial conditions \( (q_0, p_0) \) such that for instance \( (J(q_0, p_0, \delta), H(q_0, p_0, \delta)) \in C_1 \), and the integral manifold on which the solution, starting in \( (q_0, p_0) \), lives is compact. Hence, the solutions itself are bounded. For \( 2/3 < \delta < 1 \), only \( C_2 \) ‘corresponds’ with bounded solutions. For \( \delta > 1 \), there remains
merely one straight line with gradient $2\delta$ in the $(H(\delta), J(\delta))$-plane on which bounded solutions live. This line corresponds with purely real, periodic solutions. All bounded solutions are either periodic or quasi-periodic (see for instance Arno’ld and Avez (1968)).

One of the conclusions of the analysis of this Chapter is, that a result of the previous Chapter is confirmed: for $\delta$ sufficiently large, there are only real solutions possible. Further, the analysis reveals some new features with respect to the topological structure of solutions, which have not been found directly in the previous Chapter.

### 6.2 The symmetric problem and some basic remarks

Before we answer the questions posed in the Introduction for the non-symmetric problem, we first look at the symmetric ($\delta \equiv 0$) problem. This problem is well known and has been extensively studied, see for instance Doelman (1989). There, the attention was focussed on slow time periodic solutions of the Ginzburg-Landau equation where the coefficients were complex with small imaginary parts. The analysis presented there is a perturbation analysis in the small imaginary parts of the coefficients and as a natural start, the unperturbed problem (which corresponds with the same unperturbed problem as we consider here) is analysed. The forthcoming observations are therefore partially based on Doelman (1989).

We already mentioned that (6.1.4)-(6.1.7) is integrable. For $\delta = 0$, a second integral is found to be

$$ J = (q_1 p_2 - p_1 q_2)^2 $$

(6.2.1)

Note that $\hat{J} = (q_1 p_2 - p_1 q_2)$ is of course also an integral, but for reasons which will become clear in the next section, we consider $J$ (instead of $\hat{J}$) as second integral. The existence of the second integral suggests that there is a symmetry in the problem. In this case it is fairly easy to see that this is a circle symmetry, by analysing the dynamical system induced by the second integral $J$. Because the problem is integrable, there holds for $H = h$, $J = j$, that the flow of the Hamiltonian system takes place on a 2-dimensional manifold in $\mathbb{R}^4$, say $M_{(h,j)}$. We will need information about the topological properties of this manifold in order to say something of the flow on the manifold. For instance, if $M_{(h,j)}$ is closed, bounded, oriented and contains no fixed points of the vector field, then $M_{(h,j)}$ is a 2-torus (see for instance Hirsch (1976)), and the flow on the torus is either periodic or quasi periodic. However, if $M_{(h,j)}$ is unbounded, we must be careful in drawing conclusions about the flow: the flow can be bounded as well as unbounded. One might think for instance of a cylinder. On the cylinder, there are periodic solutions possible as well as solutions which spiral towards $\pm \infty$ for $t \to \pm \infty$.

### 6.2.1 The main tool: The Energy-Momentum Map

To answer the first question posed in the Introduction, we make use of the so-called energy-momentum map $EM$. This map is defined as

$$ EM : \mathbb{R}^4 \to \mathbb{R}^2 $$

$$ EM(x) = (\nabla H(q,p), \nabla J(q,p))(x) $$

(6.2.2)
where \( \mathbf{q} = (q_1, q_2), \mathbf{p} = (p_1, p_2), H \) and \( J \) are defined in (6.1.1) and (6.2.1) and 
\( (\nabla H(\mathbf{q}, \mathbf{p}), \nabla J(\mathbf{q}, \mathbf{p})) \) is a \( 4 \times 2 \) matrix. The Energy-Momentum map turns out to be an important tool in the study of the topology of the integral manifolds \( M_{(h,j)} \) which can now be defined as \( M_{(h,j)} = EM^{-1}(h,j) \): the manifolds have the property that for \( (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^4 \), the orbit through \( (\mathbf{q}, \mathbf{p}) \) (which is called a fiber) is contained in \( M_{(h,j)} \). The mathematical details related to \( EM \) as well as to the question why studying this map provides information about the integral manifolds can be found in the classical paper by Smale (1970) and in Abraham and Marsden (1978). In the context of this Chapter, we recall below some of the facts that are described in Smale (1970). Note that \( EM \) is associated with points \( \mathbf{q}, \mathbf{p} \), rather than with solutions \( \mathbf{q}(t), \mathbf{p}(t) \) of the Hamiltonian system.

The central question that we want to answer will be: For what \( (h, j) \in \mathbb{R}^2 \) does the topological character of \( M_{(h,j)} \) change. To this end, we define:

\[
\partial C \text{ is the set of points in } \mathbb{R}^2 \text{ where the rank of } EM \text{ equals one or zero.}
\]

For almost all points \( (\mathbf{q}, \mathbf{p}) \), \( EM \) has rank 2. Whenever the rank changes, something changes in the topology of \( M_{(h,j)} \) and thus, there is a possibility that the nature of solutions changes. It is precisely the set \( \partial C \) which can be interpreted as the set of values \( (h, j) \) where the topological character of \( M_{(h,j)} \) changes which is of course strongly related to the character of the fibers. This set \( \partial C \) forms a one-dimensional subspace in the \((H, J)\)-plane. A further action to determine the geometric properties of the integral manifolds would be to analyse how the fibers ‘foliate’ a given energy level set \( H \) is constant. Thus, putting things together, the first aim is to denote in the \((H, J)\)-space the subspace where the rank of \( EM \) equals 1 or 0. Having done this, we can determine the topology of \( M_{(h,j)} \) for \( (h, j) \in \partial C \) and \( (h, j) \notin \partial C \) and subsequently, study the flow on the manifolds for the different cases.

In Smale (1970), the terminology is somewhat different. There, \( \partial C \) is called the bifurcation set, and is defined it to be the set of points in \( \mathbb{R}^2 \) over which \( EM \) fails to be ‘locally trivial’ (see Smale (1970)). It is not guaranteed that whenever the rank of \( EM \) changes, the topology of the manifolds also changes. The following simple example shows this. Suppose we have a dynamical system of which the integral manifolds form a cone in \( \mathbb{R}^3 \):
\[
x_1^2 + x_2^3 + x_3^2 = a^2,
\]
where \( a \) is used as bifurcation parameter. It is obvious that for \( a = 0 \), the topology of the manifolds changes from a circle into a point, but the character of solutions remains the same for \( a > 0 \) and \( a < 0 \). On the other hand, if the rank of \( EM \) does not change, the topology of the integral manifolds remains the same.

### 6.2.2 Analysis in polar coordinates

For the symmetric problem, it is convenient to transform the equations using polar coordinates. This is indicated by the flow of \( J \), which induces a circle-symmetry (and which is a conserved quantity for the system). Thus, we set \( q_1 = \Gamma \cos \phi, q_2 = \Gamma \sin \phi \). This is a canonical transformation, which means that it does not destroy the Hamiltonian properties. Besides \( \Gamma \) and \( \phi \), we define another set of coordinates, namely \( \gamma = \dot{\Gamma} \) and \( \Omega = \dot{\phi}^2/\Gamma^2 \).
In these new coordinates, the Hamiltonian $H$ and the second integral $J$ read:

\begin{align}
H(\Gamma,\gamma,\phi,\Omega) &= \frac{1}{2} \left( \gamma^2 + \frac{\Omega^2}{\Gamma^2} + \Gamma^2 - \frac{1}{2} \Gamma^4 \right) \\
J(\Gamma,\gamma,\phi,\Omega) &= \Omega
\end{align}

and the by $H$ induced equations read:

\begin{align}
\dot{\Gamma} &= \gamma \\
\dot{\gamma} &= \frac{\Omega}{\Gamma^3} - \Gamma + \Gamma^3 \\
\dot{\phi} &= \frac{\Omega}{\Gamma^2} \\
\dot{\Omega} &= 0
\end{align}

The Energy-Momentum map $EM$ is in this case defined as:

\[
EM(x) = \begin{pmatrix}
-\frac{\Omega}{\Gamma^3} + \Gamma - \Gamma^3 & \gamma & 0 & \frac{1}{\Gamma^2} \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

It is now easy to determine the points where the rank of $EM$ equals 1. For those points, the following condition must hold:

\[
-\frac{\Omega}{\Gamma^3} + \Gamma - \Gamma^3 = 0 \text{ and } \gamma = 0
\]

These conditions define $\partial C$, which can be parametrized by

\[
\partial C = \{(H,J)| J = Y^4 - Y^6, H = Y^2 - \frac{3}{4} Y^4, -1 \leq Y \leq 1\} \cup \{(H,J)| J = 0\}
\]

For a plot of $\partial C$, see figure 6.1. We note that $\partial C$ is the boundary of a closed set in the $(H,J)$-space which is thus divided into three parts: $\partial C$, the interior $C$ and the exterior.

Now, we determine the topology of $M(h,j)$ for $(h,j) \in \partial C$, $(h,j) \in C$ and $(h,j) \notin \overline{C}$ where $\overline{C}$ is defined by $\overline{C} = C \cup \partial C$. Recall that $M(h,j)$ denote integral manifolds and are defined by $EM^{-1}(h,j)$. To determine the topology of $M(h,j)$, we look at the reduced phase-space $(\Gamma,\gamma)$ i.e. we ‘forget’ for the moment the rôle of the angle $\phi$ which does not return in the equations for $\dot{\Gamma}$ (6.2.5) and $\dot{\gamma}$ (6.2.6). It is readily seen from (6.2.5)-(6.2.8) that the fixed points in the reduced phase space correspond exactly with the right and left part of $\partial C$ and it can easily been shown that the left boundary corresponds with center points while the right boundary corresponds with saddle points. Inside the homoclinic loop that connects the saddle point with itself are of course periodic solutions. The trajectories in the reduced phase-space are then as indicated in the subfigures in figure 6.1. Looking at the phase portraits in the figure one should realise that they have a symmetric counterpart for negative $\Gamma$, but the figures have only been drawn for $\Gamma > 0$. In the cusp (which corresponds with $J = \Omega = 4/27, H = 1/3$), the center and saddle point coalesce. For $J = 0$, the two homoclinic loops (for $\Gamma > 0$ and $\Gamma < 0$) ‘merge’ into two heteroclinic loops. Taking the symmetry into account, it is clear from figure 6.1, subfigure 3, that
6.2 The symmetric problem and some basic remarks

Figure 6.1: The region $C$. The subfigures 1-6 give the flow in the reduced phase-space $(\Gamma, \gamma)$.

$M_{(h,j)}$ is the union of two disjunct 2-tori and an two unbounded manifolds for $(h, j) \in C$. On the tori, the flow is (quasi) periodic while on the unbounded manifolds, the flow is unbounded. On the left part of $\partial C$, each of the tori degenerates into a point in the reduced phase-space (which is a circle in the full, four dimensional phase-space), while on the right part of $\partial C$ they degenerate into tori which cross-section is a homoclinic loop. The interpretation of $\partial C$ is then clear: The topology of $M_{(h,j)}$ changes when we cross $\partial C$. Keeping $J$ fixed and letting $H$ increase, starting at zero, this is is reflected by the creation (crossing the left boundary) and the vanishing (crossing the right boundary) of the torus. Formally, it is possible that after the degeneration, there exist again bounded solutions on compact manifolds. However, this is not the case: for $(h,j) \notin C$, there are no bounded solutions. This can be observed from the equations (6.2.5)-(6.2.8) by the following argument.

For $\Omega > 4/27$ (recall that in polar coordinates, the second integral $J$ is identical to $\Omega$) it is readily seen from (6.2.6) that the right hand side of $\dot{\gamma}$ is always positive, which means that all solutions are unbounded (which implies of course that the corresponding integral manifolds are also unbounded). Recall that the cusp in figure 6.1 corresponds with $J = \Omega = 4/27$. Hence, if for $\tilde{\Omega} > 4/27$ the integral manifolds are unbounded, we can draw the same conclusion for all points $(h, j)$ which can be reached from $(\tilde{h}, \tilde{\Omega})$, for an arbitrary $\tilde{h}$ by some smooth path in the $(H, J)$-plane without crossing $\partial C$. This is precisely the complement of $\overline{C}$. This shows that for $(h, j) \notin \overline{C}$, there are no bounded solutions.

**Remark**

In the next section, we will analyse the non-symmetric case $\delta \neq 0$. Although the problem remains fully integrable, we can no longer associate the second integral with a circle symmetry. This has as result, that changing to polar coordinates does not simplify the
problem. However, the results obtained so far do not depend on the coordinate system and can therefore directly be transported to a Cartesian coordinate system. The idea is now to repeat the calculations for $\delta = 0$ with respect to a Cartesian coordinate system as a guideline for the case $\delta \neq 0$ with the ‘advantage’ that we can compare them for $\delta = 0$ with the case of the polar coordinates.

### 6.2.3 Analysis in the Cartesian coordinates

For the Cartesian coordinate system, $EM$ reads as follows:

$$EM(x) = \begin{pmatrix} -q_1(q_1^2 + q_2^2) + q_1w_1 & -q_2(q_1^2 + q_2^2) + q_2w_2 & p_1 & p_2 \\ 2\dot{J}p_2 & -2\dot{J}p_1 & -2\dot{J}q_2 & 2\dot{J}q_1 \end{pmatrix} x$$

with $\dot{J} = \sqrt{J}$ and $x \in 4$. Now, we try to re-find the parametrization of $\partial C$, which is to say that we consider those points $(q, p)$ where the rank of $EM$ equals 1. Thus, either one of the three conditions below must hold:

$$\nabla H(q, p) = \mu \nabla J(q, p) \text{ or } (6.2.14)$$

$$\nabla H(q, p) = 0, \nabla J(q, p) \neq 0 \text{ or } (6.2.15)$$

$$\nabla J(q, p) = 0, \nabla H(q, p) \neq 0 \text{ (6.2.16)}$$

Once we have found a triple $(\mu, q, p)$ for which the rank of $EM$ equals one, we know also that for $(\mu, q(t), p(t))$ the rank equals 1, where $q(t), p(t)$ is a trajectory of the dynamical system induced by the second integral $J$. This can be seen as follows. Consider the flow with initial point $(q_0, p_0)$ and generated by the second integral $J$. We denote this flow by $G_{(q_0, p_0)}(q, p)$, and the corresponding vector field is given by

$$\dot{q} = \frac{\partial J}{\partial p} \quad (6.2.17)$$

$$\dot{p} = -\frac{\partial J}{\partial q} \quad (6.2.18)$$

Now, because $J$ is an integral of the flow, generated by $H$, we have:

$$H(q_0, p_0) = H(G_{(q_0, p_0)}(q, p)) \quad (6.2.19)$$

Of course, (6.2.19) holds also for $J$:

$$J(q_0, p_0) = J(G_{(q_0, p_0)}(q, p)) \quad (6.2.20)$$

Then, we take the (total) derivative on both sides of (6.2.19) and (6.2.20). This yields:

$$\nabla H(p_0, q_0) = \nabla H(G_{(q_0, p_0)}(q, p))DG_{(q_0, p_0)}(q, p) \quad (6.2.21)$$

$$\nabla J(p_0, q_0) = \nabla J(G_{(q_0, p_0)}(q, p))DG_{(q_0, p_0)}(q, p) \quad (6.2.22)$$

where $D$ denotes the total derivative, i.e. $DG_{(q_0, p_0)}(q, p)$ is a $4 \times 4$ matrix. Multiplying (6.2.22) with $\mu$ and subtraction yields (taking into account (6.2.14))

$$\nabla H(G_{(q_0, p_0)}(q, p)) = \mu \nabla J(G_{(q_0, p_0)}(q, p)) \quad (6.2.23)$$
which proves the statement. Thus, in other words for all the points \((q, p)\) generated by the flow of \(J\), starting in \((q_0, p_0)\) we find the same \(\mu\) satisfying \(\nabla H = \mu \nabla J\). This means that we can limit ourselves to points which correspond to different trajectories with respect to the flow induced by \(J\). The reverse is not true: We can find triples \((\mu, q_1, p_1), (\mu, q_2, p_2)\) (i.e. the same \(\mu\) but different \(q\) and \(p\)) which correspond with different points in the \((H, J)\)-plane. Let us now continue with the procedure to find the area in the \((H, J)\)-plane for which the rank of \(EM\) equals 1. Evaluating equation (6.2.14) gives four equations:

\[
\begin{align*}
q_1(1 - s) &= 2\mu p_2\hat{J} \\
q_2(1 - s) &= -2\mu p_1\hat{J} \\
p_1 &= -2\mu \hat{J} q_2 \\
p_2 &= 2\mu \hat{J} q_1
\end{align*}
\]

where \(s = q_1^2 + q_2^2\). For this system of equations, we can find three families of ‘special’ solutions. They are:

\[
\begin{align*}
(q_1, p_1, q_2, p_2) &= \left(\sqrt{\frac{1}{2\mu}}, 0, 0, \sqrt{\frac{2\mu - 1}{4\mu^2}}\right) \\
(q_1, p_1, q_2, p_2) &= \left(0, \sqrt{\frac{2\mu - 1}{4\mu^2}}, \sqrt{\frac{1}{2\mu}}, 0\right) \\
q_1 &\neq 0, q_2 = 0 \text{ and } p = 0
\end{align*}
\]

Thus, for every \(\mu \geq 1/2\), we can find particular \((q, p)\) such that the rank of \(EM\) equals 1. These points can be substituted in \(H\) and \(J\). Using (6.2.28) and (6.2.29), this gives:

\[
(H, J) = \left(\frac{1}{2\mu} - \frac{3}{16\mu^2}, \frac{1}{4\mu^2} - \frac{1}{8\mu^3}\right)
\]

We can then use the parameter \(1/(2\mu)\) to parametrize the curve in the \((H, J)\)-plane which denotes where the rank of \(EM\) equals 1 and which describes (a part of) \(\partial C\). It should be noted that the first two families of solutions (6.2.28) and (6.2.29) give rise to the same parametrization. We already remark that this will no longer be the case if \(\delta \neq 0\). The third family of special solutions corresponds in the \((H, J)\)-plane with the \(J = 0\)-axis.

Putting things together, we find again the plot given in figure 6.1. The conclusions are taken from the case of the polar coordinates: for \((h, j) \in \overline{C}\) there are bounded solutions which live on (degenerate) 2-tori (as well as unbounded solutions), outside \(\overline{C}\) no bounded solutions exist. Note that this plot is ‘complete’: it is not possible to find other solutions which satisfy one of the equations given in (6.2.24) - (6.2.27), and which give contributions to \(\partial C\), which are not considered yet. This can be seen as follows. We will show that for every value \(h\) and for \(j > 0\), the integral manifold \(M_{(h,j)}\) contains the point \((q_1, p_1, q_2, p_2) = (0, \overline{p}_1, \overline{q}_2, 0)\) where \(\overline{p}_1, \overline{q}_2\) depend on \(h\) and \(j\). This means that the first family of what we called ‘special’ solutions, covers actually all the possible cases. Putting it different, a limitation to solutions (6.2.28) is no restriction. To show this, we consider arbitrary \(h\) and positive \(j\) as values for the energy \(H\) and the second integral \(J\). Then, we note that from the expression for \(J\) (see 6.2.1) follows

\[
p_1^2 = \frac{j}{q_2^2}
\]
and substituting this in $H$ gives:

$$j = 2hq_2^2 - q_2^4 + \frac{1}{2}q_1^6$$  \hspace{1cm} (6.2.33)

Now, we observe that $j$ as function of $q_2$ is surjective on $\mathbb{R}^+$ for every $h$. Thus, given $h$ and $j > 0$, we solve (6.2.33) to find $q_2$ and find $p_1$ from (6.2.32). Hence, we have shown that $(q_1, p_1, q_2, p_2) = (0, p_1, q_2, 0)$ is for every $h$ and positive $j$ an element of $M(h,j)$. This completes the argument.

### 6.3 The non-symmetric problem

Let us now look at the non-symmetric problem, i.e. $\delta \neq 0$. The first very remarkable fact is that the problem is still integrable! Besides the Hamiltonian given in (6.1.8), a second integral is given by

$$J = (q_1p_2 - p_1q_2)^2 + \delta \left( (p_1^2 - p_2^2) - \frac{1}{2}(q_1^4 - q_2^4) + w_1q_1^2 - w_2q_2^2 \right)$$  \hspace{1cm} (6.3.1)

where $w_1$ and $w_2$ are given in (6.1.9)-(6.1.10). The suspicion that the problem is integrable arose because we found that equation (6.1.1) satisfies the Painlevé property (for a description of this property, see Tabor (1989)). The expression given in (6.3.1) was then established using suggestions about the possible structure of the second integral, given in Whittaker (1917). Note that for $\delta \rightarrow 0$, we recover the second integral of the unperturbed problem.

In the cases that a small perturbation of the Hamiltonian has as result that the problem is no longer integrable, it also means that most of the tori that exist for the unperturbed problem are destroyed. Apparently, the additional term $\sim \delta A$, which can be considered as a perturbation if $\delta$ is small, is sufficiently ‘neat’ to preserve the Hamiltonian character. Due to this property, it is expected that the area $\mathcal{C}$ (which corresponds to initial conditions such, that the solutions are bounded) will deform slightly in a continuous way as a function of $\delta$. We expect that in this deformed area $\mathcal{C}(\delta)$, we can draw the same conclusions as we did for the symmetric case: $\mathcal{C}(\delta)$ is associated with compact integral manifolds on which bounded solutions live; $\partial \mathcal{C}(\delta)$ corresponds to bifurcations. However, as already indicated in the Introduction, the situation is more complicated. The area $\mathcal{C}$ splits into two new areas, each of which must be studied to analyse the topological properties of the integral manifolds. It turns out that one of the areas can be seen as a deformation of ($\mathcal{C}$ for $\delta = 0$) and hence, the integral manifolds contain compact parts. The other area must be treated more carefully.

For a start, we try to answer the same questions as posed for the symmetric ($\delta \equiv 0$) problem: Can we denote an area in the $(H, J)$-plane which corresponds with bounded solutions and can we determine the character of those solutions (periodic, quasi-periodic, etc.). We follow the way of analysing which we used in the Cartesian coordinate system. This is explained in the Remark made in the previous section. Recall that the main aim is to see how $\mathcal{C}$ deforms as a function of $\delta$. The answer lies again in the Energy-Momentum map and as before, we determine when the rank of $EM$ equals one or zero, which is to say
that solve $\nabla H = \mu \nabla J$ for $\delta \neq 0$. We make use of the three special families of solutions that we found in case of the symmetric problem. They satisfy the non-symmetric problem as well. In the calculations, performed below, one should keep in mind the equivalent of equation (6.2.13) for $\delta \neq 0$, which can easily be derived from (6.1.8) and (6.3.1).

The first case from the symmetric problem, i.e. $q_1 = 0, p_2 = 0$, yields $p_1^2 = (q_2^2 + 2\delta)(w_2 - q_2^2)$ and $\mu = \frac{1}{2q_1^2 + 2\delta}$, as solution of $\nabla H = \mu \nabla J$. The condition for $p_1$ restricts $q_2$: $0 \leq q_2^2 \leq w_2 = 1 - \delta$. The latter condition on his turn gives a condition on $\mu$: $1/(2\delta) < \mu < 1/2$. Thus, for a certain $q_2$ in the interval, we find $p_1$ and $\mu$ such that $\nabla H = \mu \nabla J$. Substitution of $p_1$ and $q_2$ in $H$ and $J$ (with $q_1 = 0, p_2 = 0$) gives a parametrization of (a first part of) $\partial C$ in the $(H,J)$-plane (where $q_2$ within the beformentioned interval is used as the parameter). Note that for $\delta > 1$, the interval for $q_2$ vanishes. This degeneration is interpreted lateron.

Then we consider the second family of solutions from the symmetric problem, i.e. we put $q_2 = 0, p_1 = 0$. The equation $\nabla H = \mu \nabla J$ is satisfied for $p_2^2 = (q_1^2 - 2\delta)(w_1 - q_1^2)$ and $\mu = \frac{1}{2q_1^2 - 2\delta}$. Again, substitution of $q_1$ and $p_1$ (setting $q_2 = 0, p_1 = 0$) in $H$ and $J$ gives a parametrization of (a second part of) $\partial C$ in the $(H,J)$-plane where now, $q_1$ is used as parameter. The equation for $p_2$ indicates that for $0 \leq \delta \leq 1/2$, we must choose $2\delta \leq q_1^2 \leq 1$, while for $\delta \geq 1/2$ we must choose $1 \leq q_1 \leq 2\delta$. This means that the length of the parametrized curve becomes shorter for larger $\delta$ until it vanishes altogether for $\delta = 1/2$. For $\delta > 1/2$ the length increases again. This degeneration is also explained lateron.

The two mentioned cases give rise to two curves in the $(H,J)$-plane, which partly co-incide with each other (see figure 6.2). Both curves have a cusp and start in $s_1$. For the first case, the curve ends in $s_2$, while for the second case, the curve ends in $s_3$. Note that $s_1, s_2$ and $s_3$ are functions of $\delta$. Finally, there are four more special solutions to $\nabla H = \mu \nabla J$. They can be regarded as the equivalents of the third family of the unperturbed problem (see (6.2.30) and they are listed below.

$$(q_1, 0, 0, 0) \quad \text{and} \quad \mu = 1/(2\delta) \quad (6.3.2)$$

$$(0, p_1, 0, 0) \quad \text{and} \quad \mu = 1/(2\delta) \quad (6.3.3)$$

Figure 6.2: The regions $C_1$ and $C_2$, for $\delta \neq 0$. 


(0, 0, q_2, 0) and \( \mu = -1/(2\delta) \) \hspace{1cm} (6.3.4)
(0, 0, 0, p_2) and \( \mu = -1/(2\delta) \) \hspace{1cm} (6.3.5)

These four solutions, substituted in \( H \) and \( J \), parametrize two straight lines in the \((H, J)\)-plane. Substitution of (6.3.2) and (6.3.3) into \( H \) and \( J \) parametrize \( l_1 \), while substitution of (6.3.4) and (6.3.5) parametrize \( l_2 \). These lines, are the analogon of the \( J = 0 \)-axis in case of the symmetric problem. In a way, we can say that the \( J = 0 \)-axis splits in two lines with gradients \( 2\delta \) and \(-2\delta \). Note that these lines pass either through \( s_1 \) and \( s_2 \) (and the origin) or the origin and \( s_3 \) and thus, together with the two curves already found, define the equivalent of the area \( C \) (see again figure 6.2). It can be seen from (6.1.4)- (6.1.5) that for initial conditions such that we start on \( l_1 \) (i.e. \( q_1(t_0) \neq 0, p_1(t_0) = q_2(t_0) = p_2(t_0) = 0 \)), the \( q_2, p_2 \)-mode gets not excited: \( q_2 = 0, p_2 = 0 \) is a fixed point of the subsystem \( \dot{q}_2, \dot{p}_2 \) and the solutions in the \( q_1, p_1 \)-plane are periodic. This solution \((q_1, p_1 \neq 0, q_2, p_2 = 0)\) is called the \( q_1, p_1 \)-normal mode and it corresponds with a real solution of (6.1.1). Using an analogous reasoning, it should be clear that \( l_2 \) is associated with a \( q_2, p_2 \)-normal mode which is a purely imaginary solution of (6.1.1).

For \( \delta > 0 \), \( C \) is divided by \( l_1 \) into \( C_1 \) and \( C_2 \) which ‘evolve’ as functions of \( \delta \). This is, because the procedure to describe \( \partial C \) which bounds \( C_1 \) and \( C_2 \) depends on \( \delta \). It turns out that there are four different cases with respect to \( \delta \) to consider (see figure 6.3).

For \( 0 < \delta < 2/3 \), \( C_1 \) is bounded by the parametrised cusped-curve, starting in \( s_1 \) and ending in \( s_2 \), and by the line segment \( s_1 - s_2 \). The area \( C_2 \) is bounded by the parametrised curve starting in \( s_2 \) and ending in \( s_3 \), and by the line segments \( o - s_1 \) and \( o - s_3 \). \( C_1 \) ‘decreases’ for increasing \( \delta \), \( C_2 \) ‘increases’. The area \( C_1 \) can be considered as the deformation
of \( C \) in case of the symmetric problem. Thus, by means of continuity, we can expect that in \( C_1 \), there are still bounded solutions on tori (as well as unbounded solutions). These solutions can be considered as the ‘perturbed’ solutions of the case \( \delta = 0 \). The area \( C_2 \) is ‘new’, but contains also bounded solutions. Using Morse theory, this is almost immediately clear; we come back on this shortly. For now, we continue with describing the evolution of \( C_1 \) and \( C_2 \) as a function of \( \delta \). We observe that there is a significant degeneration in the evolution for \( \delta = 2/3 \): \( C_1 \) vanishes. This is ‘caused’ by the fact that \( s_1 \) and \( s_2 \) (see figure 6.2) which are functions of \( \delta \), coincide for \( \delta = 2/3 \). Thus, the part of \( \partial C \) which connects \( s_1 \) and \( s_2 \) does not exist anymore.

For \( 2/3 < \delta < 1 \), \( C_1 \) appears again, and increases for increasing \( \delta \) while \( C_2 \) decreases. For \( \delta = 1 \), there occurs another degeneration: \( C_2 \) has vanished. This is because \( s_3 \) coincides with the origin, and thus, the part of \( \partial C \) which connects \( s_3 \) and the origin vanishes.

For \( \delta > 1 \), only \( C_1 \) remains and grows for increasing \( \delta \).

Now that we have described how \( \partial C \) evolves as a function of \( \delta \), we can study the manifolds associated with the compact areas \( C_1(\delta) \) and \( C_2(\delta) \), which are bounded by \( \partial C(\delta) \). To do this, we remark that there are essentially two principal cases to consider. The first one is \( 0 < \delta < 1 \). In that case, there are two compact areas \( C_1(\delta) \) and \( C_2(\delta) \). This case should actually be divided into two. The first is \( 0 < \delta < 2/3 \), in which case the boundary between \( C_1(\delta) \) and \( C_2(\delta) \) has a line segment in common, the second is \( 2/3 < \delta < 1 \), in which case \( C_1(\delta) \) and \( C_2(\delta) \) have only a single point in common. The other principal case is \( \delta > 1 \); then there is only one compact area, \( C_1(\delta) \).

We will show successively that:

1. \( C_1(\delta) \) corresponds to compact (parts of) integral manifolds for \( 0 < \delta < 2/3 \)
2. \( C_1(\delta) \) corresponds to unbounded integral manifolds for \( \delta > 2/3 \)
3. \( C_2(\delta) \) corresponds to compact (parts of) integral manifolds for \( 0 < \delta < 1 \)

An important conclusion that can then be drawn is the following. We recall that for all \( \delta \), there exist real, periodic solutions. This can immediately be observed from (6.1.4)-(6.1.7): we set \( q_2 = 0, p_2 = 0 \) and note that \( \dot{q}_1, \dot{p}_1 \) is independent of \( \delta \) and allows periodic solutions. In the \((H(\delta), J(\delta))\)-plane, these solutions correspond with \( l_1 \) (see figure 6.2). The implication is then that for \( \delta > 1 \), these are the only bounded solutions, because \( C_1 \) corresponds with unbounded integral manifolds (on which unbounded solutions live) and \( C_2 \) does not exist anymore. From the complement of \( C_1 \), it was already shown that it was associated with unbounded solutions. This conclusion is completely in agreement with the observation of the Remark made in Chapter 4, section 4.4.3 and with the analysis, presented in Chapter 5: for \( \delta \) large enough, the imaginary part of \( A \) is exponentially damped and there exist only real solutions.

Now, we will show the three assertions that we made previously.

Ad. 1. This is the easiest case and has already be analysed. The area \( C_1 \) can be considered as the deformation of \( C \) in case of the symmetric problem. Thus, by means of continuity, we
know that in $C_1$, there are still bounded solutions on tori (as well as unbounded solutions). These solutions can be considered as the ‘perturbed’ solutions of the case $\delta \equiv 0$.

Ad. 2. Suppose $q_2 > 0$ and $\delta > 1$, i.e. $C_1$ is the only ‘closed’ area, bounded by $l_1$ and some parametrized curve. Suppose that we take initial conditions such that $(h, j) \in C_1$. Then, consider (6.1.3):

$$\ddot{q}_2 = \left( (\delta - 1) + (q_1^2 + q_2^2) \right) q_2 > (\delta - 1)q_2$$  \hfill (6.3.6)

Now, there are two essentially different situations:

- $q_2(0) > 0$ and $\dot{q}_2(0) > 0$. Then, it is immediately clear from (6.3.6) that $\dot{q}_2$ increases and thus, $q_2 \to \infty$, independently of the initial conditions for the other variables. Using the same arguments, one shows that for $q_2(0) < 0$ and $\dot{q}_2(0) < 0$, necessarily $q_2 \to -\infty$.

- $q_2(0) > 0$ and $\dot{q}_2(0) < 0$. Because we still have that $\dot{q}_2$ decreases, there are again two possibilities. Either there is a $t_0$ for which $q_2(t_0) > 0$ and $\dot{q}_2(t_0) > 0$ and we are back in the first situation: $q_2 \to \infty$. Or there is a $t_0$ for which $q_2(t_0) < 0$ and $\dot{q}_2(t_0) < 0$ and we must, also according to the first item, conclude that $q_2 \to -\infty$.

The remaining case $(q_2(0) < 0$ and $\dot{q}_2(0) > 0$) can be treated analogous. Thus, all solutions with initial conditions corresponding to $(h, j) \in C_1(\delta)$ are unbounded for $\delta > 1$ and consequently, the manifolds itself are unbounded. To conclude the argument, we remark that $C_1$ deforms smoothly as a function of $\delta$ and exists also for $2/3 < \delta < 1$. This means that we can conclude that also for $2/3 < \delta < 1$, $C_1$ corresponds with unbounded manifolds on which unbounded solutions live.

Ad. 3. We will show that around the origin in the four dimensional phase-space (the origin is a double elliptic point for $0 < \delta < 1$, with eigenvalues $\pm i, \pm i\sqrt{1 - \delta}$) there are solutions, living on compact manifolds (2-tori) which are close to the origin. As before, these manifolds can be identified with $EM^{-1}(h, j)$ for some $(h, j)$. The crux is now to decide which area $(C_1, C_2$ or the complement of $C_1 \cup C_2$) contains $(h, j)$. It turns out that necessarily, $(h, j)$ is an element of $C_2$. As soon as we know that the compact integral manifolds close to the origin are contained in $C_2$, we can conclude that $C_2$ corresponds with compact integral manifolds for all positive $\delta, \delta < 1$, and all $(h, j) \in C_2$. This is again because the topological character of the manifolds does not change within $C_2$ as function of $\delta$.

To show that $(h, j)$ is an element of $C_2$, we first note that the origin in phase-space corresponds with $H = 0, J = 0$. Thus, the invariant tori around the origin correspond to values of $H$ and $J$ which are in a sufficiently small neighbourhood $E \subset \subset 2$ of the origin. Now, we note that the distance $d$ of $C_1$ to the origin, defined as $\min_{(h, j) \in C_1} \text{dist}((h, j), 0)$ is determined by $s_2$ (see figure 6.1) and it is easy to see that $d = O(\delta)$. Thus, for $\delta$ sufficiently large, we can achieve that the intersection of $E$ and $C_1$ is empty and hence, $(h, j) \notin C_1$. Then, the only remaining candidates which can contain $(h, j)$ are $C_2$ and the complement of $C_1 \cup C_2$. But we already showed that the complement of $C_1 \cup C_2$ ‘corresponds’ only with unbounded integral manifolds. Hence, $(h, j)$ must be in $C_2$. 
To complete the argument, we must show that around the origin, there exist bounded solutions. This however, is immediately clear from Morse-theory (see section 6.1.1), for it should be clear that for $0 < \delta < 1$ the origin is still a non-degenerate critical point with index zero of the Hamiltonian $H(\delta)$. Hence, the integral manifolds are locally diffeomorphic with a 3-sphere on which the flow is quasi-periodic.

**Remark**

We can also show directly that for solutions, living on the tori near the origin, the values of $H$ and $J$ are in $C_2$. Therefore, we linearise (6.1.4)-(6.1.7), and consider the linearised solutions: $q_1(t) = r_1 \sin t, q_2(t) = r_2 \sin(\sqrt{1 - \delta} t)$. Then, substituting these into the expressions for $H$ and $J$, we find:

\[
H = \frac{1}{2} (r_1^2 + (1 - \delta) r_2^2) + O(r_1^4, r_2^4, r_1^2 r_2^2) \tag{6.3.7}
\]

\[
J = \delta (r_1^2 - (1 - \delta) r_2^2) + O(r_1^4, r_2^4, r_1^2 r_2^2) \tag{6.3.8}
\]

Thus, we see that $J = 2\delta H - 2\delta (1 - \delta) r_2^2$, and for $r_2 = 0$, we recover the boundary $l_1$ (see figure 6.2) of $C_2$ in the first quadrant, in the neighbourhood of the origin. Alternatively, we can write $J = -2\delta H + 2\delta r_1$ and we recover the boundary $l_2$ of $C_2$ in the fourth quadrant.

To conclude this section, we show some numerical calculations to get an insight in the bifurcations that occur at $l_1$ and $l_2$. To do this, we consider the intersection with $3$ of a Poincaré map of the flow for $p_1 - p_2 = 0$. First of all, we consider the situation for $\delta = 0$, in which case $l_1$ and $l_2$ coincide which each other and are actually the $J(0) = 0$-axis. For initial values $\mathbf{p}, \mathbf{q}$ such that $J(\mathbf{p}, \mathbf{q}, 0)$ is slightly positive, the plot is given in figure 6.4. From the preceding theory, we know that the compact parts of the integral manifolds

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Figure 6.4: In (a), the intersection of the Poincaré map at $p_1 - p_2 = 0$ with $H = 0.25, \delta = 0$ and $J$ slightly larger than zero. In (b), we plotted for clarification only half the information that is contained in (a).
Figure 6.5: In (a), the intersection of the Poincaré map with \( p_1 - p_2 = 0 \) on \( 3 \), for \( H = 0.2, \delta = 0.1 \) and \( J = 0.042, 0.04 \) and 0.038 is plotted, in (b), the various projections are shown.

are tori. Hence, the intersection of the Poincaré map is topological a circle. As already mentioned, in this case we have two 2-tori. Therefore, the map should consist out of two circles. The reason that we see four circles is that both the inward directed trajectories, as well as the outward directed trajectories have been used. At \( J = 0 \), there occur two things: simultaneously, the two tori merged, and the map reduces to a single fixed point, corresponding to a periodic solution (at \( J = 0 \), the Poincaré map consist only of the two heavy dots in figure 6.4). In the hope to clarify the figure, we plotted in (b) only half of the information, so that one sees the position of the sections of the tori in \( 3 \).

In figure 6.5, the situation is shown for \( \delta = 0.2 \). Figure 6.5 (a) corresponds with the intersection of the map with \( 3 \), figure 6.5 (b) are the projections on the different planes in \( 3 \) of figure 6.5 (a). For a fixed value of \( H \), we have shown the situation for \( J \) slightly above, right on and slightly below \( l_1 \). We observe that the two phenomena that happened simultaneously for \( \delta = 0 \) are now separated. For \( J \) slightly above \( l_1 \), the compact parts of the manifolds are still two 2-tori, visualised by the four circles located in the twisted ‘eight-shaped’ curves. At \( l_1 \), the two tori merge and hence, the Poincaré map consist of the two twisted eight-shaped curves. The two curves will exist for decreasing \( J \) until eventually, at \( l_2 \), the map consists out of merely two points: we have ‘reached’ the \( q_2, p_2 \) normal mode, which is, as already mentioned, a periodic solution.
References


Samenvatting

In dit proefschrift proberen we te voorspellen hoe de bodem van rivieren en kanalen zich gedraagt onder invloed van de stroming in de rivier. Het onderzoek probeert een bijdrage te geven aan het formuleren van een model dat in staat is het meanderende gedrag van rivieren te begrijpen en te voorspellen.

Het modelleren van rivieren in hun natuurlijke omgeving is een lastige opdracht. We hebben te maken met een zeer gecompliceerde wisselwerking tussen het water in de rivier, de erodeerbare bodem en zijwanden en de direkte omgeving van de rivier (bomen, grasland, struikgewas). De dwarsdoorsnede is grillig, de bodem verandert, de oevers eroderen en al deze processen beïnvloeden elkaar op een ingewikkelde manier. Om hier het hoofd aan te bieden zijn er wiskundige modellen ontwikkeld die een ‘vereenvoudigde werkelijkheid’ van een rivier beschrijven. In de praktijk blijken deze modellen, ondanks de vereenvoudigingen, in staat te zijn goede voorspellingen te geven voor de ‘echte’ wereld. Zo wordt een kanaal of rivier vaak gemodelleerd als een in dwarsdoorsnede rechthoekige bak, die oneindig lang is en waarvan de zijwanden niet erodeerbaar zijn. Er is slechts interactie mogelijk tussen de bodem en het snelheidsveld van de stroming (dat gemakshalve tweedimensionaal is verondersteld): de vorm van de bodem beïnvloedt het snelheidsveld, dat op zijn beurt weer zorgt voor sedimenttransport, dat de bodem vervormt, etc. In die rechte, oneindige lange rivier bekijken we in hoofdstuk 2 kleine verstoringen van een vlakke bodem en een eenvoudig snelheidsveld. In bepaalde omstandigheden, waarbij de breedte-diepte verhouding van de rivier een belangrijke parameter is, blijken kleine verstoringen te gaan groeien, en zijn we in staat om de evolutie van deze verstoringen in de tijd en ruimte te volgen. Er onstaat een patroon van alternerende zandbanken, ‘heuvels en dalen’ die elkaar afwisselen in de loop van de rivier. De golfslag van deze banken wordt bepaald door de neutrale kromme die lokaal de vorm heeft van een (dal)parabool en aangeeft wanneer de basis oplossing (de vlakke bodem) stabiel is. Boven de kromme is de basisoplossing instabiel, onder de kromme is deze stabiel. Het minimum van de parabool is een belangrijk punt: het bepaalt welk bodem patroon we initiëel zien. De amplitude van de banken kan zowel op een regelmatige als een onregelmatige manier veranderen in de tijd en de ruimte (zie bijvoorbeeld figuur 2.3 op pagina 30 en 2.10 op pagina 42 voor een tweetal (geïdealiseerde) voorbeelden). De wiskundige theorie waarmee deze resultaten behaald worden is ontwikkeld eind jaren zestig, begin jaren zeventig. De essentie van deze nietlineaire theorie is dat het ‘initiele’ (lineaire) profiel van alternerende banken gemoduleerd wordt door rekening te houden met nietlineaire effecten van het model. Het blijkt noodzakelijk te zijn om een amplitude (of modulatie)functie te introduceren die ‘leeft’ op een lange tijd- en ruimteschaal. Deze amplitude functie voldoet aan de Ginzburg-Landau vergelijking, een partiële, nietlineaire differentiaalvergelijking. De oplossingen van deze vergelijking zijn een veel bestudeerd wiskundig onderwerp en geven in dit geval aan, hoe de zandbanken zich in de tijd en ruimte gedragen. Dit kan ‘netjes’ periodiek zijn of ‘wilder’ quasi-periodiek.

Een volgende logische stap in de analyse lijkt om de aanname van rechte rivieren los te laten en te kijken naar slingerende rivieren (ook wel meanderen genoemd), om daar vervolgens dezelfde vraag te stellen: wat kunnen we zeggen over de bodemontwikkeling. Deze vraag
blijkt een stuk moeilijker te beantwoorden te zijn dan in het geval van de rechte kanalen. Een deel van de problemen wordt veroorzaakt doordat de basisoplossing, een eenvoudige oplossing van het wiskundige model dat de meanderende rivieren beschrijft, al tamelijk ingewikkeld is. Uitspraken over de stabiliteit zijn nog een stuk gecompliceerder. Vanwege de complexiteit van het ‘echte’ fysische model bestaat de mogelijkheid dat de onderliggende instabiliteitsmechanismen overschaduwd dreigen te worden door problemen van een andere aard (b.v. rekenkundige). Daarom wordt in hoofdstuk 3 een modelprobleem ontwikkeld, een eenvoudige wiskundige vergelijking die in principe alle kenmerken van het echte fysische probleem van de meanderende rivieren in zich draagt. Dit hoofdstuk vormt de verbinding tussen de fysica (met veel wiskunde) en de wiskunde (met de fysica als achterliggende motivatie).

In hoofdstuk 4 ontwikkelen we de stabiliteitsanalyse voor dit modelprobleem. Daarbij laten we ons leiden door de theorie van hoofdstuk 2: we proberen modulatie vergelijkingen te vinden die de ontwikkeling in tijd en ruimte beschrijven van de basis oplossing (het equivalent van de vlakke bodem in de rechte rivieren). In plaats van één neutrale kromme vinden we er nu in essentie twee, die wel dicht bij elkaar liggen. Belangrijk in de theorie is de onderlinge verhouding tussen twee kleine parameters, $\varepsilon$ en $\nu$. De eerste, $\varepsilon$, beschrijft hoe ver we van het minimum van de neutrale kromme zijn, de tweede, $\nu$, geeft de mate van ‘slingeren’ van de rivier aan. Als $\varepsilon$ kleiner is dan $\nu^2$ zien we feitelijk nog steeds maar één neutrale kromme; de andere is te ver weg om zijn invloed te laten gelden. Als $\varepsilon$ groter is dan $\sqrt{\nu}$ zijn beide krommes relatief zó dicht bij elkaar dat we geen onderscheid meer kunnen maken tussen de twee. In het tussengebied treden interessante ‘nieuwe’ modulatie vergelijkingen op.

In hoofdstuk 5 en 6 worden oplossingen van deze Ginzburg-Landau-achtige vergelijkingen bestudeerd. In hoofdstuk 5 kijken we naar constante en eenvoudige ruimteperiodieke, stationaire oplossingen en bekijken we de stabiliteits eigenschappen van deze oplossingen. Feitelijk kijken we dan naar eindigdimensionale Galerkin benaderingen van oplossingen. Voor een gedeelte zijn de resultaten analytisch, voor een gedeelte zijn ze numeriek. In hoofdstuk 6 kijken we meer algemeen naar de (topologische) structuur van stationaire oplossingen. Dit doen we dan niet door naar Galerkin benaderingen te kijken, maar door te observeren dat de te bestuderen vergelijking een integreerbaar Hamilton systeem is met twee vrijheidsgraden. Begrensde oplossingen ‘leven’ op tweedimensionale variëteiten in een vierdimensionale ruimte. Ook nu is de onderlinge verhouding tussen $\varepsilon$ en $\nu$ weer een belangrijke parameter.

Hoofdstuk 4.5 en 6 staan op het eerste gezicht ver af van de oorspronkelijke vraagstelling met betrekking tot rivieren. Toch zijn ze een noodzakelijke stap in de ontwikkeling van de theorie. Door het bestuderen van het modelprobleem zijn we immers op de hoogte van de verschijnselen die ook aanwezig zullen zijn in een ‘realistisch’ riviermodel. Bij de analyse van dát model kunnen we ons dan concentreren op andere moeilijkheden, daarbij de verschijnselen uit het modelprobleem in het achterhoofd houdend. Het bestuderen van het realistisch riviermodel, waarbij de wiskundige resultaten naar de ‘echte’ wereld vertaald zullen worden, zal een gezamenlijke inspanning moeten zijn van de onderzoeker en de belanghebbende instituten.
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**Herinnering aan Holland**

Denkend aan Holland
zie ik brede rivieren
traag door oneindig
laagland gaan,
rijen ondenkbaar
ijle populieren
als hoge pluimen
aan den einder staan;
en in geweldige
ruimte verzonken
de boerderijen
verspreid door het land,
boomgroepen, dorpen,
geknotte torens,
kerken en olmen
in een groots verband.
de lucht hangt er laag
en de zon wordt er langzaam
in grijze veelkleurige
dampen gesmoord,
en in alle gewesten
wordt de stem van het water
met zijn eeuwige rampen
gvreedsd en gehoord.

H. Marsman.