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Introduction

Perturbation Theory

Considers the effect of small disturbances in the equation to the solution of the equation.
A first example

Let $x - 2 = 0$. Then $x = 2$. What about

$$x - 2 = \epsilon \cosh(x)$$

Assume $\epsilon \ll 1$ and substitute $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$

$$(x_0 + \epsilon x_1 + \ldots) - 2 = \epsilon \cosh(x_0 + \epsilon x_1 + \ldots)$$

and solve separately for each order of $\epsilon$:

$$\ldots \Rightarrow x \approx 2 + \epsilon \cosh(2)$$

Conclusions

The effect of the small parameter $\epsilon$ is small; therefore the perturbation is said to be regular. Otherwise we call it singular.
Order symbol notation

Let \( f(x), g(x) \) be two functions defined around \( x = x_0 \).

We say, \( f = o(g) \) as \( x \to x_0 \) iff \( \lim_{x \to 0} f/g = 0 \).

We say, \( f = O(g) \) (of order) as \( x \to x_0 \) iff \( \lim_{x \to 0} f/g = \text{const} \neq 0 \).

We say, \( f \sim g \) (goes like) as \( x \to x_0 \) iff \( \lim_{x \to 0} f/g = 1 \).

Examples:

1. \( x^2 = o(x) \) as \( x \to 0 \), b/c \( x^2/x \to 0 \).
2. \( 3x^2 = O(5x^2) \) as \( x \to 0 \), b/c \( (3x^2)/(5x^2) \to 3/5 \).
3. \( \sin(x) \sim x \) as \( x \to 0 \), b/c \( \sin(x)/x \to 1 \).
4. \( \sin(x) = x + O(x^3) \) as \( x \to 0 \), b/c \( (\sin(x) - x)/x^3 \to 1/6 \).
5. \( \sin(x) \sim x - \frac{x^3}{3!} \) as \( x \to 0 \), b/c \( \sin(x)/(x - x^3/3!) \to 1 \).

So the solution to \( x - 2 = \epsilon \cosh(x) \) is \( x \sim 2 + \epsilon \cosh(2) \) as \( \epsilon \to 0 \).
The Fundamental Theorem of Perturbation Theory

If

\[ A_0 + A_1 \epsilon + \cdots + A_n \epsilon^n + O(\epsilon^{n+1}) = 0 \]

for \( \epsilon \to 0 \) and \( A_0, A_1, \ldots \) independent of \( \epsilon \), then

\[ A_0 = A_1 = \cdots = A_n = 0. \]

That is why we could solve separately for each order of \( \epsilon \):
Regular Perturbations

\[ x^2 - 3x + 2 + \epsilon = 0, \quad \epsilon \ll 1. \]

Assume the roots have expansion \( x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots : \)

\[
(x_0 + \epsilon x_1 + \ldots)^2 - 3(x_0 + \epsilon x_1 + \ldots) + 2 + \epsilon = 0,
\]

\[ \ldots \Rightarrow x = 1 + \epsilon + \epsilon^2 + O(\epsilon^3), \quad 2 - \epsilon - \epsilon^2 + O(\epsilon^3). \]

Compare with exact solution,

\[ x = \frac{3 \pm \sqrt{1 - 4\epsilon}}{2} = \frac{3 \pm (1 - 2\epsilon - 2\epsilon^2 + \ldots)}{2} \]
Singular Perturbations

\[ \epsilon x^2 - 2x + 1 = 0, \quad \epsilon \ll 1. \]

Assume the roots have expansion \( x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots : \)

\[ \epsilon (x_0 + \epsilon x_1 + \ldots)^2 - 2(x_0 + \epsilon x_1 + \ldots) + 1 = 0. \]

\[ \Rightarrow \quad x = \frac{1}{2} + \frac{1}{8\epsilon} + \frac{1}{16\epsilon^2} + O(\epsilon^3) \]

But where is the second solution? Compare with exact solution,

\[ x = \frac{1 \pm \sqrt{1 - \epsilon}}{\epsilon} = \frac{1 \pm (1 - \epsilon/2 - \epsilon^2/8 + \ldots)}{\epsilon} \]

\[ \Rightarrow \quad x = \frac{1}{2} + \frac{1}{8\epsilon} + O(\epsilon^2), \quad 2/\epsilon - 1/2 + O(\epsilon). \]

We can find the second solution by solving for \( w = \epsilon x, \) so

\[ w^2 - 2w + \epsilon = 0. \]
Ordinary differential equations
\[ \dot{x} + x = \epsilon x^2, \quad x(0) = 1. \]

Assume the roots have expansion \( x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots : \)

\[ (\dot{x}_0 + \epsilon \dot{x}_1 + \ldots ) + (x_0 + \epsilon x_1 + \ldots ) = \epsilon (x_0 + \epsilon x_1 + \ldots )^2, \]
\[ x_0(0) + \epsilon x_1(0) + \cdots = 1. \]

\[ \ldots \Rightarrow x = e^{-t} + \epsilon (e^{-t} - e^{-2t}) + O(\epsilon^2). \]
Singular Perturbations

Heat equation: Temperature $x$ for a stone in water

$$\epsilon \dot{x} = 1 - x, \ x(0) = 0.$$ 

Assume the roots have expansion $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$: 

$$(\dot{x}_0 + \epsilon \dot{x}_1 + \ldots) = 1 - (x_0 + \epsilon x_1 + \ldots).$$

$$\ldots \Rightarrow x_0 = 1 \ (\text{Does not satisfy initial conditions}).$$

Again, We can find a solution by transforming, $\tau = \epsilon^{-1} t$,

$$\frac{d}{d\tau} x = 1 - x, \ x(0) = 0.$$ 

$$\ldots \Rightarrow x = 1 - e^{-2\tau} = 1 - e^{-2t/\epsilon}.$$
Singular in the domain

\[ \dot{x} + \epsilon x^2 = 1, \quad x(0) = 0. \]

Assume the roots have expansion \( x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots \):

\[
(\dot{x}_0 + \epsilon \dot{x}_1 + \ldots) + \epsilon(x_0 + \epsilon x_1 + \ldots)^2 = 1.
\]

\[ \ldots \implies x \sim t - \epsilon \frac{t^3}{3} - \epsilon^2 \frac{2t^5}{15}. \]

This solution is non-uniform (blows up for large \( t \)).

\[ x \sim t - \epsilon \frac{t^3}{3} \] is an asymptotic expansion for \( t < \sqrt{3/\epsilon} \).
The non-linear spring
Duffing’s equation

Consider a spring with a small non-linear perturbation

\[ \ddot{x} + \omega^2 x + \epsilon x^3 = 0, \quad x(0) = a, \quad \dot{x}(0) = 0. \]

with \( \omega^2 = k/m_1 \) and \( \epsilon \ll 1 \). Expand:

\[ \ddot{x}_0 + \omega^2 x_0 = 0 \quad \Rightarrow \quad x_0 = a \cos(\omega t) \quad (1) \]

\[ \ddot{x}_1 + \omega^2 x_1 = -\frac{a^3}{4} (3 \cos(\omega t) + \cos(3\omega t)). \quad (2) \]

... \Rightarrow \quad x \sim a \cos(\omega t) + \epsilon \frac{a^3}{32\omega^2} (\cos(3\omega t) - \cos(\omega t) - 12t\omega \sin(\omega t)).

Again, this blows up for large \( t \). How can we find a uniform solution?
Linstead’s Method

1. Assume the system has a natural frequency $\Omega$.
2. Transform to a strained time variable $\tau = \Omega t$.
3. Expand solution $x = x_0 + \epsilon x_1 + \ldots$ as well as frequency $\Omega = \Omega_0 + \epsilon \Omega_1 + \ldots$.
4. Use the new variables $\Omega_n$ to remove secular terms.

For Duffing’s equation we obtain

$$x = a \cos(\tau) + \epsilon \frac{a^3}{36\omega^2}(\cos(3\tau) - \cos(\tau)) + \ldots$$

with $\Omega = \omega + \frac{3a^2}{8\omega} \epsilon + \ldots$.
Phase-space diagram

Any autonomous (no explicit $t$-dependence) second-order system

$$\ddot{x} = f(x, \dot{x}).$$

can be written as

$$\dot{x} = y, \quad \dot{y} = f(x, y)$$

Then

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{f(x, y)}{y}.$$ 

A plot of in the $xy$ plane is called a phase-space diagram. Solution plots are called orbits.
For the linear spring, $\ddot{x} + \omega^2 x = 0$, we obtain $\frac{1}{2}y^2 = -\frac{1}{2}\omega^2 x^2 + c$.

**Figure:** Phase space diagram for the linear spring with $\omega = 1$. 
Phase-space diagram

Properties of phase-space diagrams

- A closed orbit corresponds to a periodic system
- Points where $\dot{x} = \dot{y} = 0$ are called critical points and represent equilibrium points of the system.
- Orbits cannot intersect or cross
- This plot shows only the orbit structure, all temporal information is lost.
For the linear spring with dissipation, $\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0$, we obtain 

$$x = e^{-\gamma t} \left( a \cos(\omega t) + b \sin(\omega t) \right), \quad \omega^2 = \omega_0^2 - \gamma^2.$$ 

**Figure:** Phase space diagram for the linear damped spring with $\omega_0 = 2\gamma = 1$. 
For the Hertian spring, \( \ddot{x} + \omega^2 |x|^{3/2} = 0 \), we obtain
\[
\frac{1}{2} y^2 = -\frac{2}{5} \omega^2 |x|^{5/2} + c.
\]

**Figure:** Phase space diagram for the Hertzian spring with \( \omega = 1 \).
For Duffing's equation, $\ddot{x} + \omega^2 x + \epsilon x^3 = 0$, we obtain

$$\frac{1}{2} y^2 = -\frac{1}{2} \omega^2 x^2 - \frac{1}{4} \epsilon x^4 + c.$$

**Figure**: Phase space diagram for the non-linear spring with $\omega = \epsilon = 1$. 
Classification of Equilibrium Points

We classify equilibrium points by linearizing the system around them. We obtain the linear system

\[ \dot{x} = ax + by, \]
\[ \dot{y} = cx + dy. \]

Writing this as a second order system, we get

\[ \ddot{x} - (a - d)\dot{x} + (ad - bc)x = 0 \]
\[ x = ae^{\lambda t} \text{ are solutions to this equation if } \lambda \text{ satisfies} \]
\[ \lambda^2 - (a - d)\lambda + (ad - bc) = 0 \]

Thus if solutions \( \lambda_1, \lambda_2 \) are distinct,

\[ x = a_1e^{\lambda_1 t} + a_2e^{\lambda_2 t} \]

We distinguish six types of critical points: [blackboard]