APiE Exercise – ordinary differential equations (ODE)

Exercise 1 (Ex.1–5 account for 2 points each; total 10 pts.)

The motion of a harmonic oscillator is described by the differential equation:

$$m\frac{d^2x(t)}{dt^2} = m\ddot{x} = -kx(t) \tag{1}$$

where m is the mass, k is the spring-constant, $x(t)$ is the position and d^2x/dt^2 is the second time-derivative of $x(t)$.
a) Get *analytically* the solution of Eq. (1) for initial conditions $x_0 = x(0)$ and $v_0 = \dot{x}(0) = v(0)$ and plot it as function of time for $x_0 = 0\,\text{m}$ and $v_0 = 0.1\,\text{m/s}$. What is the period T and the frequency ω? For the plot use $m = 1\,\text{kg}$ and $k = 0.1\,\text{N/m}$.
b) Now compute *analytically* the trajectory $x(t)$ and then plot the kinetic, the potential, and the total energy of the harmonic oscillator as function of time (together).
c) Modify mass m and/or spring-constant k and describe how T changes.

Exercise 2.

a) Solve Eq. (1) numerically using the Euler algorithm (as introduced in the course). Plot solution and energies together with the analytical solution of the previous Exercise 1. Compare them for different time-steps. Which time-step do you find satisfactory?
b) Solve Eq. (1) numerically using the Euler-Cromer algorithm as introduced in the course. Plot solution and energies together with the analytical solution of the previous exercises 1. and 2.a) (use different colors). Compare them for different time-steps. Which time-step do you find satisfactory for Euler-Cromer? in comparison to Euler?

Exercise 3.

In order to solve this differential equation numerically, you can use the so-called Verlet integration algorithm. It was derived in class, but we also show the steps here - repeat them for yourself!

1. Write down the Taylor-series for $x(t+\Delta t)$ and $x(t-\Delta t)$ up to second order in time (terms higher than Δt^2 are ignored).
2. Add $x(t-\Delta t)$ to $x(t+\Delta t)$ and rewrite the equation such that you have $x(t+\Delta t)$ on the left-hand side of the equation.
3. This scheme can now be used to compute the trajectory of the mass (with given $x(t)$, $x(t-\Delta t)$, and acceleration $d^2x/dt^2 = f(t)/m$, with force $f(t) = -kx(t)$).
   For the old position use the estimate: $x(t-\Delta t) \approx x(t) - v(t)\Delta t$.

4. Plot the old time-x-series and the new results, using the same amplitude, phase-angle = 0, $m = 1$, $k = 0.1$, $\Delta t = 0.1$, and $t_{\max} = 200$, $x(0) = 0$ and $v(0) = 0.1$.
5. How big is the error of the calculation? How do you quantify it? Or, with other words, which time-step would you use to have a reasonably small error here? (define reasonable)

Plot also the total energy of the numeric harmonic oscillator as function of time. Plot separately kinetic and potential energy. Describe the differences to the analytical solution.

Exercise 4

Add a damping force $f_d = -\gamma v(t)$ to the spring-force and plot the new results for a γ such that the motion is damped completely at $t = 200$. Which γ-value did you use?

Solve the new corresponding differential equation *analytically* first - and then compare it to one of your numerical solution algorithms. How did the period change when you added the damping force? Modify γ and describe how T changes with γ. Implement the damping force in your Verlet numerical solution from exercise 3. Discuss the problems that occur.

Exercise 5

Solve the differential equation (1) with a built-in Matlab function of your choice and compare to the previous examples/solutions – and answer the questions also for this integrator.

Which (power) class of error e have the different methods? (Definition: The error increases with power α for time-step h such that $e \propto h^\alpha$.)

Exercise 6 (Voluntary +3 points)

Generalize the spring-mass system by adding a driving force $f_d \sin(\omega_d t)$ to the right hand side of equation (1). Plot a "Poincare-cut" for three different ω_d (smaller, similar and larger than ω) and f_d (very small, small, and large) and describe your observations.

Derive the differential equation for a pendulum with a mass-less rigid rod that can swing around a fixed point with a mass m at the other end. Show the same plots as for the driven harmonic oscillator and describe the differences.

Exercise 7 (Voluntary +2 points)

Generalize the spring-mass system to a linear chain of N masses as will be detailed in the Lecture on Debugging and Efficient Programming.