Mixed Lubrication of Line Contacts

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MIXED LUBRICATION OF LINE CONTACTS

PROEFSCHRIFT

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Assistent-promotor: dr.ir. D.J. Schipper
voor Bernadet
Abstract

This thesis deals with friction. The effects of friction can be experienced everyday. Not much is possible without friction. In almost all things we do daily friction is involved. In industry friction is very important as well. For many applications, as e.g. bearings, friction is undesirable, whereas for other applications, as e.g. traction drives friction is essential. The friction in lubricated line contacts can be calculated, by the model described in this thesis. Firstly, using this model an estimate can be made, in which lubrication regime a system operates. Secondly, within a lubrication regime an estimate can be made how much friction is to be expected. Also, the influence of the different parameters involved can be determined.

The lubrication regimes can be determined using the model by calculating the friction in the different regimes and, thus, a Striebeck curve can be constructed. The model is a combination of a contact model for rough line contacts and a film thickness equation for the full film lubricated contact situation, it is based on the fact that the pressure is a potential. Different pressures can therefore be added, The total pressure causes the deformation of the surfaces. The pressure distribution for highly loaded rough contacts without lubrication or operating in the Boundary Lubrication regime (BL), is equal to the pressure distribution for a highly loaded Elasto- Hydrodynamic (EHL) contact. Therefore the results of calculations on rough surfaces and EHL calculations can be combined in the Mixed Lubrication (ML) regime.

Already much has been published about lubrication and friction in the EHL regime. For smooth surfaces the film thicknesses can be calculated accurately. These calculations can be used in the EHL regime and for the part of the load that is carried by the hydrodynamic component in the ML regime. Much less research has been done on the BL regime. In this thesis a simple model is used to calculate the coefficient of friction in the BL regime. The pressure distributions and separations are calculated for rough contacts.

Based on a reference case, the influence of the different parameters on the generalized Striebeck curve can be calculated. The results of the model are compared with experiments published in literature. These experiments are done on a two-disk machine, which is a highly loaded contact situation. Also calculations are compared with measurements done under conditions of sheet metal forming. This is a contact with a relatively low load. In both cases of the model shows good agreement with the results of the measurements.

A different lowly loaded contact situation is a mechanical face seal. The measurements done are carried out with water. For the mechanical face seals a new film
thickness equation is derived, based on a small-bearing equation. The results show that the calculations and measurements are in good agreement, using this new film thickness equation in combination with the contact model.

The model can also be used to calculate traction curves. For this purpose an elastic-plastic shear stress model for boundary layers (BL) is derived for the situation around 0% slip, to calculate the friction around 0% slip. Again, the calculations are compared with measurements, and a good agreement is found.

An alternative contact model is used to examine the influence of the distance between the mean height of the summits and the mean height of all points on a surface. Also, the influence of a different summit distribution, caused by wear, is calculated.
Samenvatting

Dit proefschrift gaat over wrijving. De gevolgen van wrijving zien we elke dag om ons heen. Zonder wrijving is er maar weinig mogelijk. Bij bijna alle dagelijkse handelingen speelt wrijving een grote rol. Ook in de industrie is wrijving van groot belang. Voor veel toepassingen, zoals lagers, is wrijving ongewenst, terwijl voor andere toepassingen, zoals wrijvingsoverbrengingen, wrijving juist weer essentieel is. In dit proefschrift wordt een model beschreven, waarmee de wrijving in gemengd gesmeerde lijncontacten kan worden berekend. Met dit model kan in de eerste plaats een uitspraak gedaan worden in welk smeringsregime een systeem werkt. Binnen de smeringsregimes kan vervolgens een voorspelling worden gemaakt hoe hoog de wrijving is. Ook de invloed van de verschillende parameters die op de wrijving van invloed zijn, kan worden vastgesteld.

De smeringsregimes kunnen aan de hand van het model worden vastgelegd door de wrijving in de verschillende smeringsregime te berekenen, waarna een Stribeck curve kan worden geconstrueerd. Het model is een combinatie van een contactmodel voor ruwe lijncontacten en een filmdikte vergelijking voor de volle film smerings-situatie, hetgeen is gebaseerd op het feit dat de druk een potentiaal is. Verschillende drukken mogen daarom bij elkaar opgeteld worden. De som van de drukken zorgt voor de vervorming van de oppervlakken. De drukverdeling van hoog belaste ruwe oppervlakken zonder smering of in het grens-smeringsregime (BL) is gelijk aan de drukverdeling van een hoog belast Elasto- Hydrodynamisch gesmeerd (EHL) contact. De resultaten van berekeningen aan ruwe oppervlakken en EHL berekeningen kunnen daarom gecombineerd worden in het gemengde smeringsregime (ML).


Uitgaande van een referentie-situatie is de invloed van de verschillende parameters op de generaliseerde Stribeck curve berekend. De uitkomsten van het model zijn vergeleken met experimenten uit de literatuur uitgevoerd met een twee-schijvenmachine; dit is een hoog belaste contact situatie. Ook zijn de berekeningen vergeleken met metingen gedaan onder plaatvervormings-condities. Dit is een relatief laag belast contact. Het blijkt dat het model onder beide contactcondities goede resultaten geeft.
Een ander laag belaste contactsituatie is een mechanische asafdichting. De metingen die hieraan gedaan zijn, zijn uitgevoerd met water. Voor mechanische asafdichtingen is een nieuwe filmdikte vergelijking afgeleid, gebaseerd op een smallager vergelijking. Het blijkt dat met deze filmdikte vergelijking in combinatie met het contactmodel de berekeningen en metingen zeer goed overeenkomen.

Het model kan ook gebruikt worden om tractiecurven te berekenen. Hiervoor is voor de situatie rond 0% slip een elastisch-plastisch afschuifmodel voor grenslagen (BL) opgesteld, om de wrijving daar te beschrijven. De berekeningen zijn weer vergeleken met metingen en een goede overeenkomst wordt gevonden.

Een alternatief contactmodel is gebruikt om de invloed van de afstand tussen de gemiddelde hoogte van de toppen en de gemiddelde hoogte van alle punten op een oppervlak te onderzoeken. Ook is de invloed van een andere distributieverdeling van de toppen, veroorzaakt door slijtage, berekend.
Acknowledgements

Many people have contributed to this thesis, either by physical, intellectual or mental support.

First of all I would like to thank the members of the Tribology group of the faculty of Mechanical Engineering of the University of Twente for their support, cooperation and discussions during the project. Most of all, however, I thank them for creating a pleasant working atmosphere: Ton de Gee, Johan Lijtmer, Hans Moes, Wijtze ten Napel, Matthijn de Rooij, Dik Schipper, Kees Venner, Laurens de Boer, Gerrit van der Bult, Willy Kerver, Walter Lette, Willy Olthof, Erik de Vries, Jan Bos, Bernd Brolle, Rob Cuperus, Rudi ter Haar, Qiang Liu, Harald Lubbinge, Henk Metselaar, Elmer Mulder, Daniël van Odyck, Patrick Pirson, Jan Willem Sloetjes, Ronald van der Stegen, Harm Visscher, André Westeneng and Ysbrand Wijnant.

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Lochem, november 1999
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## Nomenclature

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<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>Hertzian contact radius for smooth spheres</td>
<td>m</td>
</tr>
<tr>
<td>$a_i$</td>
<td>fit parameter ($i = 1, 2$)</td>
<td>[-]</td>
</tr>
<tr>
<td>$A_C$</td>
<td>real area of asperity contact</td>
<td>m$^2$</td>
</tr>
<tr>
<td>$A_H$</td>
<td>total area of the hydrodynamic component in contact</td>
<td>m$^2$</td>
</tr>
<tr>
<td>$A_{nom}$</td>
<td>nominal or apparent contact area</td>
<td>m$^2$</td>
</tr>
<tr>
<td>$A_r$</td>
<td>real contact area</td>
<td>m$^2$</td>
</tr>
<tr>
<td>$A$</td>
<td>dimensionless real contact area</td>
<td>[-]</td>
</tr>
<tr>
<td>$b$</td>
<td>half Hertzian width</td>
<td>m</td>
</tr>
<tr>
<td>$b^*$</td>
<td>half width of deformed line contact</td>
<td>m</td>
</tr>
<tr>
<td>$b^*$</td>
<td>normalized half width of deformed line contact</td>
<td>[-]</td>
</tr>
<tr>
<td>$B$</td>
<td>length of the cylinder</td>
<td>m</td>
</tr>
<tr>
<td>$C^*$</td>
<td>normalized half width of deformed line contact as defined by Lo (1969)</td>
<td>[-]</td>
</tr>
<tr>
<td>$d_d$</td>
<td>distance between mean plane of the heights of the summits and mean plane of the surface heights</td>
<td>m</td>
</tr>
<tr>
<td>$D_d$</td>
<td>dimensionless distance between mean plane of the heights of the summits and mean plane of the surface heights</td>
<td>[-]</td>
</tr>
<tr>
<td>$E'$</td>
<td>combined elasticity modulus</td>
<td>Pa</td>
</tr>
<tr>
<td>$E_i$</td>
<td>elasticity modulus of contacting surface $i$ ($i = 1, 2$)</td>
<td>Pa</td>
</tr>
<tr>
<td>$f$</td>
<td>coefficient of friction</td>
<td>[-]</td>
</tr>
<tr>
<td>$f(s)$</td>
<td>Gaussian heights distribution</td>
<td>[1/m]</td>
</tr>
<tr>
<td>$F_f$</td>
<td>friction force</td>
<td>N</td>
</tr>
<tr>
<td>$F_j$</td>
<td>integral identity (with a $j$ number)</td>
<td>[-]</td>
</tr>
<tr>
<td>$f_C$</td>
<td>coefficient of friction in the BL regime</td>
<td>[-]</td>
</tr>
<tr>
<td>$F_N$</td>
<td>normal force</td>
<td>N</td>
</tr>
<tr>
<td>$F_C$</td>
<td>load carried by the asperities</td>
<td>N</td>
</tr>
<tr>
<td>$F_H$</td>
<td>load carried by the hydrodynamic component</td>
<td>N</td>
</tr>
<tr>
<td>$G$</td>
<td>lubricant number</td>
<td>[-]</td>
</tr>
<tr>
<td>$G$</td>
<td>shear modulus of the lubricant</td>
<td>Pa</td>
</tr>
<tr>
<td>$h$</td>
<td>film thickness, separation</td>
<td>m</td>
</tr>
<tr>
<td>$h$</td>
<td>dimensionless film thickness</td>
<td>m</td>
</tr>
<tr>
<td>$h_{00}$</td>
<td>integration constant</td>
<td>m</td>
</tr>
<tr>
<td>$h_{cen}$</td>
<td>central film thickness</td>
<td>m</td>
</tr>
<tr>
<td>$h_{dry}$</td>
<td>separation in the dry contact situation</td>
<td>m</td>
</tr>
<tr>
<td>$h_{min}$</td>
<td>minimum film thickness</td>
<td>m</td>
</tr>
<tr>
<td>$H$</td>
<td>dimensionless separation</td>
<td>[-]</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
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<td>--------</td>
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</tr>
<tr>
<td>$H_{00}$</td>
<td>integration constant</td>
<td></td>
</tr>
<tr>
<td>$H_c$</td>
<td>dimensionless central separation</td>
<td></td>
</tr>
<tr>
<td>$H_{cen}$</td>
<td>dimensionless central film thickness</td>
<td></td>
</tr>
<tr>
<td>$H_h$</td>
<td>dimensionless separation</td>
<td></td>
</tr>
<tr>
<td>$H_{EI}$</td>
<td>elastic/viscous asymptote</td>
<td></td>
</tr>
<tr>
<td>$H_{EP}$</td>
<td>elastic/piezoviscous asymptote</td>
<td></td>
</tr>
<tr>
<td>$H_{RI}$</td>
<td>rigid/viscous asymptote</td>
<td></td>
</tr>
<tr>
<td>$H_{RP}$</td>
<td>rigid/piezoviscous asymptote</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>lubricant number (Moes parameter)</td>
<td></td>
</tr>
<tr>
<td>$\tilde{L}$</td>
<td>lubrication number (Schipper parameter)</td>
<td></td>
</tr>
<tr>
<td>$M$</td>
<td>load number</td>
<td></td>
</tr>
<tr>
<td>$N$</td>
<td>number of asperities in contact</td>
<td></td>
</tr>
<tr>
<td>$\tilde{N}$</td>
<td>normalized number of asperities in contact</td>
<td></td>
</tr>
<tr>
<td>$n$</td>
<td>density of asperities</td>
<td></td>
</tr>
<tr>
<td>$\hat{n}$</td>
<td>density of asperities number</td>
<td></td>
</tr>
<tr>
<td>$n_X$</td>
<td>degrees of freedom ($\chi^2$-distribution)</td>
<td></td>
</tr>
<tr>
<td>$p$</td>
<td>pressure</td>
<td></td>
</tr>
<tr>
<td>$\tilde{p}$</td>
<td>dimensionless contact pressure</td>
<td></td>
</tr>
<tr>
<td>$p_{ov}$</td>
<td>average contact pressure</td>
<td></td>
</tr>
<tr>
<td>$p_h$</td>
<td>maximum Hertzian pressure</td>
<td></td>
</tr>
<tr>
<td>$p_c$</td>
<td>central pressure</td>
<td></td>
</tr>
<tr>
<td>$p_C$</td>
<td>asperity pressure</td>
<td></td>
</tr>
<tr>
<td>$p_H$</td>
<td>hydrodynamic pressure</td>
<td></td>
</tr>
<tr>
<td>$p_r$</td>
<td>constant ($p_r = 196.2 \cdot 10^6$ Pa)</td>
<td></td>
</tr>
<tr>
<td>$p_T$</td>
<td>total pressure</td>
<td></td>
</tr>
<tr>
<td>$P$</td>
<td>dimensionless nominal contact pressure</td>
<td></td>
</tr>
<tr>
<td>$P_c$</td>
<td>dimensionless central contact pressure</td>
<td></td>
</tr>
<tr>
<td>$R$</td>
<td>reduced radius of cylinder</td>
<td></td>
</tr>
<tr>
<td>$R_a$</td>
<td>CLA surface roughness</td>
<td></td>
</tr>
<tr>
<td>$R_{a0}$</td>
<td>reference CLA surface roughness</td>
<td></td>
</tr>
<tr>
<td>$R_q$</td>
<td>RMS surface roughness</td>
<td></td>
</tr>
<tr>
<td>$R_e$</td>
<td>equivalent radius for a point contact</td>
<td></td>
</tr>
<tr>
<td>$s$</td>
<td>argument of the height distribution $\phi(s)$</td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td>Slip</td>
<td></td>
</tr>
<tr>
<td>$S_{ep}$</td>
<td>elastic-plastic Slip transition parameter</td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
<td></td>
</tr>
<tr>
<td>$U_2$</td>
<td>velocity number</td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td>velocity</td>
<td></td>
</tr>
<tr>
<td>$v^+$</td>
<td>sum velocity</td>
<td></td>
</tr>
<tr>
<td>$v_{av}$</td>
<td>average velocity</td>
<td></td>
</tr>
<tr>
<td>$v_{sq}$</td>
<td>sliding velocity</td>
<td></td>
</tr>
<tr>
<td>$W$</td>
<td>load number</td>
<td></td>
</tr>
<tr>
<td>$w$</td>
<td>deformation</td>
<td></td>
</tr>
</tbody>
</table>
Nomenclature

\[ w_0 \] integration constant \[ m \]  
\[ \bar{w} \] dimensionless deformation \[ - \]  
\[ \bar{w}_0 \] dimensionless integration constant \[ - \]  
\[ x \] spatial coordinate \[ m \]  
\[ X \] dimensionless spatial coordinate \[ - \]  
\[ y \] spatial coordinate \[ m \]  
\[ z \] viscosity-pressure index (Roelands) \[ - \]

Greek symbols

\[ \alpha \] viscosity-pressure coefficient (Barus) \[ 1/Pa \]  
\[ \alpha^* \] adapted viscosity-pressure coefficient \[ 1/Pa \]  
\[ \alpha_0 \] roughness comparison ratio for line contacts \[ - \]  
\[ \alpha_s \] roughness comparison ratio for point contacts \[ - \]  
\[ \beta \] radius of asperities \[ m \]  
\[ \gamma_1 \] adaption parameter for hydrodynamic component in ML \[ - \]  
\[ \gamma_2 \] adaption parameter for asperity contact component in ML \[ - \]  
\[ \dot{\gamma} \] shear rate \[ s^{-1} \]  
\[ \eta \] viscosity \[ Pa\cdot s \]  
\[ \eta_0 \] viscosity at ambient pressure \[ Pa\cdot s \]  
\[ \eta_{\infty} \] constant \( \eta_{\infty} = 6.315 \cdot 10^{-5} Pa\cdot s \) \[ Pa\cdot s \]  
\[ \lambda_s \] dimensionless separation \[ - \]  
\[ \nu \] Poisson’s ratio \[ - \]  
\[ \rho \] density \[ kg/m^3 \]  
\[ \rho_0 \] density at ambient pressure \[ kg/m^3 \]  
\[ \sigma_s \] standard deviation of the asperity height distribution \[ m \]  
\[ \sigma_s' \] dimensionless roughness \[ - \]  
\[ \tau \] shear stress \[ Pa \]  
\[ \tau_C \] shear stress of asperity contact \[ Pa \]  
\[ \tau_H \] shear stress of the hydrodynamic component \[ Pa \]  
\[ \tau_l \] the lubricant limiting shear stress \[ Pa \]  
\[ \tau_0 \] Eyring shear stress \[ Pa \]  
\[ \phi(s) \] distribution of the asperities \[ - \]  
\[ \psi \] liquid fraction \[ - \]
sub-, superscripts

1, 2 surface 1, 2
a entrance
av average
b outlet
cen central
dif differential
dry dry, without lubrication
max maximum
min minimum
x x-direction
y y-direction
+ sum

Dimensionless numbers

\[ \tilde{A} = \frac{A_r}{bB} \]
\[ \tilde{b}^* = \frac{b^*}{b} \]
\[ D_d = \frac{\pi d_d B E'}{8 FN} \]
\[ G = \alpha E' \]
\[ \tilde{h} = \frac{h}{R} \]
\[ H = \frac{h}{R} \sqrt{\frac{E' R}{\eta_0 u^+}} \]
\[ H_h = \frac{\pi h B E'}{8 \frac{E'}{FN}} \]
\[ L = \alpha E' \left( \frac{\eta_0 u^+}{E' R} \right)^{1/4} \]
\[ \mathcal{L} = \frac{\eta_0 u^+}{p_{av} R_a} \]
\[ M = \frac{F_N}{B E' R} \sqrt{\frac{E' R}{\eta_0 v^+}} \]
\[ \tilde{N} = \frac{N}{B b \eta} \]
\[ \bar{n} = \frac{32 n F_N}{\pi E' B} \sqrt{3 R} \]
\[ \bar{p} = \frac{p}{p_h} \]
\[ P_c = \frac{p_c}{p_h} \]
\[ U_\Sigma = \frac{\eta_0 v^+}{E' R} \]
\[ W = \frac{F_N}{B E' R} \]
\[ X = \frac{x}{b} \]
\[ \lambda_s = \frac{h}{\sigma_s} \]
\[ \bar{\sigma}_s = \frac{\pi \sigma_s B E'}{8 F_N} \]

**Abbreviations**

- BL Boundary Lubrication
- (E)HL (Elasto) Hydrodynamic Lubrication
- G&W Greenwood and Williamson
- ML Mixed Lubrication
Nomenclature
Chapter 1

Introduction

1.1 Tribology and lubrication

In industry most moving contacts are lubricated, because lubrication effectively reduces friction and wear. Most applications require low friction between the relatively moving surfaces. Examples are gears, roller bearings, seals and cam & tappet mechanisms. Some applications, however, demand high friction, such as traction drives and clutches.

Since materials tend to improve in quality, the need for tribological knowledge in industry increases as well. Due to improved knowledge of materials and the mechanics of solids, machine components fail less, because the yield stress (fracture) is exceeded or by fatigue, and often last as long as the whole machine. The next step in machine design is that tribo-systems are optimized for friction and wear. In this thesis the emphasis lies on friction. In general, if there is no need for traction, friction means energy consumption.

Often the following question is asked by technicians: "What is the coefficient of friction in this system." The reason why this question can not be answered unambiguously is that the coefficient of friction depends on many parameters. Very important is the question whether the system is lubricated or not. Further, the values of the material properties at the interface of the contacting surfaces and the environment are dominating factors. In case the system is lubricated, the properties of the fluid, become important. The forces acting on the contacting surfaces as well as the shape of the contact matter. Also very important are the velocities of the interacting surfaces, especially when the system is lubricated. In that case the fluid in the system can wedge the surfaces apart and thus the surfaces become separated. Usually fluid friction is less than the friction induced when the surfaces have direct contact.

In 1902 Strubeck presented his paper on the influence of the velocities of the interacting surfaces and the load on the coefficient of friction for plain journal bearings as well as for roller bearings. Three lubrication modes can be distinguished: Boundary Lubrication (BL), Mixed Lubrication (ML) and Hydrodynamic Lubrication (HL). The latter is often referred to as Elasto-Hydrodynamic Lubrication (EHL). In this
case elastic deformation of the interacting surfaces becomes important. Those three lubrication modes or regimes can be found back in the curve named after Striebeck. Section 1.2 will deal with different lubrication regimes and the so-called generalized Striebeck curve. In Section 1.3 the objective of this thesis will be explained and in the last section of this chapter the outline of this thesis will be discussed.

1.2 The generalized Striebeck curve

The different lubrication regimes are schematically represented in Fig. 1.1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{lubrication_regimes.png}
\caption{The lubrication regimes.}
\end{figure}
1.2. The generalized Strubeck curve

At high velocities the surfaces are fully separated by a fluid film (Fig. 1.1a). The surfaces are fully separated due to pressure build-up, which is caused by the motion of the surfaces and (Elasto-)Hydrodynamic Lubrication (EHL) occurs. When the velocity decreases, pressure build up in the fluid decreases and as a result the asperities on the surfaces begin to touch and a part of the load is carried by the asperities, resulting in an increase of the coefficient of friction. Friction is controlled by the interacting asperities as well as by the fluid between the surfaces. This regime is called Mixed Lubrication (ML), Fig. 1.1b. Further decrease of the velocities or increase of the load of the surfaces leads to further decrease of the hydrodynamic action. More asperities will touch. When all load is carried by the asperities, friction is controlled by shearing of the boundary layers, which are usually present on the solid bodies. This regime is referred to as Boundary Lubrication (BL), Fig. 1.1c. In this regime the coefficient of friction is nearly independent of load and velocity. Together these three regimes form the Strubeck curve.

Figure 1.1 shows boundary layers. These boundary layers can, for instance, be formed on the surfaces when additives from the fluid react with the surface. Lubricants often contain additives in order to protect the surfaces from wear.

A frictional situation which is not shown in Fig. 1.1 is “ineffective lubrication”. In “ineffective lubrication” the lubricant supplied to the contact is not able to form a protective boundary layer between the interacting asperities. For instance because the (local) temperature is too high. In that case the coefficient of friction approaches the dry (unlubricated) value, which can be very high (e.g. 0.5 - 1.0) and the tribosystem will fail. The coefficient of friction is defined as: $f = F_f/F$, with $F_f$ the friction force and $F$ the normal force. For comparison: the coefficient of friction in the BL regime is for oil lubricated systems in the order of 0.1 - 0.15. This thesis will deal with lubricated contacts, having protective boundary layers on the surfaces.

The generalized Strubeck curve is shown in Fig. 1.2. On the vertical axis of the Strubeck curve the coefficient of friction is plotted as a function of a lubrication number on the horizontal axis. The curve in Fig. 1.2 is denoted as generalized since the original curves measured by Strubeck (1902) only show a part of the ML-regime and the (E)HL-regime. In Fig. 1.2 the horizontal axis has a logarithmic scale, other authors, like Strubeck himself, use a linear scale.

The lubrication number is defined in many ways in literature. A lot of definitions have in common that the lubrication number is proportional to the angular velocity $\omega$ (in rad/sec) or the number of revolutions per minute (rev/min) in case of journal bearings (Strubeck, 1902; Hersey, 1915; McKee, 1927; Altrogge, 1950), the sliding velocity (Biel, 1920; Lenning, 1960) or the sum velocity $v^+$ (Schipper, 1988), and the viscosity $\eta$ of the lubricant (Hersey, 1915; Biel, 1920; McKee, 1927; Lenning, 1960; Schipper, 1988). In the denominator of the lubrication number various quantities are used. Some authors use the average pressure over the contact (Schipper, 1988), some prefer the load (Lenning, 1960), and others use the load per unit projected area.
(Hersey, 1915; McKee, 1927; Altrogge, 1950). Finally, a surface roughness parameter (e.g. the CLA surface roughness $R_a$ (Schipper, 1988)) is sometimes included. What quantities are used, often depends on the nature of the subject the authors are dealing with.

In principle, all three regimes in the generalized Striebeck curve can also be encountered when the load is increased, instead of decreasing the velocity of the surfaces. As pointed out by Landheer, Faessen and de Gee (1990) this may, however, lead to an unacceptable increase in (local) temperature, because the production of frictional heat is proportional to the product of load and velocity. In Fig. 1.2 the transitions between the different lubrication regimes are indicated by dashed lines. The transitions depend on various parameters, e.g. the micro-geometry, the mode of deformation (elastic and/or plastic) of the surfaces, viscosity, the macro-geometry of the surfaces and load. Prior to the design of a machine-element it is very useful to know the position of these transitions as function of the above parameters. For instance when a low friction is required, the best operational condition is near the transition of ML to (E)HL. If high traction is needed, operating conditions in or close to the BL-regime are preferred.

1.3 Objective of this thesis

In Section 1.1 it has been stated that tribo-systems (components) in machine design need to be optimized with respect to friction (as well as wear). This thesis will
focus on friction. As follows from the above, it is very important to know in which lubrication regime a tribo-system operates (see Section 1.2). Therefore, in this thesis a model will be developed to predict the Stribeck curve on the basis of the different parameters, mentioned in Section 1.2. The basis of this model is a mixed lubrication model. When one is able to predict the Stribeck curve, the transitions (E)HL-ML and ML-BL can be determined. Different parameters can be varied to study their influence on the Stribeck curve.

Some restrictions are made:

- In practice most contacts are line contacts or very wide elliptical contacts, therefore modeling and calculations are restricted to elastically deforming line contacts. The model set up in this thesis may be adapted to point contacts, but this is beyond the scope of this thesis.

- A parameter that is not considered in this thesis is temperature. For all calculations isothermal conditions are supposed.

The thesis will not deal with macroscopic wear. However, for the situation that wear only changes the surface height- and asperity height distribution (running-in wear) the effect of wear on friction will be studied.

1.4 Outline

In this chapter the objective of this thesis has been stated. The next chapter will give an overview of available literature, concerning the three lubrication regimes. At low loads a rough surface deforms less than a smooth surface. In Chapter 3 the influence of the surface roughness on the elastic deformation of the surface will be described. The contact model for un lubricated contacts will be outlined.

In Chapter 4 the mixed lubrication model will be presented, which is the core of this thesis. By modeling and predicting the ML-regime, the transitions to the other lubrication regimes can be calculated.

In Chapter 5 the Stribeck curves are calculated numerically and the results of these calculations will be shown. A parameter study is carried out, using a Gaussian surface roughness distribution. Also a comparison of the Stribeck curves with experiments in literature will be made.

In Chapter 6 it will be explained what traction curves are. The difference with Stribeck curves will be explained and traction curves will be calculated.

In Chapter 7 a discussion on the results and an extension on the model will be made. Finally, in Chapter 8, conclusions are drawn and recommendations are given.
Chapter 2

Literature

In the study of Mixed Lubrication it is important to know what happens in the other lubrication regimes. In this chapter first a short overview of the literature on Elasto-Hydrodynamic Lubrication is given, followed by a discussion of the literature on Boundary Lubrication. In these discussions, the frictional behaviour is included. In Section 2.3 the literature on Mixed Lubrication is reviewed.

2.1 (Elasto) Hydrodynamic Lubrication

So far, the field of Elasto-Hydrodynamic Lubrication has received most attention in literature. Especially with respect to theoretical understanding of tribology and model building, (E)HL is the most developed part of the tribology. Most of the theory of (E)HL is based on the so-called Reynolds equation, published by Reynolds in 1886. This equation gives the relation between the pressure in the fluid film, the geometry of the surfaces and the velocities of the moving surfaces. The equation is, in fact, a simplification of the Navier-Stokes equations, derived by assuming a narrow gap and demanding conservation of mass. Reynolds’ equation has been written in many forms; in cartesian coordinates it can be written as:

$$
\frac{\partial}{\partial x} \left( \frac{\rho h^3 \partial p}{\eta \partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\rho h^3 \partial p}{\eta \partial y} \right) = 6 \left( v^+ \right) \frac{\partial (\rho h)}{\partial x} + 6 \rho h \frac{\partial (v^+)}{\partial x} + 12 \frac{\partial (\rho h)}{\partial t} \tag{2.1}
$$

with:

- $v^+$ : the sum velocity of the moving surfaces ($v^+ = v_1 + v_2$)
- $x, y$ : spatial cartesian coordinates
- $t$ : time
- $p$ : pressure
- $h$ : film thickness
- $\rho$ : density
- $\eta$ : viscosity

The left-hand-side terms are generally referred to as the Poiseuille terms. The three terms on the right-hand-side of eq. (2.1) denote three possible effects that generate
pressure. The first term is referred to as the wedge term, the second as the stretch term and the last as the squeeze term. The wedge term is also often referred to as the Couette term. The stretch term can be omitted in this thesis, i.e. the sum velocity is assumed to be constant in the direction of the motion. As the direction of motion the \( x \)-direction is chosen, therefore no \( y \)-terms are included on the right-hand-side of eq. (2.1). Since this thesis deals with line contacts, the pressure is constant over the length of the cylinder and thus the second term on the left hand side is zero.

To solve a differential equation one needs boundary conditions. Generally, the pressure is defined as zero on the edges of the domain, i.e. the pressure, calculated in the domain, is in fact the pressure difference with the ambient pressure. Thus:

\[
p(x_a) = p(x_b) = 0, \quad (2.2)
\]

with \( x_a \) and \( x_b \) the boundaries at, respectively, the inlet and the outlet of the domain. This condition is valid at any moment in time.

An important phenomenon that must be accounted for in (E)HL is cavitation. In a diverging gap the pressure in the liquid diminishes, but it must always be larger than, or equal to the vapour pressure of the liquid. When the pressure is equal to the vapour pressure, cavitation will occur. Cavitation is accounted for in (E)HL by introducing an additional parameter, the liquid fraction, in the Reynolds equation (eq. (2.1), see e.g. Kostreva (1984)).

For the non-steady-flow problem a start condition is needed for the time-derivative. Often the zero-pressure condition is taken:

\[
p(t = 0) = 0, \quad (2.3)
\]

but it also possible to take e.g. the pressure distribution of a steady-flow distribution. The start condition depends on the problem that is being considered, in this thesis the steady-flow case. The squeeze term in Reynolds’ equation can thus be left out and a start condition is not needed.

The remaining equation thus reads:

\[
\frac{\partial}{\partial x} \left( \frac{\rho h^3}{\eta} \frac{\partial p}{\partial x} \right) = 6 \left( v^+ \right) \frac{\partial (\rho h)}{\partial x}, \quad (2.4)
\]

with boundary conditions as given in eq. (2.2).

Reynolds’ equation can be solved, resulting in a pressure distribution. To solve Reynolds’ equation it is, however, useful to know the load applied to the contact:

\[
F_N = B \int_{-\infty}^{\infty} p(x)dx, \quad (2.5)
\]
with $B$ the length of the contact (length of the cylinders).

Reynolds’ equation yields the pressure distribution, if the geometries of the surfaces (i.e. the film thickness $h$), the viscosity $\eta$, and the density $\rho$ are known. Since these three parameters depend on the pressure, relationships have been derived in literature which give their relation with pressure. Section 2.1.1 describes which relations relate to the deformation of contacting surfaces, Section 2.1.2 deals with the viscosity-pressure relation and Section 2.1.3 with the density-pressure relation.

### 2.1.1 Deformation

When a pressure is applied to a surface, this surface will deform. The extent of the deformation is greatly determined by the elasticity modulus $E$ of the surface. A great contribution to the theory of deforming surfaces has been made by Hertz (1881). On the basis of Hertz’ theory the deformation of two cylinders in an (E)HL contact can be calculated (see e.g. Timoshenko and Goodier (1982)):

$$ w(x) = -\frac{4}{\pi E'} \int_{-\infty}^{\infty} p(s) \ln |x - s| ds, \quad (2.6) $$

with:

- $w$: deformation
- $E'$: reduced elasticity modulus.

The reduced elasticity modulus is defined as:

$$ \frac{2}{E'} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}, \quad (2.7) $$

with $E_1$ and $E_2$ the elastic moduli and $\nu_1$ and $\nu_2$ Poisson’s ratio of surface 1 and 2, respectively.

The film thickness between two cylinders can now be written in the following form (parabolic approximation):

$$ h(x) = h_00 + \frac{x^2}{2R} + w(x), \quad (2.8) $$

with:

- $h_00$: constant
- $R$: reduced radius of the cylinders
The reduced radius is defined by:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2},$$

with $R_1$ and $R_2$ the radii of the cylinders 1 and 2, respectively. Equation (2.8) is also valid for one cylinder pressing against a flat surface. The radius of the flat surface equals infinity and, thus, the reduced radius is equal to radius of the cylinder. The constant $h_{00}$ is the minimum film thickness, in case the surfaces do not deform.

### 2.1.2 Viscosity-pressure index

In the literature several viscosity-pressure relations can be found. In this thesis only two of them will be used: Barus’ equation (Barus, 1893) and Roelands’ relation (Roelands, 1966). These two relations have received most attention in literature. The first because it is the oldest one and because of its simplicity and the latter because it has been proven to be quite accurate for many lubricants.

The fact that the viscosity increases almost exponential with pressure is a very important effect for fluid film formation in (E)HL contacts. This exponential behaviour is contained in the Barus viscosity-pressure relation (Barus, 1893):

$$\eta(p) = \eta_0 \exp(\alpha p),$$

with:

- $\eta_0$ : viscosity at ambient pressure
- $\alpha$ : viscosity-pressure coefficient

This equation is one of the most widely used viscosity-pressure relations, because it is easy to use in analytical derivations. Use of Barus’ equation should be restricted to rather low pressures only, since for high pressures (higher than approximately 0.1 GPa.) the predicted viscosities are too high.

A viscosity-pressure relation that holds for higher pressures is the one proposed by Roelands (1966). He performed a lot of measurements on viscosities. His relation is accurate for pressures up to 1 GPa. Roelands’ relation reads:

$$\eta(p) = \eta_0 \cdot \exp \left\{ \left( \frac{1}{2} \left( \frac{p}{P_r} \right)^2 - 1 \right) \cdot \ln \left( \frac{\eta_0}{\eta_{\infty}} \right) \right\},$$

or

$$\eta(p) = \eta_0 \cdot \left( \frac{\eta_{\infty}}{\eta_0} \right)^{\frac{1-\frac{3}{2}p}{P_r^2^{\frac{3}{2}}}},$$

with:
\( \eta_\infty \) : constant \( (\eta_\infty = 6.315 \cdot 10^{-5} \text{ Pa}\cdot\text{s}) \)

\( p_r \) : constant \( (p_r = 196.2 \text{ MPa}) \)

\( z \) : viscosity-pressure index

For most mineral oils \( 0.6 \leq z \leq 1.0 \). Equation (2.11) is a simplified version of the full Roelands relation, the original relation also has a term for the dependence of the viscosity on the temperature.

For low pressures Barus’ equation and Roelands’ equation yield the same results, it is therefore possible to express the viscosity-pressure index \( z \) of Roelands in the viscosity-pressure index \( \alpha \) of Barus. This is done by defining:

\[
\alpha = \frac{1}{\eta} \left( \frac{\partial \eta}{\partial p} \right)_{p=0},
\]

thus \( \alpha, \eta_0 \) and \( z \) are mutually dependent:

\[
\frac{\alpha p_r}{z} = \ln \left( \frac{\eta_0}{\eta_\infty} \right). \tag{2.13}
\]

### 2.1.3 Density-pressure relation

In (E)HL the compressibility of the lubricant must be taken into account for high pressures. In the literature the most frequently used density-pressure relation is the relation proposed, by Dowson and Higginson (1966):

\[
\rho(p) = \rho_0 \frac{0.59 \text{ GPa} + 1.34p}{0.59 \text{ GPa} + p}, \tag{2.14}
\]

with: \( \rho_0 \) the density at ambient pressure. Dowson and Higginson (1966) performed their measurements up to 0.4 GPa, and more recent measurements (see e.g. Hamrock (1994)) proved that use of the fit of Dowson and Higginson must be restricted to pressures up to approximately 1 GPa The result shown in this thesis are based on calculations done with the fit of Dowson and Higginson. According to Hamrock (1994) the fit of Dowson and Higginson is only valid for mineral oils.

More recently Jacobson and Vinet (1987) presented a high-quality density-pressure relation, which is valid for pressures up to 3 GPa.

### 2.1.4 Results of (E)HL calculations

In the past many authors have contributed to solving the (E)HL-problem for line contacts. In general, all equations and parameters are made dimensionless in order to reduce the number of parameters for the calculations. In the literature two sets
of dimensionless parameters are used. The first set has four parameters (Dowson and Higginson, 1966):

\[
\begin{align*}
\bar{h} &= \frac{h}{R} & W &= \frac{F_N}{BE'R} \\
U_\Sigma &= \frac{\eta_0 v^+}{E'R} & G &= \alpha E'
\end{align*}
\]  
(2.15)

with:

\begin{align*}
\bar{h} & : \text{dimensionless film thickness} \\
W & : \text{dimensionless load number} \\
U_\Sigma & : \text{dimensionless velocity number} \\
G & : \text{dimensionless lubricant number}
\end{align*}

The second set has only three parameters and follows from the first set. The three parameters are called the Delft- or Moes dimensionless numbers (Moes, 1992):

\[
H = \bar{h} U_\Sigma^{-\frac{1}{4}} \quad M = W U_\Sigma^{-\frac{1}{4}} \quad L = G U_\Sigma^{\frac{1}{4}}
\]  
(2.16)

with:

\begin{align*}
H & : \text{dimensionless film thickness} \\
M & : \text{dimensionless load number} \\
L & : \text{dimensionless lubricant number}
\end{align*}

Using these dimensionless numbers the film thickness and pressure distributions can be calculated. In Fig. 2.1 an example of the results of such a calculation is given \((M = 50, L = 15)\). The values on the X and Y axis are in dimensionless parameters, the film thickness according to eq. (2.16). The pressure is made dimensionless using the maximum Hertzian pressure \(p_h\). In Appendix B the Hertzian theory is discussed and the maximum Hertzian pressure is given in that appendix as well. The same applies to the half Hertzian width \(b\), which is used to make the spatial coordinate \(x\) dimensionless.

The shape of the pressure distribution resembles the Hertzian pressure distribution (given in Appendix B as well). A conspicuous feature of a highly loaded line contact is the pressure spike or “Petrusevich spike” (after Petrusevich (1951)), the local maximum close to the outlet of the contact zone. Near the pressure spike the film contracts and the minimum film thickness of the contact is reached. In Fig. 2.1 the central film thickness and the minimum film thickness are plotted for this specific case. It is clear from this figure that for high loads the central film thickness is a better parameter to specify the separation between the surfaces than the minimum film thickness. In the mixed lubrication model, to be discussed in Chapter 4, the
central film thickness is also the preferred parameter and therefore results of the central film thickness will be given here.

To make a fit of the central film thickness use can be made of the asymptotes available in (E)HL. Moes (1997) proposed a function fit for the central film thickness in a line contact. The asymptotes for the rigid situations (isoviscous and piezoviscous) are based on the film thickness at the position of the maximum pressure. This location is also approximately the location of the centre of the pressure distribution. Moes used this definition for the central film thickness instead of using the real centre of the contact to be able to make smooth curve fits for the central film thickness.

Four regimes (or asymptotes) can be distinguished, i.e. the Rigid-Isoviscous (RI), the Rigid-Piezoviscous (RP), the Elasto-Isoviscous (EI) and the Elasto-Piezoviscous (EP) regimes. The asymptotes for the central film thickness are defined by:

\[
\begin{align*}
H_{RI} & = 3 M^{-1} & \text{RI-asymptote} \\
H_{RP} & = 1.287 L^{2} & \text{RP-asymptote} \\
H_{EI} & = 2.621 M^{-\frac{1}{6}} & \text{EI-asymptote} \\
H_{EP} & = 1.311 M^{-\frac{1}{6}} L^{\frac{3}{2}} & \text{EP-asymptote}
\end{align*}
\]
Using the asymptotes from eqns. (2.17) to (2.20), the function fit for the central film thickness is (Moes, 1997):

\[
H_{cm} = \left[ \left( H_{\text{RI}}^{3^2} + H_{\text{EI}}^{3^2} \right)^{3^s} + \left( H_{\text{RP}}^{3^2} + H_{\text{EP}}^{3^2} \right)^{-3^s} \right]^{s^{-1}},
\]

with \( s \) as auxiliary variable defined as:

\[
s = \frac{1}{5} \left( 7 + 8 e^{-2 \frac{M}{M_{\text{RI}}}} \right). \tag{2.22}
\]

Moes (1997) also made a plot of this function (Fig. 2.2), the ’Moes-diagram’. In this figure the different asymptotes have been marked.

Koets (1962) has shown, that the solution for the Elasto-Pie佐viscous regime is valid for the following conditions only:

\[
L \sqrt{M} > 13.3 \quad \text{and} \quad M > 0.1L^{\frac{2}{3}}. \tag{2.23}
\]

These conditions are also plotted in Fig. 2.2. Often, the Elasto-Pie佐viscous regime is called the Ertel-Grubin regime, after Ertel (Ertel, 1939) and Grubin (Grubin and Vinogradova, 1949). Most concentrated (point- or line) contacts do operate in the EP-regime.
2.1.5 Friction in EHL

Friction in (E)HL can be divided into three components, i.e. rolling-or-squeezing friction, sliding friction and geometric friction. Rolling-or-squeezing friction is caused by the shear stress of the lubricant that has to be squeezed into the contact. Rolling friction is therefore built up in the inlet zone of the contact. Mathematically, the rolling friction force \( F_{f,\text{rolling}} \) is described by:

\[
F_{f,\text{rolling}} = B \int_{-\infty}^{\infty} \frac{1}{2} h \frac{\partial p}{\partial x} \, dx. \tag{2.24}
\]

If the pressure distribution is symmetric, the rolling friction force is zero.

Sliding friction is caused by the shearing of the lubricant in the contact zone, due to the relative motion of the surfaces. The sliding friction \( F_{f,\text{sliding}} \) is written as:

\[
F_{f,\text{sliding}} = B \int_{-\infty}^{\infty} \eta \frac{\nu_{\text{slip}}}{h} \, dx. \tag{2.25}
\]

The third component, which is often omitted, is the geometric friction \( F_{f,\text{geometric}} \) (Moes, 1997; Patir and Cheng, 1979):

\[
F_{f,\text{geometric}} = B \int_{-\infty}^{\infty} p \frac{\partial h}{\partial x} \, dx. \tag{2.26}
\]

This component is due to the normal stresses, \( p \), acting on an inclined surface. It is of importance when surface textures are taken into account.

Since the shear of the lubricant increases exponentially with pressure, the rolling pressure is only a very small portion of the total friction. For highly loaded, smooth surfaces only sliding friction is of importance. In eq. (2.25) Newtonian behaviour of the fluid is assumed (the viscosity-pressure dependence can be included in this equation). Generally, the hydrodynamic shear stress can be written as a function of the shear rate \( \dot{\gamma} \):

\[
\tau_H = f(\dot{\gamma}). \tag{2.27}
\]

So, friction in an (E)HL contact is in general defined by:

\[
F_f = \iint_{A_H} \tau_H(\dot{\gamma}) \, dA_H, \tag{2.28}
\]
in which $A_H$ denotes the hydrodynamic area and $\dot{\gamma}$ is defined by:

$$\dot{\gamma} = \frac{v_{\text{avg}}}{h}. \quad (2.29)$$

The different relations between the hydrodynamic shear stress and the shear rate are given in Fig. 2.3 (Evans, 1983). These friction curves are obtained when the sum velocity is kept constant and the sliding velocity is varied. A particular lubricant may change its $\tau_H/\dot{\gamma}$ behaviour under the influence of pressure.

In all four curves of Fig. 2.3 three regions can be distinguished. At low sliding velocities the shear stress increases linearly with the shear rate. This behaviour is isothermal. In the second region the shear stress deviates from the linear curve; it increases less rapidly, until a maximum is reached. With increasing sliding velocity thermal effects become more important. When the maximum is reached the shear stress decreases in the third region, due to shear heating.

In Fig. 2.3 the line $D=1$ is plotted. The Deborah number $D$ is defined as:

$$D = \frac{\eta v_{\text{avg}}}{G b}, \quad (2.30)$$

with $v_{\text{avg}}$ the average velocity ($= \frac{1}{2} v^+)$, $b$ the half Hertzian width of the contact and $G$ the shear modulus, not to be confused with $G$, the lubricant number. The
Deborah number is the ratio of the relaxation time of the lubricant \( (\eta/G) \) and the characteristic process velocity of the lubricant \( (b/v_{av}) \). For Deborah numbers much smaller than unity \( (D \ll 1) \) the lubricant behaves viscous and for \( D \gg 1 \) elastic behaviour is dominant.

The four curves in Fig. 2.3 represent different rheological behaviour. Curve I is applicable for linear viscous cases (Newtonian):

\[
\tau_H = \eta \dot{\gamma}.
\]  

(2.31)

Curve II is the non-linear viscous relation. Now, the fluid model of Eyring as proposed by Bell, Kannel and Allen (1964), is applicable:

\[
\dot{\gamma} = \frac{\tau_0}{\eta} \sinh\left(\frac{\tau_H}{\tau_0}\right)
\]  

(2.32)

The Eyring shear stress, \( \tau_0 \), is shown in Fig. 2.3. It is a threshold parameter, above which a transition from Newtonian to non-Newtonian behaviour occurs.

Curve III, the elastic - non-linear viscous curve, was described by Johnson and Tevaarwerk (1977):

\[
\dot{\gamma} = \dot{\gamma}_e + \dot{\gamma}_v = \frac{\dot{\tau}_H}{G} + \frac{\tau_0}{\eta} \sinh\left(\frac{\tau_H}{\tau_0}\right).
\]  

(2.33)

This equation is, in fact, a non-linear Maxwell description of the lubricant. The stress rate \( \dot{\tau}_H \) can be eliminated from this equation by writing: \( \dot{\tau}_H = d\tau_H/dt = (v_{av}/b) \cdot d\tau_H/d(x/b) \). Thus:

\[
\eta \dot{\gamma} = D \frac{d\tau_H}{d(x/b)} + \tau_0 \sinh\left(\frac{\tau_H}{\tau_0}\right),
\]  

(2.34)

with the Deborah number \( D \) defined as in eq. (2.30). Equations (2.33) or (2.34) describe the elastic non linear behaviour of the lubricant as well as the Newtonian behaviour. The last behaviour is described in case \( D \ll 1 \) and \( \tau_H \ll \tau_0 \), since in that case \( \sinh(\tau_H/\tau_0) \approx \tau_H/\tau_0 \) and eq. (2.34) then reduces to \( \eta \dot{\gamma} = \tau_H \).

In Fig. 2.3 a fourth curve (IV) has been drawn. This curve describes the elastic/plastic behaviour of lubricants. In the description of the other three curves the shear stress can increase without limitations. However, in reality, like solid materials, also lubricants have an ultimate strength. This behaviour is described by the so-called Prandtl-Reuss equations (Johnson and Cameron, 1967; Evans and Johnson, 1986):

\[
\tau_H < \tau_1 \rightarrow \dot{\gamma} = \frac{\dot{\tau}_H}{G}
\]

\[
\tau_H = \tau_1 \rightarrow \dot{\gamma} = \frac{\dot{\tau}_H}{G} + \frac{\tau_1 \dot{\gamma}}{\tau_H},
\]  

(2.35)
where $\tau_l$ is the limiting shear stress (see Fig. 2.3). The limiting shear stress was already proposed by Smith (1959).

The elastic/plastic behaviour is also described by other authors. Based on measurements (Bair and Winer, 1979b), Bair and Winer (1979a) proposed a model based on the limiting shear stress ($\tau_l$):

$$\dot{\gamma} = \frac{\tau}{G} - \frac{\tau_l}{\eta} \ln \left( 1 - \frac{\tau}{\tau_l} \right). \quad (2.36)$$

Gecim and Winer (1980) introduced a modified Bair and Winer model, based on a study on the effect of lubricant limiting shear stress on the nominal film thickness:

$$\dot{\gamma} = \frac{\tau_l}{\eta} \arctanh \left( \frac{\tau}{\tau_l} \right). \quad (2.37)$$

In principle, the friction in an (E)HL contact is determined by four parameters, $\eta$, $\tau_0$, $G$ and $\tau_l$. Of the four parameters, $G$ is the most difficult to measure and often the approximation $G = 30\tau_l$ is used.

## 2.2 Boundary Lubrication

Amontons (1699) discovered that over a wide range of normal forces the coefficient of friction is independent of the normal force. In fact, according to Dowson (1979), Amontons studied “the frictional characteristics of greased surfaces, which would now be described as Boundary Lubrication”. Although Boundary Lubrication is one of the first fields of tribology, which received attention of researchers, the theory describing Boundary Lubrication is not as developed as the theory for (E)HL.

One of the characteristics of BL is that the coefficient of friction is independent of the applied normal force over a wide range of contact pressures, as described by Amontons. It is, however, still not possible to predict the coefficient of friction in the BL-regime. This is caused by the fact that the knowledge of what happens at the boundaries of the surfaces is still inadequate. The chemical composition of the boundary layers and the physical properties of these layers under contact conditions (high pressure, high temperature, high shear rates) are still subject of research and will be for a long time. There are a lot of combinations of surface materials, with or without a coating, different lubricants, with different additives, which remain to be studied. Therefore, friction in the BL-regime is mainly an experimental study.

A very fine overview of the role of chemical reactions in the lubrication of concentrated contacts is given by e.g. Godfrey (1968) and Sakurai (1981). Hardy and Doubleday (1922) were ones of the first to give a definition of BL: the condition
where the “friction depends not only on the lubricant, but also on the chemical nature of the solid boundaries.” Three fundamental mechanisms of interaction between lubricant and solid are generally distinguished (Godfrey, 1968):

- physical adsorption: the relative weak binding of lubricant molecules to the solid by Van der Waal’s forces.

- chemical adsorption: lubricant molecules are held to the surface by chemical bonds. Unlike in physical adsorption, the formation of bonds is irreversible and the bonds are stronger.

- chemical reaction: an exchange of valence electrons between lubricant and solid surfaces occurs. A new chemical compound is formed. The reaction films are more stable than physically or chemically adsorbed films.

In practice a combination of these mechanisms will occur. For a more detailed description of the influence of the different operation conditions in Boundary Lubrication the reader is referred to Godfrey (1968).

### 2.2.1 Friction in BL

From Briscoe and co-workers a number of articles have appeared on the subject of shear stress in Boundary Lubrication. In these articles, e.g. Briscoe (1981) and Briscoe, Scruton and Willis (1973), the results of measurements in the BL regime are presented. It is proven that the shear stress is not entirely proportional to the pressure. And, thus, the coefficient of friction in the BL regime is not entirely constant. The shear stress and the coefficient of friction are related by definition through:

\[
f \equiv \frac{F_f}{F_N} = \frac{\tau}{p}
\]  

(2.38)

Georges, Mazuyer, Loubet and Tonck (1992), who studied surface films of colloidal carbonate, found three distinct regions in the curve which shows shear stress \( \tau \) as a function of mean contact pressure curve \( p_{\text{av}} \) (Fig. 2.4). For all three regions the relation \( \tau \propto p^c \) is valid, with \( c \) a constant which differs for the three regions. For the region at higher pressures the shear stress is proportional to the pressure, i.e. \( c = 1 \). In most practical cases, this region is the relevant region for BL.

### 2.3 Mixed Lubrication

The mixed lubrication regime can be seen as the intermediate regime between regimes (E)HL and BL. ML therefore has properties of both regimes. The coefficient of friction of ML is intermediate between the coefficients of friction of (E)HL and BL. The wear rate is also between those of the other two regimes.
In Section 1.2 it has already been pointed out that on the horizontal axis of the generalized Stribeck curve many different lubrication numbers are used. Most work on mixed lubrication has been done experimentally. The work of Stribeck (1902), Hersey (1915), Biel (1920), McKee (1927), Altrogge (1950), Lenning (1960) and Schipper (1988) has already been mentioned in Section 1.2.

The theoretical work that has received most attention in the literature is the average flow model of Patir and Cheng (1978, 1979). Patir and Cheng introduced the average Reynolds equation. Using flow factors, they were able to analyze the effects of roughness on the mean hydrodynamic load, the mean viscous friction and the mean bearing inflow. The analysis of Patir and Cheng, however, is only valid for separations, larger than, approximately, three times the combined root mean square surface roughness \( R_g \).

An article, that has received less attention, is that of Johnson, Greenwood and Poon (1972). In Chapter 4 their model will be explained and it will be extended to calculate the Stribeck curve.
2.4 Definition of transitions and lubrication number

A lot of different lubrication numbers are in use. The aim of the different lubrication numbers is to characterize the transitions from (E)HL to ML and from ML to BL by one single number. The lubrication number $L$ used in this thesis is introduced by Schipper and defined as:

$$L = \frac{\eta_0 v^+}{P_{av} R_a},$$

with:

- $\eta_0$: viscosity at ambient pressure
- $v^+$: effective sum velocity
- $P_{av}$: average (Hertzian) pressure of the contact
- $R_a$: CLA surface roughness

Based on measurements, performed with lubricated concentrated contacts the values $L$ for both transitions were obtained, based on a large number of measurements, with lubricated concentrated contacts, performed on a pin-on-disc machine, a reciprocating tribometer and a two-disk machine. Different oils were used, both mineral and synthetic, with different viscosities and additive packages. Also, the surface roughness was varied and different materials were used. The average pressure was in the range of 100 MPa to 3 GPa.

There are a few ways to define the transitions between the different lubrication regimes. Schipper (1988) defines them by drawing straight lines through points, measured in the different lubrication regimes (Fig. 2.5). The transitions are then defined as the intersection of these lines representing friction in the different lubrication regimes.

A different definition of the transition between ML and BL is e.g. the one-percent criterion. In this definition the BL/ML transition is defined by the point where the coefficient of friction is 99% of the coefficient in the BL regime. This definition can work when the generalized Strubeck curve is very smooth. However, when a generalized Strubeck curve is measured, the error in the measurements will be in the order of, or larger than, one percent.

An alternative criterion for the (E)HL to ML transition of a rough surface is the comparison of the generalized Strubeck curve for the rough surface with that for a smooth surface. At the point where the two curves start to differ (starting from the (E)HL regime) the transition is defined. This criterion is not very practical either, since generalized Strubeck curves for the rough and the smooth surface must both be available.
In this thesis the criterion based on the intersections will be used, since it is the most practical one. The transition between two regimes can thus be characterized with one single value, but it must be kept in mind that a transition is never as abrupt as a single value implies. In reality, a transition is a smooth process (see Fig. 2.5).

2.5 Summary

In the literature, most theoretical work has been done on (E)HL. In the BL-regime experiment has been most important. This is also valid for the ML-regime. Only Patir and Cheng have tried to develop a model for the mixed lubrication regime. However, this model is only valid for separations, which are about three times the surface roughness. At that point the first asperities begin to touch and thus the mixed lubrication regime is about to start. The ML-regime is therefore hardly covered by the model of Patir and Cheng (1978). Since the model of Johnson et al. (1972) covers the ML regime better, their model will be extended in Chapter 4 in order to calculate the Strubeck curve.

The surface roughness plays an important role in the ML regime. It is therefore very important to get a good overview of the deformation of a rough surface. In
the literature some work has already been done in this area. In the next chapter an overview of this work is given, together with an extension to make it more useful for the mixed lubrication model.
Chapter 3

Deformation of a rough line contact

3.1 Introduction

In Subsection 2.1.1 it has already been pointed out that a surface will deform when a pressure is applied. This is not only true when a fluid pressure is applied, but also when the pressure is caused by asperity contact. For the mixed lubrication model it is important to know the deformation of the line contact under asperity pressure. In this chapter the model for describing the deformation of a rough line contact will be given.

Only elastic deformation of the surface and the asperities will be considered, which is reasonable since Kagami, Yamada and Hatazawa (1983) have shown that, for point contacts, the effect of the mode of deformation is small.

In this chapter firstly the contact model of Greenwood and Williamson (1966) for un lubricated contacts will be discussed. Then, the deformation model for un lubricated contacts will be dealt with. In Section 3.4 a similarity analysis will be applied to the deformation model. Section 3.5 shows an example of the calculations, and Section 3.6 the results. In Section 3.7 comparison will be made with the work of Lo (1969). Greenwood, Johnson and Matsubara (1984) propose a criterion for establishing if the deformation model is to be used or whether the Hertzian theory is adequate enough. Greenwood et al. (1984) studied point contacts. In Section 3.7 a similar criterion for line contacts is proposed. In Section 3.8 the conclusions concerning this topic are given.

3.2 Contact Model

For the un lubricated contact situation three main contact models are in use. The most frequently cited and the most simple one is that of Greenwood and Williamson (1966). The models of Whitehouse and Archard (1970) and Nayak (1971) are a bit more complex. The model which will be used in this thesis is that of Greenwood
and Williamson (hereafter referred to as the G&W-model), not only because its simplicity, but also because it has been shown to be quite accurate. G&W developed a model for the contact of real surfaces that is based on the stochastic or random nature of the roughness of a surface. Their model was developed for the contact of two flat surfaces of which one surface is rough and the other one is smooth. Greenwood and Tripp (1970-71) extended this model to the case of two rough surfaces.

The most important assumption in the G&W-model is that all summits have the same parabolic radius of curvature. Each summit is replaced by a parabola with radius $\beta$. The summit height are supposed to be randomly variable and the summits to be uniformly distributed over the rough surface with a density of asperities $n$. Appendix A deals with the definition of what a summit is and how to calculate $n, \beta$ and $\sigma_s$ from a measured surface. ($\sigma_s$ is the standard deviation of the heights distribution of the summits.)

In their calculations G&W assume a Gaussian or normal probability distribution of the summits. Many manufacturing processes result in surface height distributions that are indeed Gaussian. In that case, the probability $P$ that a randomly chosen summit is in contact with the opposing surface is:

$$P(s > h) = \int_h^\infty f(s) \, ds$$  \hspace{1cm} (3.1)

with $h$ the separation, i.e. the distance from the opposing (smooth) surface to the plane through the mean of the summit heights, and $f(s)$ the Gaussian distribution:

$$f(s) = \frac{1}{\sigma_s \sqrt{2\pi}} \exp\left(-\frac{s^2}{2\sigma_s^2}\right).$$  \hspace{1cm} (3.2)

In many calculations the integral identity $F_j$ is used:

$$F_j(h) = \int_h^\infty (s - h)^j \phi(s) \, ds,$$

with $j$ a real number and $\phi(s)$ a distribution function. For the Gaussian heights distribution, the normalized function $\phi(s)$ is:

$$\phi(s) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right).$$  \hspace{1cm} (3.4)

In Figure 3.1 the summits of one surface are drawn, with the mean plane through the summits and the Gaussian distribution.

In the theory of G&W the summits deform independently, according to the Hertzian theory. This means, that the contact radius $a_1$, area $A_1$ and the load $F_{N,1}$,
FIGURE 3.1: Gaussian height distribution of the summits.

carried by a single summit are given by:

\[ a_1 = \beta^{1/2}w^{1/2}, \quad A_1 = \pi \beta w, \quad F_{N,1} = \frac{2}{3} E' \beta^{1/2} w^{3/2}, \]  

with \( w \) the compliance of the summit, i.e. the amount the summit deforms. The modulus of elasticity \( E' \) is already defined in eq. (2.7). It should be noted that G&W use a different definition of \( E' \), with the left-hand-side of eq. (2.7) replaced by \( 1/E' \) and the factor \( 2/3 \) on the right-hand-side of the third equation of eq. (3.5) replaced by \( 4/3 \). The notation of G&W is customary in the field of contact mechanics (Johnson, 1985). In the field of (E)HL, however, the present notation is conventional.

Using eq. (3.5), G&W derived expressions for the number of summits in contact \((N)\):

\[ N = n A_{\text{nom}} \int_{h}^{\infty} f(s) \, ds = n A_{\text{nom}} F_0 \left( \frac{h}{\sigma_s} \right), \]  

the total area of real contact \((A_r)\):

\[ A_r = \pi n \beta A_{\text{nom}} \int_{h}^{\infty} (s-h) f(s) \, ds = \pi n \beta \sigma_s A_{\text{nom}} F_1 \left( \frac{h}{\sigma_s} \right), \]  

and the applied normal load \((F_N)\):

\[ F_N = \frac{2}{3} n \sqrt{\beta E'} A_{\text{nom}} \int_{h}^{\infty} (s-h)^{3/2} f(s) \, ds = \frac{2}{3} n \beta \sigma_s \sqrt{\frac{\sigma_s}{\beta}} E' A_{\text{nom}} F_\tau \left( \frac{h}{\sigma_s} \right), \]  

with \( A_{\text{nom}} \) the nominal contact area.
For an un lubricated contact eq. (3.8) makes it possible to calculate the separation
$h$ as a function of $F_N$. In mixed lubrication eq. (3.8) can be used to calculate the
load $F_C$, carried by the asperities in contact with the opposing surface, as a function
of separation $h$ (see Chapter 4). The model of G&W is based on a statistical
distribution of the summits. One of the assumptions is that the summits do not
interact upon deflection. Therefore the separations, i.e. the distance between the
mean height through the summits and the opposing surface, may not be too small.

3.3 Deformation Model

In 1881 Hertz formulated his theory on the deformation of elastic bodies. His the-
eory is based on smooth surfaces. A summary of the Hertzian theory is given in
Appendix B. The Hertzian theory does not hold, however, for rough surfaces. The
higher asperities, outside the zone which would deform according to Hertz, will touch
the opposing surface. Those asperities will carry a part of the applied load. The
Hertzian theory is based on continuous contact, while in a rough surface the force
is transmitted through an array of discrete contact spots. Further away from the
centre of the contact the surfaces widen and there will be fewer asperities which will
bridge the gap. The total effect of all individual forces, transmitted through the
summits, can be treated statistically as has been done in the theory of Greenwood
and Williamson (1966). This theory was extended to the contact of a rough sphere
and a smooth surface by Greenwood and Tripp (1967). They also calculated the
elastic deformation of two rough spheres in contact.

Lo (1969) presented an article on the deformation of rough cylinders. The work
of Lo is based on the theory of G&W as well. However, Lo used an approximation
for the pressure distribution, instead of employing the real pressure distribution as
found with the model of G&W. Especially at the edges of the pressure distributions
large differences with the pressures are found when actually applying the model of
G&W. The approximated pressure distribution of Lo also gives problems for high
loads, since it can not deal with pressure distributions close to the Hertzian pressure
distribution. In this chapter no assumptions are made regarding the pressure distri-
bution. In Section 3.7 comparison between the current calculations and the results
of Lo will be made.

By adding plots and function fits of the maximum (central) pressure, the real
area of contact, the number of contacts and the effective half width of the contact,
it is easier to calculate the parameters which affect the deformation of rough line
contacts.

As the G&W theory applies to contact between flat surfaces, $A_{nom}$ in eqns. (3.6),
(3.7) and (3.8) is simply equal to length times width. In a line contact situation
$A_{nom}$ is a function of the degree of deformation. The inherent difficulty can be
3.4. Similarity Analysis

circumvented by assuming that the G&W theory also applies to line contacts. Then, the pressure \( p(x) \) in rough line contact can be found from:

\[
p(x) = \frac{F_N}{A_{\text{nom}}} = \frac{2}{3} \eta \beta \sigma_s \sqrt{\frac{\sigma_s}{\beta}} E' F_s \left( \frac{h(x)}{\sigma_s} \right).
\]  

(3.9)

Since pressure is a potential it makes no difference for the deformation of the surfaces whether the surfaces are deformed by fluid pressure or by asperity pressure. The relation between separation of the surfaces and pressure applied to the surfaces is therefore also given by the film thickness-pressure relation in (E)HL (eq. (2.6) and eq. (2.8)). Also the force balance equation, eq. (2.5), is needed to solve the contact problem.

The separation \( h(x) \) can now be calculated as function of the pressure distribution applied (eq. (2.6) and eq. (2.8)) and the pressure distribution can be calculated as a function of the separation (Eq. (3.9)). This problem has to be solved iteratively. This is done using multigrid techniques (see e.g. Venner, 1991). Using multigrid one can use many gridpoints and thus obtain good accuracy, while keeping fast convergence. In order to achieve convergence, underrelaxation has been applied to \( p \).

### 3.4 Similarity Analysis

To reduce the number of parameters all parameters are scaled with respect to the Hertzian parameters (Appendix B): the Hertzian half contact width \( b \) and the Hertzian maximum pressure \( p_h \) for unlubricated and smooth surfaces. Use of the Hertzian parameters for scaling makes comparison with the Hertzian theory easy. The following dimensionless numbers are used:

\[
\bar{p} = \frac{p}{p_h}
\]

(3.10)

\[
X = \frac{x}{b}
\]

(3.11)

\[
H_h = \frac{hR}{b^2} = \frac{\pi hBE'}{8 \frac{F_N}{b^2}}
\]

(3.12)

\[
\bar{\sigma}_s = \frac{\sigma_s R}{b^2} = \frac{\pi \sigma_s BE'}{8 \frac{F_N}{b^2}}
\]

(3.13)

\[
\bar{n} = 4nb^2 \sqrt{\beta R} = \frac{32 nF_N}{\pi BE' \sqrt{\beta R}}
\]

(3.14)

The dimensionless number for the separation \( H_h \) has received the subscript \( h \) to distinguish it from \( H \) (eq. (2.16)), since \( H_h \) is made dimensionless using Hertzian parameters. Using this set of numbers, all numbers are in the order of magnitude of
unity. Two new dimensionless numbers are introduced: the dimensionless standard deviation \( \bar{\sigma}_s \) and the dimensionless density of asperities \( \bar{n} \). Notice that \( \bar{n} \) is proportional to the load \( F_N \), whereas \( \bar{\sigma}_s \) is proportional to \( F_N^{-1} \). The product of \( \bar{n} \) and \( \bar{\sigma}_s \) is independent of \( b \) and of the applied load \( F_N \):

\[
\bar{n} \cdot \bar{\sigma}_s = 4n\sigma_s \sqrt{\beta R}.
\]  

(3.15)

In Johnson et al. (1972) it is shown that \( n_0 \beta \sigma_s \) is approximately constant, with a value ranging between 0.03 and 0.05, so, since \( R \gg \beta \), the product \( \bar{n} \cdot \bar{\sigma}_s \) will be greater than one.

Equation (3.9) can now be expressed in dimensionless numbers:

\[
\bar{p} = \frac{2}{3} \bar{n} \bar{\sigma}_s \sqrt{\bar{\sigma}_s} F_3 \left( \frac{H_h}{\bar{\sigma}_s} \right)
\]  

(3.16)

The equations for the shape of the surface, eq. (2.6) and eq. (2.8), are combined and also made dimensionless:

\[
H_h(X) = H_0 + \frac{X^2}{2} - \frac{1}{\pi} \int_{-\infty}^{\infty} \bar{p}(S) \ln(|X - S|) dS,
\]

(3.17)

with \( H_0 \) a dimensionless constant.

The equilibrium of forces in dimensionless numbers is, eq. (2.5):

\[
\frac{\pi}{2} = \int_{-\infty}^{\infty} \bar{p}(X) dX.
\]

(3.18)

The problem can now be solved using the given dimensionless numbers and equations. The calculations of the deformation of a line contact can be split up in two parts. In the next section an example of the calculations will be shown and in Section 3.6 the results of the final calculations will be presented.

### 3.5 Example

To show how the surface roughness influences the pressure distribution and deformation of a surface an example of a calculated pressure distribution and accompanying separation is given. For that purpose, a reference case has been defined in Table 3.1.

Results, shown in Fig. 3.2 have been obtained for this reference case, varying the load. Thus, the product \( \bar{n} \cdot \bar{\sigma}_s \) has been kept constant. In Fig. 3.2 only one half of the pressure distribution is shown, because of the symmetry in the problem. In this section negative separations have been allowed, in spite of the fact that large
negative separations have no physical meaning. The curves are plotted to show that, from a mathematical point of view, calculation of these cases does not present a problem.

From Fig. 3.2 it can be concluded that for high values of $F_N$ the shape of the pressure distribution is fairly similar to that of the Hertzian pressure distribution.

![Image of pressure distribution graph](image)

**Figure 3.2**: Pressure distribution for $\tilde{n} \cdot \tilde{\sigma}_s = 2$ and different loads. Parameters from Table 3.1.
The pressure distribution for loads larger than \( F_N = 500 \text{kN} \) practically coincides with the Hertzian distribution. When the plot is enlarged, a small difference can be seen for instance at \( X = 1 \) (the Hertzian width). This is not a numerical error, but physical, because a small part of the load is carried by the asperities outside the Hertzian contact area for smooth surfaces. For lower loads the pressure distribution widens and the maximum pressure decreases.

In Fig. 3.3 the separations, corresponding to the pressure distribution of Fig. 3.2, have been plotted. It can be observed that for high loads the separations in the centre become highly and unacceptably negative.

**FIGURE 3.3:** Separations over surface roughness \( (H/R)/\bar{\sigma}_s \) for \( \bar{h} \cdot \bar{\sigma}_s = 2 \) and different loads. Parameters from Table 3.1.

When the G&W model is used, it is important to know its restrictions. One of the most important features of the model is that it is based on a statistical approach. Therefore within the nominal contact area, a very large number of asperity contacts are supposed to occur. Accordingly, the model deals with the part of the roughness spectrum, in which the wavelengths are small compared to the contact width.

Another assumption, made in the G&W model, is that the asperities deform independently of each other. The asperities may therefore not merge. The criterion that has been used for this problem is that the separations may not become too negative. In this section the calculations are not stopped at zero separation to show that they can be performed for large negative separations. However, physical interpretation does not allow for large interpenetration of the rough surfaces. In the
next section the calculations regarding the real area of contact and the number of summits in contact will therefore be stopped at separations equal to zero.

3.6 Results

Using the deformation model, a number of parameters has been calculated, i.e. the pressure in the centre of the contact, \( \bar{p}_c \) and the corresponding separation \( H_{h,c} \) (Section 3.6.1), the effective half width \( b^* \) (Section 3.6.2), the real contact area \( A_r \) (Section 3.6.3) and the real number of contacts \( N \) (Section 3.6.4). From the calculations it can be concluded, that the central separation is very good parameter to describe the deformation of a rough line contact.

In order to calculate \( b^* \) a definition has been formulated for the half width. All parameters have been plotted and, except for the central separation, fitted as well.

3.6.1 Pressure and separation

For wide ranges of \( \bar{n} \) and \( \bar{\sigma}_s \) the maximum pressure (\( = \) central pressure) and the corresponding separation have been calculated. The central pressure, \( \bar{p}_c \), has been plotted against \( \bar{n} \) in Fig. 3.4. The corresponding separations have been plotted in Fig. 3.5.

![FIGURE 3.4: Central pressure as a function of \( \bar{n} \) for different values of \( \bar{n} \cdot \bar{\sigma}_s \).](image-url)
From Fig. 3.4 it can be concluded that for all values of \( \bar{n} \cdot \bar{\sigma}_s \) the dimensionless central pressure approaches one \((\bar{p}_c = 1)\) for high values of \( \bar{n} \). This means that for high loads (large \( \bar{n} \)) the pressure distribution becomes equal to the Hertzian pressure distribution. For low values of \( \bar{n} \) the slopes of all curves are parallel. Therefore, an accurate curve fit can be made:

\[
\bar{p}_c = \left( 1 + (a_1 \cdot \bar{n}^a \cdot \bar{\sigma}_s^a)^{a_4} \right)^{(1/a_4)},
\]

(3.19)

with \( a_i \) as fit parameters.

\[
a_1 = 0.953, \\
a_2 = 0.0337, \\
a_3 = -0.442, \text{ and} \\
a_4 = -1.70.
\]

This fit is accurate up to 4%. Since \( a_2 \) is very small, the central pressure is nearly independent of \( \bar{n} \). Rewriting eq. (3.19) into dimensional parameters, it appears that for high values of \( \bar{\sigma}_s \) (which implies low loads) the central pressure increases almost proportional to the total applied load, i.e. \( p_c \propto F_N^{0.976} \). This is caused by the fact
that the load is not only carried in the Hertzian contact width but by the asperities outside the Hertzian contact width as well.

In the following paragraphs the contact width, the real contact area and the number of summits in contact are calculated, plotted and fitted. The plots of the real contact area and the number of summits in contact are restricted to separations larger than zero.

3.6.2 Contact width

Besides the maximum pressure in the contact, the width of the pressure distribution is also a measure of the influence of the surface roughness on the contact. Thus an effective dimensionless half width, $\tilde{b}^*$, may be defined similar to Greenwood and Tripp (1967):

\[
\tilde{b}^* = \frac{b^*}{\bar{b}} = \frac{3\pi}{4} \frac{\int_0^\infty X\bar{p}(X)\,dX}{\int_0^\infty \bar{p}(X)\,dX}.
\]  

To make easy comparison possible, the factor $3\pi/4$ has been implemented in eq. (3.20) in order to scale $\tilde{b}^*$ to unity in case of the Hertzian pressure distribution.

![Figure 3.6: Pressure distribution for the reference case with the effective dimensionless half width, $\tilde{b}^*$, Compared with Lo (1969).](image-url)
In fact the integral in the numerator needs to be calculated only, since the integral in the denominator is equal to \( \pi/4 \) (eq. (3.18)).

Figure 3.6 gives the calculated effective dimensionless half width for the reference case. In this figure the approximated pressure distribution according to Lo (1969) has been plotted as well. The central pressure has been taken equal to that in the present case. From this pressure distribution it can be concluded that the approximation by Lo results in lower pressures near the edge of the contact, as already stated in Section 3.3.

The effective dimensionless half width has been plotted as function of \( \bar{n} \) in Fig. 3.7. The results come up to expectations: \( \bar{b}^* \) is one for high values of \( \bar{n} \) and increases for lower values of \( \bar{n} \).

![Figure 3.7](image)

**FIGURE 3.7:** Effective dimensionless half width, \( \bar{b}^* \), as a function of \( \bar{n} \) for different values of \( \bar{n} \cdot \bar{\sigma}_s \).

The dimensionless effective half width has been fitted as well:

\[
\bar{b}^* = \left(1 + (a_1 \cdot \bar{n}^a \cdot \bar{\sigma}_s^a)^{1/a_4}\right)^{(1/a_4)}, \tag{3.21}
\]
with \( a_i \):

\[
\begin{align*}
a_1 &= 1.235, \\
a_2 &= -0.0345, \\
a_3 &= 0.439, \text{ and} \\
a_4 &= 1.71.
\end{align*}
\]

This fit is accurate up to 5%. The effective half width is almost independent of \( \bar{n} \) as well. For low values of \( \bar{n} \) the dimensionless effective half width \( \bar{b}^* \) is proportional to \( F_N^{-0.474} \). Therefore the dimensional effective half width is almost independent of the load \( (b^* = F_N^{0.026}) \) for low values of \( \bar{n} \).

### 3.6.3 Real contact area

According to Greenwood and Williamson (1966) in a flat surface on flat surface contact, the real total contact area \( A_r \) is given by:

\[
A_r = \pi n \beta \sigma_s A_{\text{nom}} \int_{\frac{h}{\sigma_s}}^{\infty} \left( s - \frac{h}{\sigma_s} \right) \phi(s) \, ds. \tag{3.22}
\]

Since in a line contact situation, there is no fixed value for the nominal contact area \( (h \) is a function of \( x \), in the present case the real total contact area has to be found by integration:

\[
A_r = \pi n \beta \sigma_s B \int_{-\infty}^{\infty} \int_{\frac{h(x)}{\sigma_s}}^{\infty} \left( s - \frac{h(x)}{\sigma_s} \right) \phi(s) \, ds \, dx. \tag{3.23}
\]

It is not possible to write eq. (3.23) in the dimensionless numbers of Section 3.4, therefore \( A_r \) in eq. (3.23) is made dimensionless by dividing \( A_r \) by the Hertzian half width \( b \) and the width of the contact \( B \) and by replacing \( h, \sigma_s \) and \( X \) by their dimensionless forms:

\[
\bar{A} = \frac{A_r}{bB} = \pi n \beta \sigma_s \int_{-\infty}^{\infty} \int_{\frac{H(X)}{\sigma_s}}^{\infty} \left( s - \frac{H(X)}{\sigma_s} \right) \phi(s) \, ds \, dX, \tag{3.24}
\]

with \( \bar{A} \) the dimensionless real total contact area. In eq. (3.24) the product \( n \beta \sigma_s \) appears. As stated above for many surfaces this product has a characteristic value of 0.03-0.05 (Johnson et al., 1972). In this study a value of 0.05 is taken (Table 3.1).

In Fig. 3.8 the dimensionless real contact area, as defined in eq. (3.24), has been plotted as a function of \( \bar{n} \). Calculations have been performed for positive values of
the separation only. For different values of $\eta \beta \sigma$ the result can simply be divided by 0.05 and multiplied by the new value.

The curves in Fig. 3.8 have been fitted according to:

$$\tilde{A} = \left( a_1 + (a_2 \cdot \tilde{\eta}^{a_3} \cdot \tilde{\sigma}^{a_4})^{a_5} \right)^{(1/a_5)}, \quad (3.25)$$

with $a_i$:

$$a_1 = 0.292, \quad a_2 = 0.477, \quad a_3 = -0.992, \quad a_4 = -1.470, \text{and} \quad a_5 = -0.463.$$  

This fit is accurate up to 4%.

From this fit it follows that the real area of contact is almost linearly proportional to the load ($A_r \propto F_N^{0.978}$).
3.6.4 Number of summits in contact

The number of summits in contact can be calculated by applying:

\[ N = n A_{\text{nom}} \int_{\frac{h}{x_0}}^{\infty} \phi(s) \, ds. \]  

(3.26)

Since in this case there is no fixed value for the nominal contact area and the separation depends on the spatial coordinate \( x \), eq. (3.26) has to be rewritten as:

\[ N = nB \int_{-\infty}^{\infty} \int_{\frac{h}{x}}^{\infty} \phi(s) \, ds \, dx. \]  

(3.27)

Writing this equation in the numbers of Section 3.4 this becomes:

\[ \tilde{N} = \frac{N}{B b n} = \int_{-\infty}^{\infty} \int_{\frac{h}{x_0}}^{\infty} \phi(s) \, ds \, dX. \]  

(3.28)

In this equation \( \tilde{N} \) has been normalized by dividing it by the length of the contact, \( B \), the Hertzian contact width, \( b \), and the density of the asperities, \( n \).

The number of asperities \( \tilde{N} \) has been plotted in Fig. 3.9. Again a curve has been drawn for the cases where the central separation is equal to zero. The number of asperities has been calculated for positive values of the separation only.

The curves in Fig. 3.9 have been fitted according to:

\[ \tilde{N} = \left( a_1 + (a_2 \cdot \bar{n}^{a_3} \cdot \tilde{\sigma}^{a_4})^{a_5} \right)^{(1/a_6)}, \]  

(3.29)

with \( a_i \):

\[ a_1 = 0.287, \]
\[ a_2 = 6.318, \]
\[ a_3 = -0.988, \]
\[ a_4 = -1.435, \text{ and} \]
\[ a_5 = -0.443. \]

This fit is accurate up to 10%.

From this fit it follows that for low values of \( \bar{n} \), the number of asperities in contact is almost linearly proportional to the load \( (N \propto F_N^{0.947}) \).
3.7 Discussion

Greenwood et al. (1984) defined a parameter $\alpha_s$ which indicates whether or not one can neglect the effect of roughness on Hertzian deformation. For contact between a sphere and a plane, the parameter $\alpha_s$ is defined as:

$$\alpha_s \equiv \frac{\sigma_s R_s}{a_0^2},$$

(3.30)

with $a_0$ the contact radius of two smooth surfaces and $R_s$ the radius of the sphere. Greenwood et al. (1984) claim that for $0 < \alpha_s < 0.05$ an error between the rough solution and the smooth solution is less than 7%. To compare these results with the results, shown in Fig. 3.7, an equivalent parameter $\alpha_l$ is defined for line contacts:

$$\alpha_l \equiv \frac{\sigma_l R}{b^2},$$

(3.31)

$\alpha_l$, thus defined, is equal to $\tilde{\sigma}_s$, the dimensionless roughness. In Fig. 3.10 the effective half width has been plotted as a function of $\tilde{n}$, together with the curve $\alpha_l = 0.05$. From this figure it can be concluded that for line contacts a similar criterion can be applied as for point contacts. For $\alpha_l = 0.05$ the errors are of magnitude 3% for $\tilde{n} \cdot \tilde{\sigma}_s = 100$ to 11% for $\tilde{n} \cdot \tilde{\sigma}_s = 1$. Below $\alpha_l = 0.05$ the Hertzian theory can be used
safely and above this criterion the present theory is recommended.

Based on the calculations of Section 3.6 a comparison can be made with the results of Lo (1969). The most important assumption Lo made was that the pressure distribution can be expressed as:

\[ p^*(x) = p_c^* \left( 1 - x^2/C^* \right)^2, \quad 0 \leq |x| \leq C^* \]
\[ p^*(x) = 0, \quad |x| > C^*, \quad (3.32) \]

with \( C^* \) the dimensionless half width of the surface of the contact. This pressure distribution can not represent a pressure distribution which approaches the Hertzian pressure distribution for a line contact. The Hertzian pressure distribution reads \( p = p_c \left( 1 - x^2 \right)^{1/2} \), while Lo’s equation reads \( p = p_c \left( 1 - x^2 \right)^2 \). Also for low loads it appears that the results of Lo are not very accurate (see Fig. 3.6). This is due to the definition of the pressure distribution by Lo, eq. (3.32), which states that the asperity contacts for \( |x| > C^* \) are set to zero (\( p^* = 0 \)).

Lo gives relationships between the load \( F_N \) and the parameters \( p_c, C^*, A_r \) and \( N \). In Table 3.2 comparison with the current results is made. The differences found are due to the differences in pressure distribution.
TABLE 3.2: Comparison of Lo’s results with current results.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>[ \begin{array}{c} \text{Lo} \ \text{m-value} \end{array} ]</th>
<th>Present [ \begin{array}{c} \text{m-value} \ \text{m-value} \end{array} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ p_c ]</td>
<td>[ \begin{array}{c} 0.95 \ 0.055 \end{array} ]</td>
<td>[ \begin{array}{c} 0.976 \ 0.026 \end{array} ]</td>
</tr>
<tr>
<td>[ b^* \quad (C^*) ]</td>
<td>[ \begin{array}{c} 0.72 \ 0.3 \ 0.916 \end{array} ]</td>
<td>[ \begin{array}{c} 0.5 \ 0.5 \ 0.978 \end{array} ]</td>
</tr>
<tr>
<td>[ N ]</td>
<td>[ \begin{array}{c} 0.87 \ 0.68 \end{array} ]</td>
<td>[ \begin{array}{c} 0.947 \end{array} ]</td>
</tr>
</tbody>
</table>

Parameter \( (p_c, b^*, A_r, N) \) \( \propto F_N^m \)

3.8 Conclusions

The surface roughness has a clear influence on the deformation of a line contact. This is demonstrated most clearly by the maximum central pressure in the contact (Fig. 3.4). For high values of the parameter \( \bar{\sigma}_s = \sigma_sR/b^2 \) (i.e., for low loads) the maximum pressure increases linearly almost proportional to the total applied load. For high loads the Hertzian relation \( (p_c \propto F_N^{0.5}) \) applies.

An effective half contact width has been defined in analogy with the effective half contact width of Greenwood and Tripp (1967). For low loads the effective half width is almost independent of the load \( (b^* \propto F_N^{0.026}) \). For high values of the load the effective half width is equal to the Hertzian half width.

A parameter \( \alpha_l \), similar to the \( \alpha_s \) parameter of Greenwood et al. (1984), is defined, which shows that for \( \alpha_l \) values smaller than 0.05 the error compared to the Hertzian solution is less than about 11%.

The real contact area and the real number of asperities in contact are almost linearly proportional to the applied load \( (A \propto F_N^{0.978}, N \propto F_N^{0.947}) \).

Function fits are found, which can be used to calculate the various parameters.
Chapter 4
Mixed Lubrication Model

4.1 Introduction
When the deformation model for rough line contacts described in Chapter 3, is combined with the (Elasto) Hydrodynamic Lubrication ((E)HL) theory for smooth surfaces, it becomes possible to calculate Striebeck curves. The first for calculations in the Mixed Lubrication regime were made by Johnson et al. (1972). They did not calculate Striebeck curves, but used a model to calculate the “no-contact time fraction”, a parameter used for correlating measurements of wear in rolling contact bearings.

The Mixed Lubrication (ML) regime is the intermediate regime between the Boundary Lubrication (BL) regime and the (Elasto) Hydrodynamic Lubrication ((E)HL) regime. In the model of Johnson et al. (1972) the load on the ML contact is carried by a BL and an (E)HL force component:

\[ F_N = F_C + F_H, \]  \hspace{1cm} (4.1)

with \( F_C \) the load carried by the interacting asperities and \( F_H \) the load carried by the hydrodynamic (or (E)HL) component. Johnson et al. (1972) proposed that the total pressure, \( p_T \), in the contact is split up in the asperity pressure, \( p_C \) and the elasto hydrodynamic pressure, \( p_H \). (Fig. 4.1).

In Mixed Lubrication, both the film thickness equation for the (E)HL-component and the separation, resulting from asperity contact have to be adapted, since both influence each other. Johnson et al. (1972) used the concept of elastic springs to work out their model. The model proposed here will use a direct method. The load carried by the hydrodynamic action of the fluid film for smooth surfaces is known quite accurately from literature (Section 2.1), the load carried by the asperities is calculated and described in detail in Chapter 3. An adaptation is made to make both components suitable for the mixed lubrication calculations.

In the next section it will be explained how the hydrodynamic component is incorporated in the ML-model. In Section 4.3 the same is done for the asperity-deformation model. The reference planes of the hydrodynamic component and the
asperity contact component are different. This will be explained in Section 4.4. In Section 4.5 the last step will be made: calculation of the coefficient of friction in the Mixed Lubrication regime.

4.2 The hydrodynamic component

Recalling that the (E)HL-problem is calculated using eqns. (2.4), (2.5) and (2.8) (in combination with eq. (2.6)), the film thickness equation can be adapted by writing:

\[ p_T = \gamma_1 p_H, \]  

with \( \gamma_1 \) a constant, equal to or greater than unity. Reynolds’ equation (eq. (2.4)) then changes to:

\[ \frac{\partial}{\partial x} \left( \frac{\rho h^3}{\eta} \frac{\partial p_H}{\partial x} \right) = 6 \left( v^+ \right) \frac{\partial (\rho h)}{\partial x}. \]  

(4.3)

The total pressure is replaced by the hydrodynamic pressure \( p_H \), since only \( p_H \) is important in relation to the (E)HL-component. For the deformation of the contact, the total pressure of both asperities and fluid is important, therefore in the equation for the deformation (eq. (2.8)) the total pressure \( p_T \) is substituted by \( \gamma_1 p_H \) (eq. (4.2)):

\[ h(x) = h_0 + \frac{x^2}{2R} - \frac{4\gamma_1}{\pi E} \int_{-\infty}^{\infty} p_H(s) \ln|x - s| \, ds. \]  

(4.4)
4.3 The asperity contact component

The third equation that has to be adapted for the Mixed Lubrication model is the force balance equation (eq. (2.5)). The same substitution can be made in this case, since the total load has to be carried by both components:

\[ F_N = B \gamma_1 \int_{-\infty}^{\infty} p_H(x) dx. \] (4.5)

The results of the (E)HL calculations (Section 2.1.4) can now be adapted to be used in the Mixed Lubrication model. The substitution

\[ F_N \rightarrow \frac{F_N}{\gamma_1} \]

\[ E' \rightarrow \frac{E'}{\gamma_1} \] (4.6)

in the original equations (eqns. (2.4), (2.5) and (2.8)) transforms these equations to the new eqns. (4.3), (4.4) and (4.5). The substitution of eq. (4.6) implies that in Mixed Lubrication the elasticity modulus must be adapted as well. If only the load would be adapted, the width would become too small.

Making this transformation, the results of the (E)HL calculations, i.e. eq. (2.21) become:

\[ H_{\text{con}} = \left[ \gamma_1^{-s} \left( H_{R_{1z}}^{\gamma_1^{-\frac{1}{2}}} + \gamma_1^{-\frac{1}{2}} H_{E_{1z}}^{\gamma_1^{-\frac{1}{2}}} \right)^{\frac{1}{2} s} + \left( H_{R_{Pz}}^{\gamma_1^{-\frac{1}{2}}} + H_{E_{Pz}}^{\gamma_1^{-\frac{1}{2}}} \right)^{-\frac{1}{2} s} \right]^{s^{-1}}, \] (4.7)

with \( s \) again as auxiliary variable defined as:

\[ s = \frac{1}{5} \left( 7 + 8 e^{-2 \frac{H_{R_{1z}}}{H_{R_{1z}}} \gamma_1^{-\frac{1}{2}}} \right). \] (4.8)

The asymptotes in eqns. (4.7) and (4.8) are the same as given in Section 2.1.4.

A similar substitution can be made for the asperity contact component.

4.3 The asperity contact component

The results of the calculations for the asperity contact can be adapted for the calculations in the Mixed Lubrication regime analogous to the way in which the hydrodynamic component was adapted. Thus:

\[ p_T = \gamma_2 p_C. \] (4.9)
The relation between $\gamma_2$ and $\gamma_1$ is:

$$1 = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}. \quad (4.10)$$

The pressure generated by the asperities is given in eq. (3.9):

$$p_C(x) = \frac{2}{3} \eta \beta \sigma_s \sqrt{\frac{\sigma_s}{\beta}} E' F\frac{2}{3}(\frac{h(x)}{\sigma_s}), \quad (4.11)$$

the deformation equation is written as:

$$h(x) = h_0 + \frac{x^2}{2R} - \frac{4 \gamma_2}{\pi E'} \int_{-\infty}^{\infty} p_C(s) \ln(|x - s|) ds, \quad (4.12)$$

and the load equation:

$$F_N = B \gamma_2 \int_{-\infty}^{\infty} p_C(x) dx. \quad (4.13)$$

Equations (4.11), (4.12) and (4.13) give the same results as eq. (3.9), the combination of eqns. (2.6) and (2.8) and eq. (2.5) if the following substitutions are made:

$$F_N \rightarrow \frac{F_N}{\gamma_2} \quad \gamma_2 \rightarrow \gamma_2$$

$$E' \rightarrow \frac{E'}{\gamma_2}$$

$$n \rightarrow n \cdot \gamma_2 \quad (4.14)$$

For the central pressure in a rough line contact these substitutions lead to:

$$\frac{p_c}{p_h} = \left(1 + (a_1 \bar{n} a_2 \bar{\gamma} a_3 \gamma_2 ^{a_4})^{1/a_4} \right) \frac{1}{\gamma_2}, \quad (4.15)$$

with the fit parameters $a_i$ defined as for eq. (3.19). Thus, the central pressure of the asperity contact component in ML is a function of the total load as well as of the load carried by the asperities. Using eq. (4.15) the central pressure of the asperity contact can be calculated in a mixed lubricated contact. With this central pressure the separation can then be calculated using eq. (4.11).
4.4 The reference plane

The hydrodynamic component in mixed lubrication, as described in Section 4.2, is based on the separation between two smooth surfaces. The separation of the asperity contact component, however, is defined by the distance between a smooth surface and the plane through the mean height of the summits.

Generally, the mean plane through the surface will function as the reference plane for the (E)HL component. The difference between the mean plane through the surface heights and the mean plane through the summit heights is shown in Fig. 4.2.

![Diagram of different planes](image)

**Figure 4.2:** The different planes by definition of a rough surface.

The distance between the mean plane through the summits and the mean plane through the heights of the surface \( d_d \) is characterized by Whitehouse and Archard (1970) as:

\[
d_d = 0.82 \sigma, \quad (4.16)
\]

with \( \sigma \) the standard deviation of the heights of the surface. According to the model of Whitehouse and Archard (1970) \( \sigma \) and \( \sigma_s \) are related by:

\[
\sigma_s = 0.71 \sigma. \quad (4.17)
\]

Thus,

\[
d_d \approx 1.15 \sigma_s. \quad (4.18)
\]

The reference plane, used in the calculations, is the mean plane through the surface heights.

Mathematically, \( d_d \) is included by subtracting it from the separation in eq. (4.11):

\[
p_C(x) = \frac{2}{3} \eta \beta \sigma_s \sqrt{\frac{\sigma_s}{\beta}} E' \left( \frac{h(x) - d_d}{\sigma_s} \right). \quad (4.19)
\]
The parameter \( d_d \) is made dimensionless, analogous to the dimensionless separation \( H_h \):

\[
D_d = \frac{\pi d_d B E'}{8 F_N},
\]

(4.20)

with \( D_d \) the dimensionless distance between the mean plane through the summits and the mean plane of the surface. Equation (4.19) reads in dimensionless form:

\[
\bar{p} = \frac{2}{3} \bar{n} \bar{\sigma}_s \sqrt{\bar{\sigma}_s} F^*_F \left( \frac{H_h - D_d}{\bar{\sigma}_s} \right).
\]

(4.21)

4.5 Calculating the coefficient of friction

The calculation of the coefficient of friction can be split up in two steps:

1. Determination of the separation, the part of the load carried by the asperities and the part of the load carried by the hydrodynamic component.

2. The actual calculation of the friction.

The first step involves solving three equations with three unknowns:

- The first equation is eq. (4.1). The calculations are performed in dimensionless form. Equation (4.1) is therefore written as:

\[
W_N = W_C + W_H,
\]

(4.22)

with \( W_N, W_C \) and \( W_H \) the dimensionless loads, made dimensionless similar to \( W \). The subscripts \( N, C \) and \( H \) indicate the total load, the asperity component and the hydrodynamic component respectively.

- The second equality is that of eq. (4.15) and eq. (4.21):

\[
\frac{2}{3} \bar{n} \bar{\sigma}_s \sqrt{\bar{\sigma}_s} F^*_F \left( \frac{H_h - D_d}{\bar{\sigma}_s} \right) = \left( 1 + \left( a_1 \bar{n} a_2 \bar{\sigma}_s \gamma_2 a_2 \right)^{a_1} \right)^{1/a_4} \frac{1}{\gamma_2}.
\]

(4.23)

The three unknowns are \( W_H \) (or \( \gamma_1, \) since \( \gamma_1 = W_N/W_H \)), \( W_C \) (or \( \gamma_2, \) since \( \gamma_2 = W_N/W_C \)) and \( H \). Equations (4.22) and (4.7) are written in terms of the Moes- or Delft dimensionless numbers, while eq. (4.23) is written in terms of the parameters made dimensionless using the Hertzian parameters. So, care must be taken not to confuse these numbers.
4.5. Calculating the coefficient of friction

When the problem of three equations and three unknown is solved, step 2 can be made, the actual calculation of the coefficient of friction. The friction in ML can be considered as the sum of the friction of the BL and the (E)HL components. The friction in BL is described in Section 2.2.1. It was concluded that the coefficient of friction is constant over a wide range of pressures. Friction in (E)HL has been described in Section 2.1.5. Since in (E)HL the sliding friction gives the largest contribution to the friction only the sliding friction is included in the ML model. In Section 2.1.5 a summary of the different relations between the hydrodynamic shear stress and the shear rate is given.

The coefficient of friction in ML can thus be calculated by adding both friction components. In general:

\[ F_I = \sum_{i=1}^{N} \int_{A_{Ci}} \tau_{Ci}(\gamma) \, dA_{Ci} + \int_{A_H} \tau_H(\gamma) \, dA_H, \quad (4.24) \]

with:

\[ A_{Ci} : \text{the area of contact of a single asperity } i, \]
\[ \tau_{Ci} : \text{the shear stress at the asperity contact } i \text{ and} \]
\[ A_H : \text{the contact area of the hydrodynamic component}. \]

From eq. (2.28) the coefficient of friction \( f_{Ci} \) of a single asperity can be expressed as:

\[ f_{Ci} = \frac{\tau_{Ci}}{p_{Ci}}, \quad (4.25) \]

with \( p_{Ci} \) the normal pressure of a single asperity. For the Stiubeck curves the coefficient of friction can be approximated as being constant for all asperity contacts, thus the first term of eq. (4.24) can also be written as:

\[ \sum_{i=1}^{N} \int_{A_{Ci}} f_{CP_{Ci}} \, dA_{Ci} = f_{C}F_{C}, \quad (4.26) \]

with the value of \( f_{C} \) (the coefficient of friction in BL regime) determined from experiments.

The coefficient of friction can now be written as:

\[ f = \frac{f_{C}F_{C} + \int_{A_H} \tau_H(\gamma) \, dA_H}{F_N}. \quad (4.27) \]

Would the coefficient of friction \( f_{C} \) in the BL regime or the shear stress at the asperity contact \( \tau_{C} \) be known from a theoretical model, it can be incorporated
(Section 2.2.1), otherwise a value for $f_C$ will have to be found by experiments. The hydrodynamic area is the area on which the fluid under pressure is acting. This is the nominal contact area, $A_{\text{nom}}$, minus the real contact area $A_r$:

$$A_H = A_{\text{nom}} - A_r,$$  \hspace{1cm} (4.28)

In Section 3.6.3 a function fit was made of $A_r$, so a rather good approximation is available.

This chapter describes how the coefficient of friction in the mixed lubrication regime can be calculated. In the next chapter the calculation of Strubeck curves will be described.
Chapter 5

Calculating Strubeck curves

5.1 Introduction

On the basis of the contact model and the mixed lubrication model, presented in Chapters 3 and 4, the Strubeck curve can be calculated. In this chapter the results of such calculations will be shown. A parameter study is performed using a Gaussian height distribution. In Section 5.2 a reference case for these calculations will be defined. A Strubeck curve will be calculated for this reference case. The influence of the different parameters on the Strubeck curve will be shown in Section 5.3.

Section 5.4 will give a comparison between experiments and the model. The model will be applied to two oil-lubricated systems and a water-lubricated system.

In the last section conclusions from these calculations will be drawn.

5.2 Example

In order to perform a parameter study, a reference case is defined (Table 5.1). In this table the roughness parameters, the lubricant parameters and the load of the reference case are given, in both dimensional and, if applicable, dimensionless parameters (cf. Chapters 2 and 3). The dimensionless parameters $W$ and $G$ are used and not $M$ and $L$, since $M$ and $L$ depend on the velocity. The value of the product $n_\beta \sigma_a$ is given in this table as well. The value of this product is in the order of 0.03–0.05 for most surfaces, as explained in Section 3.4. Also, the value of $\bar{n} \cdot \bar{\sigma}_a$ is given. This value is larger than unity, as required (see Section 3.4).

The value for the distance between the mean plane through the surface and the mean plane through the summits $d_a$ is calculated according to eq. (4.18).

Based on the model as presented in Chapter 4 a Strubeck curve is determined as shown in Fig. 5.1, using the parameters in Table 5.1. On the horizontal axis of the generalized Strubeck curve the lubrication number $\mathcal{L}$ has been plotted. This number was defined in Section 2.4 as:

$$\mathcal{L} = \frac{\eta_0 u^+}{p_{av} R_a},$$

(5.1)
### Table 5.1: Dimensional and dimensionless parameters for the reference case

<table>
<thead>
<tr>
<th>property</th>
<th>value</th>
<th>unit</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$1.0 \cdot 10^{11}$</td>
<td>m$^{-2}$</td>
<td>density of asperities</td>
</tr>
<tr>
<td>$\beta$</td>
<td>10</td>
<td>$\mu m$</td>
<td>radius of asperities</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>0.05</td>
<td>$\mu m$</td>
<td>standard deviation of asperities</td>
</tr>
<tr>
<td>$d_d$</td>
<td>0.0575</td>
<td>$\mu m$</td>
<td>distance between planes</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.070</td>
<td>$\mu m$</td>
<td>standard deviation of heights</td>
</tr>
<tr>
<td>$R_a$</td>
<td>0.0565</td>
<td>$\mu m$</td>
<td>CLA surface roughness</td>
</tr>
<tr>
<td>$B$</td>
<td>10</td>
<td>mm</td>
<td>length of the contact</td>
</tr>
<tr>
<td>$E'$</td>
<td>231</td>
<td>GPa</td>
<td>combined elasticity modulus</td>
</tr>
<tr>
<td>$R$</td>
<td>20</td>
<td>mm</td>
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<td>viscosity pressure coefficient</td>
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<tr>
<td>$\tau_0$</td>
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<td>MPa</td>
<td>Eyring shear stress</td>
</tr>
<tr>
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<td>0.13</td>
<td>-</td>
<td>coefficient of friction in BL</td>
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</tr>
<tr>
<td>$\bar{n} \bar{\sigma}_s$</td>
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<td></td>
</tr>
</tbody>
</table>

with $R_a$ the combined CLA surface roughness. The CLA (Centre Line Average) surface roughness of the distribution $p(s)$ of one surface is defined by:

$$ R_a = \int_{-\infty}^{\infty} |s| p(s) \, ds, $$

(5.2)  

the combined CLA surface roughness then is:

$$ R_a = \sqrt{R_{a1}^2 + R_{a2}^2} $$

(5.3)
Since for a Gaussian surface heights distribution

\[ \sigma = \sqrt{\frac{\pi}{2}} R_a \approx 1.25 R_a, \]  

(5.4)

and using eq. (4.17), the value of \( R_a \) can be set to 1.13 times the standard deviation of the asperities. For the reference case: \( R_a = 0.0565 \mu m \).

![Graph showing the relationship between coefficient of friction and separation with lubrication number.

**Figure 5.1:** Generalized Strubeck curve and corresponding separation, for the reference case (Table 5.1).

Figure 5.1 shows not only the Strubeck curve, but also the corresponding dimensionless separation. The Strubeck curve corresponds to the left axis, while the separation scale is on the right axis. The separation is divided by \( \sigma_s \), since the ratio \( \lambda_s = h_c/\sigma_s \) is always in the order of unity. The separation hardly changes in the boundary lubrication regime and in the left part of the mixed lubrication regime. It is for this reason that the transition from BL to ML is more gradual than the transition from ML to (E)HL. In the (E)HL regime the separation can rise to high values. In that case asperity contact does not occur anymore.

For calculating the coefficient of friction eq. (4.27) is used. In the reference case the Eyring model has been used in order to calculate the friction of the (E)HL component. The Eyring model is described by curve II in Fig. 2.3 and by eq. (2.32).
The coefficient of friction is then calculated from:

\[
f = \frac{f_C F_C + \tau_0 \arcsinh\left(\frac{\eta \beta \sqrt{f}}{h \tau_0}\right) A_H}{F_N}.
\] (5.5)

For the viscosity \( \eta \) the Roelands' equation has been used (eq. (2.11)). In this case for the separation \( h \) a parallel gap is assumed. The value of \( h \) is calculated according to Section 4.5 (first step).

### 5.3 Striebeck curves

In this section the value of different parameters, which influence the Striebeck curve, will be varied. These parameters are: load \( (F_N) \), roughness parameters \( (n, \beta, \sigma_s) \), reference plane \( (d_d) \), combined elasticity modulus \( (E') \), viscosity \( (\eta) \) and the pressure-viscosity coefficient \( (\alpha) \).

#### 5.3.1 Influence of load

The position of the Striebeck curve depends on the load. When the load is varied, the separation in the BL regime and the film thickness equation in the (E)HL regime are both changed. In Fig. 5.2 the effect of a varying load is shown. The curves marked 3 correspond to the reference case. All other parameters are kept constant. Of course the dimensionless parameters that depend on the load and the average pressure \( p_{av} \) do change when the load is changed. It can be observed that with increasing load the Striebeck curves shift to the left. This is mainly caused by the fact that in the lubrication number \( \zeta \) the average pressure increases proportional to \( F_N^{0.5} \).

In Fig. 5.3 the effect of the load on the Striebeck curve is plotted as function of the sum velocity \( V^+ \). In this figure it is shown that for the operational conditions of Table 5.1 the transition from ML to (E)HL is almost independent of the load on the contact, i.e. they occur at nearly the same velocity. The friction in the (E)HL regime increases with load. This has two causes: 1: for larger loads the separation is smaller and 2: the viscosity increases with pressure.

With increasing load the transitions from BL to ML are more gradually. In Fig. 5.3 it can be seen that the Striebeck curves for lower loads do not differ much in the BL and ML regime, whilst for higher loads the differences increase.

In both Fig. 5.2 and 5.3 the separations \( \lambda_s \) are plotted on a logarithmic scale to stress the logarithmic character of the film thickness equation. The contact is operating in Koets' area (Fig. 2.2), therefore in the (E)HL regime the separation is proportional to \( V^+0.75 \).

In the BL regime the separations decrease with increasing loads. In fact when the asperity component is considered, the separations become negative with regard to
5.3. Striebeck curves

![Graph showing Striebeck curves](image)

**Figure 5.2:** Generalized Striebeck curves and corresponding separations, varying the load. 1: \( F_N = 125 \text{N} \), 2: \( F_N = 250 \text{N} \), 3: \( F_N = 500 \text{N} \) (reference), 4: \( F_N = 1000 \text{N} \) and 5: \( F_N = 2000 \text{N} \), other parameters as given in Table 5.1.

![Graph showing Striebeck curves](image)

**Figure 5.3:** Generalized Striebeck curves and corresponding separations, varying the load, as function of the sum velocity. 1: \( F_N = 125 \text{N} \), 2: \( F_N = 250 \text{N} \), 3: \( F_N = 500 \text{N} \) (reference), 4: \( F_N = 1000 \text{N} \) and 5: \( F_N = 2000 \text{N} \), other parameters as given in Table 5.1.
the mean line through the summits. This line is located at 1.15 times the standard deviation of the summits above the mean line through the surface, which is taken as reference for all calculations. Thus, the curves 4 and 5 in Figs. 5.2 and 5.3 do not agree with the model of Greenwood and Williamson. In Chapter 3 it has been put forward that the separation is not allowed to become negative. In the current context negative means: negative with regard to the mean line through the summits. In Chapter 7 this subject will be discussed in more detail. The real area of contact for the reference case of Table 5.1 is: \( A_r/A_H = 6.6\% \). For the curves 5 in Figs. 5.2 and 5.3 this value is \( A_r/A_H = 11.7\% \). These calculations are based on eq. (3.25).

5.3.2 Influence of roughness

The three main roughness parameters that can be varied when studying the influence of the surface roughness on the Stribeck curve are the density of asperities \( n \), the radius of the asperities \( \beta \) and the standard deviation of the summit heights \( \sigma_s \). A fourth important roughness parameter is the distance between the reference plane of the summits and the heights, the effect of which will be discussed in Section 5.3.3.

When studying the influence of the three parameters, the similarity analysis of Section 3.4 shows that a parameter study with only two parameters is sufficient, i.e. only \( \bar{n} \) and \( \bar{\sigma}_s \) are needed. In fact, doubling the number of asperities \( n \) has the same effect as increasing the radius of the asperities \( \beta \) with a factor four.

The calculations, resulting in Fig. 5.4, are done, keeping the product \( n \beta \sigma_s \) constant. Keeping this product constant, the product \( \bar{n} \bar{\sigma}_s \) will remain constant as well. In Fig. 5.4 it can be seen that the separation (or rather \( \lambda_s \)) in the BL-regime increases with increasing \( \bar{\sigma}_s \) (and thus with increasing \( \sigma_s \)). This effect has been shown already in Fig. 3.5. It is caused by the fact that in the equation for the normal load of the contact model (see eq. (3.8)) \( \sigma_s \) appears to the power 1.5. On the (E)HL side the separation \( h \) is independent of the surface roughness, therefore \( \lambda_s \) decreases with increasing value of \( \sigma_s \) \( \lambda_s \propto 1/\sigma_s \). This, together with the fact that in the BL-regime the value of \( \lambda_s \) increases with increasing \( \sigma_s \), causes the Stribeck curves to shift to the right. This effect is compensated partly by the fact that in the lubrication number \( \mathcal{L} \) the CLA roughness \( R_a \) is included in the denominator. The CLA surface roughness is assumed to be proportional to \( \sigma_s \).

The coefficient of friction in the (E)HL regime is independent of the roughness, as might be expected in the (E)HL regime.

In Fig. 5.4 both \( \bar{n} \) and \( \bar{\sigma}_s \) are varied. To study the effect of \( \bar{\sigma}_s \) and \( \bar{n} \) separately, calculations have been done varying \( \sigma_s \) (and thus \( \bar{\sigma}_s \)) and \( \bar{n} \) (\( \bar{n} \)).

The results of calculations with different \( \sigma_s \) are shown in Fig. 5.5. In this figure the reference case is plotted and three cases with the standard deviation of the summits raised by factors of, respectively, 2, 4 and 10. At first sight the differences between Fig. 5.4 and Fig. 5.5 are not very large. However, when looking at the
5.3. Striebeck curves

**Figure 5.4:** Generalized Striebeck curves and corresponding separations, varying the roughness, 1: \( \bar{n} = 98.6 \& \bar{\sigma}_s = 0.0907 \) (reference), 2: \( \bar{n} = 49.3 \& \bar{\sigma}_s = 0.181 \), 3: \( \bar{n} = 24.6 \& \bar{\sigma}_s = 0.363 \) and 4: \( \bar{n} = 9.86 \& \bar{\sigma}_s = 0.907 \), other parameters as given in Table 5.1.

Separations, differences in the BL-regime attract the attention. This is made clear in Table 5.2, which lists the values of \( \lambda_s \) of the different curves for both cases (i.e., Fig. 5.4 and Fig. 5.5).

**Table 5.2:** Separations in the BL-regime

<table>
<thead>
<tr>
<th>curve</th>
<th>( \lambda_s )</th>
<th>Fig. 5.4</th>
<th>Fig. 5.5</th>
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<tr>
<td>1</td>
<td>1.02</td>
<td>1.02</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.30</td>
<td>1.71</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.56</td>
<td>2.28</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.91</td>
<td>2.92</td>
<td></td>
</tr>
</tbody>
</table>

It can be seen that in the case of curves 2, 3 and 4 the \( \lambda_s \) values in Fig. 5.5 are consistently higher than in Fig. 5.4. This causes the Striebeck curves to shift to the right, i.e. to higher values of \( \mathcal{L} \). The ML-regimes in the Striebeck curves 2, 3 and 4 in Fig. 5.5 are also steeper than in Fig. 5.4, which is caused by the higher
Figure 5.5: Generalized Strieber curves and corresponding separations, varying the standard deviation of the summits, 1: $\sigma_s = 0.05\mu m$ (reference), 2: $\sigma_s = 0.1\mu m$, 3: $\sigma_s = 0.2\mu m$ and 4: $\sigma_s = 0.5\mu m$, other parameters as given in Table 5.1.

Figure 5.6: Generalized Strieber curves and corresponding separations, varying the density of the summits, 1: $n = 1.0 \cdot 10^{11} m^{-2}$ (reference), 2: $n = 2.0 \cdot 10^{11} m^{-2}$, 3: $n = 4.0 \cdot 10^{11} m^{-2}$ and 4: $n = 1.0 \cdot 10^{12} m^{-2}$, other parameters as given in Table 5.1.
5.3. Strubeck curves

separations in the BL-regime. Since the load is carried by relatively few summits when the separation is high, the stiffness of the asperity contact is low and therefore the lubricant will take over the load more rapidly once it is carrying load.

The shift of the Strubeck curves is moderated by the fact that \( R_0 \) is present in the denominator of the lubricant number \( \mathcal{L} \). This presence, however, is not enough to shift the Strubeck curves to the same position.

The result of varying the density of summits \( n \) is shown in Fig. 5.6. The difference between the Strubeck curves is less than that, shown in Fig. 5.5. The (E)HL branches of the Strubeck curves are identical, except that for lower values of \( n \) the (E)HL-regime starts a bit earlier. The fact that the (E)HL branches are identical is caused by the fact that only a roughness parameter is changed that has no influence on the (E)HL component. The density of summits \( n \) has no influence on \( \mathcal{L} \) unlike \( \sigma_s \) in Fig. 5.5. The increases in separation in the BL-regime are less than those, shown in Fig. 5.5. This is due to the fact that the total load in the contact model is (only) proportional to the density (eq. (3.8)). The slopes of the Strubeck curves in the ML-regime get steeper again with increasing of \( n \). This also explains why the slopes in Fig. 5.4 are approximately equal. In Fig. 5.4 \( \sigma_s \) was increased and therefore the slopes got steeper and \( n \) was decreased and thus this effect was compensated approximately.

5.3.3 Influence of reference plane

The fourth roughness parameter that can be varied is the distance between the two mean planes, i.e. the mean plane through the summits and the mean plane through the surface. In the reference case the value of \( d_d \) is set on \( 1.15\sigma_s \). This value is based on Whitehouse and Archard (1970). In Fig. 5.7 \( d_d \) is varied from a quarter, a half, once to twice this reference value of \( d_d \).

Upon shifting the reference plane of the summits, the film thickness equation and, thus, the curves in the (E)HL-regime remain unchanged. However, the ML-regime is shifted considerable as well as the slope in the ML-regime. From Fig. 5.7 it is clear that the influence of the reference plane is considerably. The transition from BL to ML shifts more than one decade and, thus, the value of the distance between the two planes is very important for the location where the transition BL/ML occurs in the Strubeck curves. The value found by Whitehouse and Archard is based on measurements. In roughness measurements the sampling interval and the sampling length are very important for the results of such measurements. It is therefore of great relevance to give the sampling interval and sampling length when measuring roughness (Greenwood, 1992). In Chapter 7 the effect of the distance between the planes will be discussed in more detail.
Figure 5.7: Generalized Stribeck curves and corresponding separations, varying the distance between the main planes, 1: \( d_d = 0.288 \sigma_s \), 2: \( d_d = 0.575 \sigma_s \), 3: \( d_d = 1.15 \sigma_s \) (reference) and 4: \( d_d = 1.725 \sigma_s \), other parameters as given in Table 5.1.

5.3.4 Influence of elasticity modulus

The combined elasticity modulus \( E' \) appears in both the film thickness equation and the asperity contact model. In Fig. 5.8 three Stribeck curves with the corresponding separations have been plotted. From the curves for the separation it can be concluded that, in all regimes, the separation increases with increasing values of \( E' \). In the BL-regime the stiffness of the asperity contact increases, while the contact pressure (and as a consequence the viscosity) in (E)HL increases with increasing values of \( E' \).

In the (E)HL-regime the friction is higher with lower values of \( E' \). This is caused by the fact that the area of contact increases with decreasing values of \( E' \). Also the lower value of the separation adds to the higher level of friction. Since the separation is lower for lower values of \( E' \), the BL-ML transition shifts to lower values of the lubrication number. When the separation is lower, the lubricant starts to carry load at lower lubrication numbers.

It should be kept in mind that the lubrication number \( \mathcal{L} \) is influenced by changing \( E' \). A higher value of \( E' \) leads to a smaller nominal contact area and thus the average pressure will be higher.
5.3. Stößeck curves

![Graph showing the coefficient of friction, \( f \), and separation, \( \lambda_s \), as a function of the lubrication number, \( \mathcal{L} \). The graph includes curves labeled 1, 2, and 3, each representing different values of \( E' \): 1: \( E' = 115 \text{GPa} \), 2: \( E' = 231 \text{GPa} \) (reference), and 3: \( E' = 462 \text{GPa} \). Other parameters are as given in Table 5.1.]

Figure 5.8: Generalized Stößeck curves and corresponding separations, varying the combined elasticity modulus, 1: \( E' = 115 \text{GPa} \), 2: \( E' = 231 \text{GPa} \) (reference) and 3: \( E' = 462 \text{GPa} \), other parameters as given in Table 5.1.

5.3.5 Influence of viscosity

In Figs. 5.9 and 5.10 the influence of the viscosity on the Stößeck curve is shown. Fig. 5.9 shows the results of varying the pressure viscosity coefficient \( \alpha \). According to eq. (2.13) the viscosity changed as well. Figure 5.10 shows the results of keeping \( \alpha \) constant and varying \( \eta_0 \).

In both figures the separations in the BL-regime are equal for all cases. The viscosity has no influence on the separation in the BL-regime. In Fig. 5.10 the separations are completely equal for all four cases. This is due to the fact that the viscosity is included in the lubrication number. In Fig. 5.9 the separations in the ML- and the (E)HL-regime increase with increasing value of \( \alpha \). The BL/ML transitions as well as the ML-(E)HL transitions shift to the left with increasing \( \alpha \).

In both figures the friction in the (E)HL-regime increases with increasing viscosity.
Figure 5.9: Generalized Strubeck curves and corresponding separations, varying the viscosity-pressure coefficient and the viscosity, 1: $\alpha = 1 \cdot 10^{-8}$ Pa$^{-1}$, $\eta_0 = 6.50$ Pa$\cdot$s, 2: $\alpha = 2 \cdot 10^{-8}$ Pa$^{-1}$, $\eta_0 = 0.362$ Pa$\cdot$s (reference), 3: $\alpha = 3 \cdot 10^{-8}$ Pa$^{-1}$, $\eta_0 = 0.0203$ Pa$\cdot$s and 4: $\alpha = 4 \cdot 10^{-8}$ Pa$^{-1}$, $\eta_0 = 1.13 \cdot 10^{-3}$ Pa$\cdot$s, other parameters as given in Table 5.1.

Figure 5.10: Generalized Strubeck curves and corresponding separations, varying the viscosity, 1: $\eta_0 = 2.03 \cdot 10^{-3}$ Pa$\cdot$s, 2: $\eta_0 = 0.0203$ Pa$\cdot$s (reference), 3: $\eta_0 = 0.203$ Pa$\cdot$s and 4: $\eta_0 = 2.03$ Pa$\cdot$s, other parameters as given in Table 5.1.
5.4 Comparison with experiments

In this section a comparison between the model and measurements is made. First two oil lubricated examples are shown, using measurements performed by Johnson and Spence (1991) (Section 5.4.1.1) and by ter Haar (1996) (Section 5.4.1.2). Next, measurements relating to a water lubricated system are shown. These have been carried out by Lubbinge (1999).

5.4.1 Oil-lubricated experiments

5.4.1.1 Two-disk machine

Johnson and Spence (1991) performed experiments on a two-disk machine, simulating the contact between gear teeth. They measured traction curves (Chapter 6). At high slip values, the coefficient of friction was found to be constant. Thus, the relevant values could be used to plot (part of) a Stribeck curve.

The roughness parameters, used in the measurements, are listed in Table 5.3, as well as other relevant parameters. Since Johnson and Spence performed their

| Table 5.3: Experimental conditions experiments of Johnson and Spence (1991) |
|---|---|---|---|
| property | value | unit | description |
| $n$ | $13 \times 10^9$ | m$^{-2}$ | density of asperities |
| $\beta$ | 2.6 | $\mu$m | radius of asperities |
| $\sigma_s$ | 0.487 | $\mu$m | standard deviation of asperities |
| $d_d$ | 0.56 | $\mu$m | distance between planes |
| $R_a$ | 0.55 | $\mu$m | CLA surface roughness |
| $B$ | 12.7 | mm | length of the contact |
| $E'$ | 231 | GPa | combined elasticity modulus |
| $R$ | 19 | mm | reduced radius of cylinder |
| $\eta_0$ | 37.4 | mPa·s | viscosity |
| $\alpha$ | $1.94 \times 10^{-8}$ | Pa$^{-1}$ | viscosity pressure coefficient |
| $\tau_0$ | 2.5 | MPa | Eyring shear stress |
| $f_C$ | 0.098 | - | coefficient of friction in BL |
| $F_N$ | 3062 | N | load 1 |
| $F_N$ | 4500 | N | load 2 |
| $F_N$ | 5800 | N | load 3 |
| $p_{av}$ | 536 | MPa | average pressure 1 |
| $p_{av}$ | 650 | MPa | average pressure 2 |
| $p_{av}$ | 738 | MPa | average pressure 3 |
measurements at three different loads, in Table 5.3 three loads $F_N$ are specified. The corresponding values of the average pressure\(^1\) fall in the high pressure regime (Koets' region), for which the model has been developed.

The coefficient of friction in the BL-regime is approximately $f_C = 0.1$. This seems rather low, but Johnson and Spence used Shell Vitrea 68 oil. Schipper (1988) used a different oil from the Shell Vitrea family, Shell Vitrea 100. For this oil he found a low coefficient of friction in the BL-regime as well.

![Graph showing coefficient of friction vs sum velocity, $v^+$](image)

**FIGURE 5.11:** Measurements by Johnson and Spence and calculated generalized Stribeck curve.

Figure 5.11 shows the three Stribeck curves, which were calculated using the three loads. Also the coefficients of friction for high values of the slip from the measurements of Johnson and Spence are shown. It can be seen that a good agreement between model and experiments is found.

### 5.4.1.2 Sheet metal forming

The experiments of ter Haar (1996) form another good example of oil lubricated measurements. The experiments were performed to simulate friction in sheet metal forming and to establish the Stribeck curve for sheet metal forming. The conditions for the experiments are listed in Table 5.4. The value of $R_a$ is not based on 1.13 times the standard deviation of the summits, but is taken from the measurements,

\[^1\text{In Johnson and Spence (1991) the maximum pressure is given instead of the average pressure.}\]
performed by ter Haar. Also, the value of \( d_d \) is calculated using the actual surface measurements of ter Haar. In Table 5.4 two values for the viscosity of two different oils (Lub1) and (Lub2) are given. The oils used by ter Haar are mineral oils. In this case the average pressure is rather low. Since

\[
\alpha \rho_{av} = \sqrt{\frac{\pi M}{32}} L
\]

the first condition of Koets \((L\sqrt{M} > 13.3)\) results in \( \alpha \rho_{av} > 4.167 \). For the current case \( \alpha \rho_{av} = 2.4 \). In Fig. 5.12 a comparison is made between the measurements with the two different oils and a Strieber curve using the viscosity of Lub2. A Strieber curve calculated with Lub1 is only slightly different, because of the low pressure in the contact. The friction in the (E)HL regime is slightly lower for Lub1.

The calculated Strieber curve agrees quite good with the measured data. The model has, in fact, been developed for highly loaded situations in Koets’ region (Fig. 2.2). The above proves that the present model is also accurate for low loads.
5.4.2 Experiments on water lubricated systems

Lubbinge (1999) performed measurements on a water-lubricated system. His research concerned the lubrication of mechanical face seals. Lubbinge has considered different geometries and operational conditions for such seals. The geometry used in this section has waviness only. The waves on the seals are sinusoidal in tangential direction. The fluid to be sealed is pressureless. The seal faces will thus operate in a purely hydrodynamic way. The crests of the waviness can be approximated by cylinders using a Taylor expansion. Since the amplitudes of the waves on the seal are small, the equivalent radius of the cylinder is very large (Table 5.5). Therefore the average pressure in the contact is low. Mechanical face seals have a relatively high elasticity modulus.

For water, the film thickness equation needs an adaption. Calculations done by Venner (1991) and a function fit by Moes (1997) are all based on a viscosity-pressure index \( z \) of Roelands' equation of \( z = 0.68 \). The viscosity-pressure index of water, however, is \( z = 0.10 \). Therefore an adaption in the calculations must be made. This adaption is described by Schipper and Ten Napel (1998). In the film thickness equation the viscosity pressure coefficient \( \alpha \) needs to be replaced by the adapted
5.4. Comparison with experiments

TABLE 5.5: Experimental conditions for the mechanical face seals measurements by Lubbinge (1999)

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</tr>
<tr>
<td>$p_{av}$</td>
<td>1.56</td>
<td>MPa</td>
<td>average pressure</td>
</tr>
</tbody>
</table>

viscosity pressure coefficient $\alpha^*$. This parameter is given by:

$$\frac{1}{\alpha^*} = \int_0^\infty \frac{\eta_0}{\eta(p)} \, dp,$$

(5.7)

with $\eta(p)$ the Roelands viscosity-pressure relation (eq. (2.12)):

$$\frac{1}{\alpha^*} = \frac{\eta_0}{\eta_\infty} \int_0^\infty \left( \frac{\eta_\infty}{\eta_0} \right) \left( 1 + \frac{p}{p_r} \right)^z \, dp,$$

(5.8)

Schipper and Ten Napel (1998) have made an approximation for this function:

$$\frac{1}{\alpha^*} = \frac{p_r}{z \, c} \left( 1 + \frac{m}{c} + \frac{m(m-1)}{c^2} + \frac{m(m-1)(m-2)}{c^3} + \text{etc.} \right),$$

(5.9)

with $c$ and $m$ constants defined as:

$$c = \ln \left( \frac{\eta_0}{\eta_\infty} \right),$$

(5.10)

$$m = \frac{1}{z} - 1.$$  

(5.11)

When using the approximation eq. (5.9), care must be taken that not too many terms are included, since in that case the approximation might diverge.
Schipper and Ten Napel (1998) have shown that for a wide range of z-values (0.6 < z < 1.2), the adaptation of \( \alpha \) is not needed. For water, however, \( z \) has a value of 0.10 and in this case such an adaption is definitely needed. In Table 5.5 the adapted viscosity pressure coefficient is given and it can be concluded that \( \alpha^* \) is so low that an (E)HL contact with water behaves almost isoviscous. For that reason a value of \( \tau_0 \) is not necessary.

The solution of the (E)HL problem, calculated by Venner (1991) and fitted by Moes (1997), is valid for an (E)HL contact of infinite width. In the case of mechanical face seals the ratio \( B/R \) is small. This fact, as well as the fact that the viscosity of water hardly depends on the pressure, causes the (E)HL solution to be invalid in this case. Since \( B/R \) is small, side leakage must be taken into account. The (E)HL line contact solution is based on an infinite long cylinder, excluding side leakage.

Wijlhuizen (Wijlhuizen, 1997; Lubbinge, Wijlhuizen and Schipper, 1998) has developed a function fit for the film thickness for a seal geometry. Instead of the (E)HL-solution (Lubbinge, 1999), this function fit can be used to calculate the Stribeck curve. The solution of Wijlhuizen is a short-bearing solution. A different short-bearing solution is the Ocvirk solution. In Appendix D the Ocvirk solution is derived and a simple result is obtained (eq. (D.9)). Since the geometry used in

![Figure 5.13: Measurements by Lubbinge (marked lines) and calculated generalized Stribeck curve (drawn line without marks).](image-url)
Appendix D is rigid, no correction on the film thickness, as in Section 4.2, needs to be made.

In Fig. 5.13 the calculated Stribeck curve, using the film thickness equation of eq. (D.9), is compared with the measurements of Lubbinge (1999). In this figure the measurements and the calculations are in good agreement. It can thus be concluded that also for this low-loaded water lubricated system the model is in good agreement with the experiments.

5.5 Conclusions

In this chapter the results of varying the different parameters that influence the Stribeck curve are shown.

- From the calculations it appears that the standard deviation of the summits $\sigma_s$ and the distance between the two mean planes $d_d$ have the largest impact on the Stribeck curve. The shift of the Stribeck curves due to the surface roughness is quite remarkable. Via the lubrication number $L$, a correction is made for the surface roughness in Fig. 5.4, but the difference is still quite remarkable. Also the distance between the mean planes is important for the Stribeck curve. The effect of this parameter will be discussed in more detail in Chapter 7.

- Figures 5.2 and 5.3 show the influence of the load on the transitions. This influence is especially significant for the BL to ML transition.

- Another parameter that affects the average pressure is the combined elasticity modulus $E'$. This parameter also affects mainly the BL/ML transition (Fig. 5.8).

- When the number of asperities $n$ or the radius of the asperities $\beta$ is varied, a small shift in the Stribeck curve can be observed.

- Parameters that have influence on the friction level are the viscosity $\eta_0$ and the pressure-viscosity coefficient $\alpha$. However, $\alpha$ causes also a small shift of the ML regime.

In Section 5.4 the model is compared with experimented results. Three cases have been studied, i.e. two oil-lubricated systems and a water-lubricated system. In Section 5.4.1.1 the experiments of Johnson and Spence (1991) are used as an example for a highly loaded system. With appropriate estimates of $n$, $\beta$ and $f_C$ the Stribeck curve match the measurements quite well. The estimated parameters can be used to calculate traction curves (next chapter).

The other two systems (Sections 5.4.1.2 and 5.4.2) operate under low average pressure and, thus, outside the Koets’ regime, showed good agreement between
experiment and the model presented in this thesis. It can therefore be concluded that the model holds very well at low pressures.

For the water-lubricated system an adaption has to be made for the pressure-viscosity parameter $\alpha$. Using this adaption it is clear that water may be considered to behave isoviscous. For the mechanical face seals an alternative film thickness equation has to be used. In Appendix D the short-bearing solution for a cylinder geometry has been derived. Using this solution a Stribeck curve has been calculated, which matches the measurements of Lubbinge (1999) accurately.

In this chapter only Stribeck curves have been calculated. The sliding velocity $v_{\text{diff}}$ has been taken equal to the sum velocity $v^+$ (200% slip). For many applications however the traction curve is of more interest. Therefore the next chapter will deal with traction curves,
Chapter 6

Traction curves

6.1 Introduction

In the calculations in Chapter 5 the sliding velocity has been taken equal to the sum velocity, \( v^\text{diff} = v^+ \). This situation of simple sliding is shown in Fig. 6.1b. In Fig. 6.1a it is shown that the velocity between two moving surfaces can be regarded as a superposition of a pure rolling- and a pure sliding motion. In Fig. 6.1b the surface velocities of the contacting surfaces \( v_1 \) and \( v_2 \) are added to obtain the sum velocity \( v^+ = v_1 + v_2 \) and subtracted to obtain the sliding velocity \( v^\text{diff} = |v_1 - v_2| \). In this figure lines for constant sum velocity, sliding velocity and constant slip value have been drawn as well. The definition for slip is:

\[
S = \frac{v^\text{diff}}{v^+} \cdot 200\% = \frac{|v_1 - v_2|}{v_1 + v_2} \cdot 200\%.
\]  

(6.1)

The slip is also often named the slide-to-roll ratio. Throughout this chapter the term slip will be used. If friction is plotted, perpendicular to the velocity field each constant slip line in the velocity field represents a Strubeck curve. In Chapter 5 Strubeck curves for the simple sliding situation are shown. Traction curves, most frequently used in (E)HL, are calculated by varying the slip. Usually this is done by keeping the sum velocity constant, while varying the sliding velocity.

Traction curves can be given in two ways:

1. the shear stress \( \tau \) as a function of the shear rate \( \dot{\gamma} \),

2. the coefficient of friction as a function of the slip.

In this thesis the second option will be used, since this will make comparison with experiments easier. Using the Eyring-RoeIands behaviour for the fluid, the hydrodynamic component of eq. (5.5) is obtained:

\[
f = \frac{\tau_0 \arcsinh\left( \frac{\eta v^\text{diff}}{h \tau_0} \right) A_H}{F_N}.
\]  

(6.2)
In eq. (6.2) the sliding velocity can be replaced by the slip \( S \) using eq. (6.1) and finally, the coefficient of friction can be calculated as a function of \( S \) (eq. 6.3).

\[
f = \frac{\tau_0 \arcsinh \left( \frac{\eta v^+ S}{2h \tau_0} \right)}{F_N} A_H. \tag{6.3}
\]

Using this equation, curve 1 in Fig. 6.2 has been calculated. In this curve, which applies to the purely hydrodynamic case \( (f_C = 0) \), the coefficient of friction \( f \) increases with increasing slip \( S \). In practice, friction usually decreases at higher slip values, say at \( S > 10\% \). This is caused by shear heating. Since in this thesis thermal effects are neglected, this is not borne out by the model.
Traction curve 2 applied to mixed lubrication, where, in calculating friction, the asperity component has to be taken into account. In this case this is done by assuming a constant value for \( f_C \) (i.e. \( f_C = 0.13 \). Curve 2 will be discussed further in Section 6.2.

When calculating traction curves the slip is varied and usually the sum velocity \( v^+ \) is kept constant. This has one big advantage: the separation and the two components of the load under conditions of mixed lubrication \( F_C \) and \( F_H \) (step 1 in Section 4.5) need to be calculated only once per traction curve. This first step is the most time-consuming. Calculations, based on the model of Greenwood and Williamson are not time consuming. However, when a more complicated contact model is used, this may be different.

### 6.2 BL-friction

In Section 6.1 the BL-component of the friction has been included, assuming a constant value for \( f_C \), equal to 0.13. The resulting traction curve (2 in Fig. 6.2) shows a step at \( S = 0\% \) (Fig. 6.2). This is contradictory to measurement practice, see e.g. Schipper (1988).

The apparent discrepancy can be explained as follows. In eq. (4.24) the friction
in the BL-regime is described as the sum of the shear stress in the different micro-contacts. The boundary layers, present in the micro-contacts, can be considered to behave in an elastic-plastic way, as in case of a solidified lubricant, and the shear stress behaviour of a boundary layer as present in the micro-contact is assumed to be similar to that of curve IV in Fig. 2.3.

In Fig. 2.3 the shear stress \( \tau_H \) in the lubricant is plotted as function of the shear rate \( \dot{\gamma} \) for the (E)HL component. When the sliding velocity \( \dot{\delta} \) and the separation \( h \) are known, the shear rate \( \dot{\gamma} \) is known and from curve IV in Fig. 2.3 the shear stress can be determined. By using eq. (6.1) the shear stress can than be obtained as a function of the slip. However, the separation in the micro-contacts is not known, therefore it is not possible to determine the shear rate \( \dot{\gamma} \) in a micro-contact. The shear stress of the BL-component can therefore not be found by simply adding up the shear stress in the different micro-contacts.

A realistic traction curve in the BL regime can be obtained by combining the elastic-plastic concept with the results of measurements (Schipper and Gelinck, 1999). Traction curves are generally measured at constant sum velocity, therefore the separation of the macro-contact remains constant with varying the slip. At low slip values the shear stress increases with the sliding velocity, since the shear stress in the micro-contacts is elastic at low shear rates. When the sliding velocity is increased further, some of the micro-contacts will begin to shear plastically. With higher sliding velocities all micro-contacts will shear plastically and the total shear stress will become constant.

Elastic-plastic traction curves for the BL-regime have been drawn schematically in Fig. 6.3. In this figure three possible fits for traction curve around zero slip are given, as well as the tangent through the possible fits. The three curves are defined as:

Curve 1:

\[
f = f_C \frac{2}{\pi} \arctan \left( \frac{\pi S}{2 S_{bp}} \right),
\]

with \( S_{bp} \) the slip at the transition from elastic to plastic behaviour of the lubricant.

Curve 2:

\[
f = \left[ f_C^a + \left( \frac{f_C}{S_{bp}} \right) \right] ^\frac{1}{a} \frac{|S|}{S},
\]

with \( a \) a fit parameter, in the plotted case equal to \(-2\). The fit parameter must be negative.
6.2. BL-friction

Curve 3:

\[ f = f_C \tanh \left( \frac{S}{S_{ep}} \right) \]  \hspace{1cm} (6.6)

The arctan function has the disadvantage that it needs a rather high \( S/S_{ep} \) ratio for \( f \) to increase to \( f_C \), i.e. the value of the BL coefficient of friction for plastically shearing boundary layers. Curve 2 also needs, besides the values of \( f_C \) and \( S_{ep} \), a fit parameter \( a \). With this fit parameter the curve can be adjusted to make a sharper angle at \( S = S_{ep} \), using a higher value for \( a \). A disadvantage is that this parameter also needs to be curve fitted. The tanh function needs, like the arctan function, a parameter less and still describes the traction around \( S=0 \) accurately. The arctan and tanh function are both used in this thesis.

The parameters \( f_C (f_C = \tau_C/p \text{ eq. (4.25)}) \) and \( S_{ep} \) are the same for all three curves. In the previous chapters \( f_C \) has been defined as “the coefficient of friction in the BL-regime”, according to the current theory, the words “caused by plastically shearing boundary layers” should be added.
6.3 Example

With the friction of the BL-component around zero slip known and applying eq. (6.4), the traction curves can now be calculated with:

\[
f = \frac{f_C F_C \arctan \left( \frac{S}{S_{\text{ep}}} \right) + \tau_0 \arcsinh \left( \frac{\eta v^+ S}{2h \tau_0} \right) A_H}{F_N}.
\]  

(6.7)

Examples of traction curves, thus calculated, are shown in Fig. 6.4. For this example, four different sum velocities and the properties from Table 5.1 are used. The value of the slip at the transition from elastic to plastic behaviour is taken as \( S_{\text{ep}} = 0.15 \), after Schipper and Gelinck (1999). The velocities are chosen from Fig. 5.3, ensuring Mixed Lubrication.

![Figure 6.4: Traction curve for the reference case (Table 5.1), at different velocities. 1: \( v^+ = 0.05 \text{m/s} \), 2: \( v^+ = 0.1 \text{m/s} \), 3: \( v^+ = 0.2 \text{m/s} \), 4: \( v^+ = 0.4 \text{m/s} \), with \( S_{\text{ep}} = 0.15\% \).](image)

From Fig. 6.4 it can be concluded that the shape of the traction curve is dominated by the BL-component. (Compare Figs. 6.2, 6.3 and 6.4.)

Comparison of traction curves with experiments is difficult, since in the literature traction curves concerning mixed lubrication are rare. Frequently, no data are available on the coefficient of friction in the BL-regime \( f_C \). Also the roughness
data needed are frequently incomplete, generally only $R_n$ or $\sigma$ values being published, without data on $n$ and $\beta$. Another problem is that it is generally not known whether the surfaces are worn. In the current analysis Gaussian distributions are assumed, the real summit distributions are not published at all in literature.

6.4 Comparison with experiments

In Section 5.4.1.1 the measurements of Johnson and Spence (1991) have been used to plot the coefficient of friction against the sum velocity at high slip values in order to verify the model. Johnson and Spence measured traction curves and therefore their experiments can be used to compare the model on traction curves with their experiments as well. The parameters, given in Section 5.4.1.1, can be used for the calculations. Johnson and Spence showed their traction curves with the torque plotted against the slide to roll ratio. A “translation” of their measurements in terms of $f$ versus $S$ curves, together with the calculated traction curves is plotted in Fig. 6.5.

![Traction curves graph](image)

**FIGURE 6.5:** Measured traction curves of Johnson and Spence (1991) and corresponding calculated traction curves (Table 5.3), with $S_{\text{ep}} = 0.25\%$, $F_N = 3062\text{N}$, curves 1 & ●: $v^+ = 2.50\text{m/s}$, curves 2 & ▼: $v^+ = 1.72\text{m/s}$, curves 3 & □: $v^+ = 0.86\text{m/s}$. Drawn line: arctan-curve, dashed line: tanh-curve.

In Fig. 5.11 in total nine points were plotted, for three different sum velocities and
three loads. In Fig. 6.5 only the traction curves, for $F_N = 3062N$, have been plotted, since the figures for the other two loads are similar. In Fig. 6.5 for each sum velocity two traction curves have been calculated, one with the arctan-function (eq. (6.4)) and one with the tanh-function (eq. (6.6)). The curves with the tanh-function have been added, as an alternative. For the measurements of Johnson and Spence the elastic-plastic transition is too abrupt to be described with the tanh-function, but for other cases (different oil-material combinations) it may well be a better description. From Fig. 6.5 it can be concluded that more study on traction curves in the Mixed Lubrication regime is needed. In Schipper and Gelinck (1999) and this section a first onset is given to a quantitative description of the elastic-plastic behaviour of the BL-component of the friction.

### 6.5 Summary

In the current chapter the theory concerning traction curves is discussed. The difference between Strieber curves and traction curves is pointed out. For the asperity component of the Mixed Lubrication regime the shear stress around zero slip (pure rolling) is dealt with. For the reference case traction curves are calculated. It is shown that for traction curves especially the interval of slip between 0% and 1% is important.

From literature an example of traction curve in the Mixed Lubrication regime is taken. The measurements of Johnson and Spence (1991) were already used in Section 5.4.1.1 to plot Strieber curves, but Johnson and Spence measured traction curves and therefore a good comparison can be made with their measurements. From the comparison it can be concluded that the arctan-function performs better than the tanh-function for the measurements of Johnson and Spence. Further research and more measurements are needed on this subject.

In the current analysis only Gaussian summit distributions are discussed. In the next chapter the theory of Nayak will be discussed. He showed that for a Gaussian height distribution the summits do not always need to be Gaussian distributed. Some calculations using his theory will be done.

In the next chapter also the influence of wear on the Strieber curve will be discussed. A non-Gaussian distribution will be used to explain the effect of the change in topography of a surface on the Strieber curve.

In Chapter 5 the lubrication number $\mathcal{L}$ is used on the horizontal axis of the Strieber curve. In the next chapter the use of this number will be discussed as well.
Chapter 7

Discussion

7.1 Introduction

In this thesis it has been indicated that a few points need more discussion:

1. The distance between the mean lines of the surface heights and the mean line through the summits was taken constant. This distance, however, has a large influence on the Stribeck curve. A theory that does not depend on this distance is Nayak’s theory (Section 7.2).

2. Many surfaces do not have a Gaussian distribution. From a worn surface the highest summits are more or less removed and as a consequence the distribution is skewed. An example of such a skewed distribution is the M-inverted $\chi^2_n$ distribution (Section 7.3).

3. The lubrication number $\mathcal{L}$, used on the horizontal axis of the Stribeck curve, does not always describe the transitions fully adequately. An attempt to cope with this problem is presented in Section 7.4.

7.2 Nayak’s theory

An interesting point, that needs some discussion, is the influence of the distance between the mean line through the summits and the mean line through the surface heights. In Section 4.4 the theoretical background on this topic is given, while in Section 5.3.3 the influence on the Stribeck curve is shown. In that section it has been shown that this influence is quite large and that, thus, the Stribeck curve is quite susceptible to the distance chosen. The theoretical value, used in Chapter 5, is based on a value given by Whitehouse and Archard (1970).

In Appendix C.1 a short summary of the theory of Nayak (1971) is given. It is based on the consideration that surfaces with a Gaussian height distribution do not necessarily have a Gaussian distribution of the summits. In Appendix C.1 the bandwidth parameter $\alpha_N$ is introduced. This parameter depends on the spectral moments $m_0$, $m_2$ and $m_4$. These moments can be measured in several ways (Thomas,
1982). A complicating factor is that the moments depend on the sampling interval of the roughness measurements. It is therefore very important to give the sampling interval when publishing roughness data. The only parameter that does not vary very much with the sampling interval is \( \sigma \) (and thus \( m_0 \)). The parameter \( \alpha_N \) also depends on the sampling interval (Sayles and Thomas, 1979).

In Appendix C.1 the summit distribution for different values of \( \alpha_N \) has been given according to the theory of Nayak (Fig. C.1). From this figure it is clear that for lower values of \( \alpha_N \) the distance between the mean line through the surface heights and that through the summits is larger. In Table C.1 the different properties of the summit distributions are shown. In Appendix C.1 the contact model for the asperities is applied to the summit distributions of Nayak. Using this contact model (eq. C.15), and applying eq. C.17 for the product \( n \beta \sigma \), Striebeck curves are calculated (Fig. 7.1). From this figure it can be concluded that the value of \( \alpha_N \) hardly influences the Striebeck curve, being it that the curve for \( \alpha_N = 100 \) is somewhat steeper than the curves for lower values of \( \alpha_N \).

The curves are also shifted with respect to the curve calculated according to the reference case of Chapter 5 (Table 5.1). From Fig. 7.1 it can be concluded that the separation in the BL-regime is smaller for the reference case than for the

![Figure 7.1: Striebeck curves using Nayak's summits distributions for different values of \( \alpha_N \) and the reference case (ref). Numbers are \( \alpha_N \) values.](image-url)
calculations with the Nayak model. It should be noted that in Fig. 7.1 the separation is scaled on \( \sigma \) and not on \( \sigma_s \) as in Chapter 5. This is done, since the summit distribution (eq. (C.6)) is scaled on \( \sigma \). In the contact model \( \sigma \) is used as well. The fact the the separation is smaller for the reference case causes the shift of the Strubeck curves. A same shift could already be seen when the distance between the mean plane through the summits and the reference plane of the hydrodynamic component was varied (Section 5.3.3). It can thus be concluded once again that the distance between the mean line through the summits and that through the surface heights is very important.

### 7.3 Worn surfaces

In the calculations in Chapter 5, Gaussian surface distributions have been assumed. Actually, most surfaces have a Gaussian surface distribution after manufacturing. However, when machine parts have operated and the roughness is measured afterwards, significant wear can be observed in many cases. During operation, the highest asperities of the surfaces come into contact with asperities of the opposing surface and are worn off. In Appendix C.2 the M-inverted \( \chi^2 \) distribution is discussed (in short: \( \chi^2 \)-distribution). This distribution is more skewed for lower \( n_x \) values (Fig. C.5).

On the basis of the \( \chi^2 \)-distribution Strubeck curves have been calculated (Fig. 7.2). Although the \( \chi^2 \)-distribution is a distribution of surface heights, it has been used in this section as a distribution of the summits. In the lubrication number \( \mathcal{L} \) the surface roughness \( R_a \) has not been adapted. In eq. (4.17) the values of \( \sigma \) and \( \sigma_s \) are related, as well as \( \sigma \) and \( R_a \) in eq. (5.4). The last relation is based on a Gaussian distribution. The relations are not adapted here, although the distributions are not longer Gaussian. The CLA surface roughness \( R_a \) is not adapted in order to make a better comparison between the different distributions possible. Thus, for the calculations in Fig. 7.2 \( \sigma_s \) and \( R_a \) are constant.

The differences in the curves in Fig. 7.2 are small. The curves in the ML-regime are a little steeper for small values of \( n_x \). This can be expected from the distributions (Fig. C.5), since for low values of \( n_x \) the distribution is steeper (larger skewness), thus having a lower effective surface roughness, \( \sigma_s \). The transition from the BL-regime to the ML-regime is nearly unaffected by the different distributions. Also the separations in the BL-regime are almost equal.

It can thus be concluded that wear (resulting in a skewed distribution) does not have much influence on the Strubeck curve, although it must be kept in mind that the scale on the horizontal axis of Fig. 7.2 is logarithmic. Going from the Strubeck curve for \( n_x = 3 \) to that for \( n_x = 100 \), the BL/ML transition is shifted by a factor of 2.
Figure 7.2: Strubeck curves using the $\chi^2_n$ distribution for different values of $n_x$, other parameters as in Table 5.1.

The distribution for $n_x = 100$ approximates the Gaussian distribution. The most important conclusion from this section therefore is that the transition from ML to (E)HL shifts to lower values of $\mathcal{L}$, with unchanged value for $R_a$, i.e. to lower velocities.

Instead of non-Gaussian distributions, as the $\chi^2$-distributions, also real summit distributions can be used, see for instance Westeneng (1996). Westeneng made 3D surface measurements, using an interference microscope. From the histogram a function fit of the summit distribution was made, using a spline and with this fit the Strubeck curve was calculated, based on the model presented in this thesis.

### 7.4 Lubrication number

In Chapter 5 most Strubeck curves have been plotted with the lubrication number $\mathcal{L}$ on the horizontal axis. However, from the calculations in Chapter 5 it appears, that this lubrication number does not describe all transitions adequately. For practical purposes it is desirable to describe the transitions with one number. Especially with regard to changes in the load and the surface roughness considerable shifts in the Strubeck curves can be noticed. It is therefore desirable to formulate a new lubrication number to describe the transitions better.
7.4. Lubrication number

Figure 7.3: Coefficient of friction \( f \) as a function of the dimensionless separation \( \lambda_s \), varying the load. 1: \( F_N = 125 \text{N} \), 2: \( F_N = 250 \text{N} \), 3: \( F_N = 500 \text{N} \) (reference), 4: \( F_N = 1000 \text{N} \) and 5: \( F_N = 2000 \text{N} \), other parameters as given in Table 5.1.

An attempt to cope with this problem is presented in Fig. 7.3. In this figure the coefficient of friction is plotted as a function of the actual separation between the opposing surfaces for various loads (see Fig. 5.2). The most interesting feature of this plot is that the BL-regime is not visible anymore. It has degraded to one point on the curve. This is caused by the fact that the separation does not alter in the BL-regime. However, the transition from ML to (E)HL is clearly visible.

The fact that the separation is not known a priori, makes \( \lambda_s \) a parameter that is difficult to use in practice. The separation has to be calculated before the friction curve can be used. It is therefore not a serious option for lubrication parameter, because it is not an independent variable.

In the literature the separation \( (\lambda = h/\sigma) \) is often used on the horizontal axis (see e.g. Priest and Taylor (1998)), but in those cases the (E)HL-separation for smooth surfaces is used. This approach does not do justice to the BL-component of the contact. In Fig. 7.3 it is shown that the ML to (E)HL-regime is not adequately described by this method as well.

It appears to be very difficult, if not impossible, to describe the transitions
BL/ML and ML/(E)HL in one single number (one for each transition). Therefore, it may be necessary to develop a number for each transition. Further research is needed on this point.
Chapter 8

Conclusions and recommendations

In this chapter the conclusions and recommendations for further research, based on the work as described in this thesis, are summarized. Sections 8.1 through 8.4 deal with, respectively: the deformation model of Chapter 3, Chapters 4 and 5, i.e. predicting Striebeck/friction curves on the bases of the mixed lubrication model, the traction curves of Chapter 6 and, finally, the discussion of Chapter 7.

8.1 Deformation

Conclusions:

- A contact model for a line contact has been developed, using the concept of Greenwood and Williamson (1966) and Greenwood and Tripp (1967).

- This model has been implemented in a computer program, which simulates the asperity contact in a rough line contact, predicting the contact pressure, the contact width, the real area of contact and the number of asperities in contact. Function fits are given, in order to make the results more readily available.

- For high loads or relatively smooth surfaces the pressure distributions are similar to the Hertzian distribution.

- The surface roughness has a clear influence on the deformation of a line contact. This is demonstrated most clearly by the maximum central pressure in the contact. For high values of the parameter $\bar{\sigma}_a = \sigma_a R/b^2$ (i.e. for low loads) the maximum pressure increases almost linearly proportionally to the total applied load. For high loads the Hertzian relation ($p_c \propto F_N^{0.5}$) applies.

- An effective half contact width $b^*$ has been defined in analogy with the effective half contact width of Greenwood and Tripp (1967). For low loads the effective half width is almost independent of the load ($b^* \propto F_N^{0.026}$). For high values of the load the effective half width is equal to the Hertzian half width.
• A parameter $\alpha_l$, similar to the $\alpha_s$ parameter of Greenwood et al. (1984), is defined, which shows that for $\alpha_l$ values smaller than 0.05 the deviation from the Hertzian solution is less than about 11%.

• The real contact area and the real number of asperities in contact are almost proportional to the applied load ($A \propto F_N^{0.978}$, $N \propto F_N^{0.947}$) for low loads.

Recommendations:

• In Chapter 3 the Gaussian distribution has been used. Other distributions, as e.g. the Nayak’s (1971) distribution and the $\chi^2$-distribution, could be of practical interest as well (e.g. in the case of worn surfaces). Using the numerical program of Chapter 3 the effect of the height distributions on the contact pressure, the contact width, the real area of contact and the number of asperities in contact can be calculated fairly simple.

8.2 Stribeck curves

Conclusions:

• In Chapter 4 a model has been worked out, which describes the mixed lubrication regime, using the deformation model of Chapter 3 and the (Elasto) Hydrodynamic Lubrication theory.

• Using this model, in Chapter 5 Stribeck curves have been calculated for different parameters. The transitions are particularly affected by the standard deviation $\sigma_s$ of the summits and the distance $d_d$ between the mean plane through the summits and the mean plane through the surface heights.

• The influence of the load on the ML to BL transition is significant.

• The reduced elasticity modulus $E'$, the number of asperities $\bar{n}$ and the radius of the asperities $\beta$ have only a small influence on the Stribeck curve.

• The oil-viscosity parameters are especially important on the level of the friction in the (E)HL regime and, to a lesser extend, in the ML regime. The influence on the transitions is only limited.

• The calculations are compared with measurements of Johnson and Spence (1991), ter Haar (1996) and Lubbinge (1999). Good agreement is found.
8.3. Traction curves

Recommendations:

• All calculations assume isothermal conditions. Thermal effects as inlet shear heating are therefore also neglected. Including thermal effects will be one of the main priorities for improving the model.

• This thesis deals with line contacts. The model, however, can also be applied to point/elliptical contacts.

• In the current calculations the model is based on separate calculations of the asperity contacts and the fluid component, the results of which are combined afterwards. However, it is also possible and, perhaps, better to combine both in one numerical algorithm.

• In the current calculations the roughness is assumed to be isotropic. For some applications it can be interesting to take the roughness orientation into account.

• For highly loaded line contacts not many measurements are available. It is therefore desirable to perform more line contact measurements.

8.3 Traction curves

Conclusions:

• The mixed lubrication model can also be applied to traction curves.

• A model was described which enables the calculation of the friction in the asperity component around zero slip.

• The calculations are compared with measurements from Johnson and Spence (1991). Although the number of measurements is limited, good agreement is found.

Recommendations:

• More traction measurements in the mixed lubrication regime are needed to validate the calculations.

• The model describing the friction around zero slip is an attempt. More research is needed on this model, in particular, with respect to the transition from elastic to plastic behaviour.

• For traction curves only surfaces with a Gaussian summit distribution are studied. It would also be interesting to study the traction curves of non-Gaussian surfaces.
8.4 Discussion

Conclusions:

- In Chapter 7 the influence of the distance between the mean plane through the summits and the mean plane through the surface \( d_g \) was studied using the model of Nayak (1971). It was again concluded that this distance is very important.

- It was concluded that the value of the bandwidth parameter \( \alpha_N \) hardly influences the Stribeck curve. For high values of \( \alpha_N \) the Stribeck curve is somewhat steeper.

- Using the model of Nayak the Stribeck curves shifted to higher values of \( L \), with regard to the reference case.

- The most important conclusion concerning the section on worn surfaces is that the transition from ML to (E)HL shifts to lower values of the lubrication number \( L \) for decreasing values of \( n_x \) (for more skewed surfaces).

- The transition from BL to (E)HL is hardly influenced by the value of \( n_x \).

- It has been found that the lubrication number \( L \) does not describe the lubrication conditions adequately in all cases. The separation \( h \) is not a serious option for substitution, because it has to be calculated and, thus, is not an independent variable.

Recommendations:

- Some attention has already been paid to non-Gaussian summit distributions. Real summit distributions need more research.

- More research is needed to see whether an improved lubrication number can be formulated. In the ideal situation the transitions are described by a single lubrication number.
Appendix
Appendix A

Determination of roughness parameters

In Chapter 5 the calculations were performed using the roughness parameters $n$, $\beta$ and $\sigma_s$ from the reference table (Table 5.1). These parameters have to be obtained from roughness height measurements. There are several types of devices available for this purpose. The reader is referred to Thomas (1982). The measurements in Sections 5.4.1.2 and 5.4.2 were made using a device based on optical interference patterns. This device is described by Lubbinge (1994).

It must first be established how a summit is defined. Only when it has been established that a point is a summit, the roughness parameters can be calculated.

A.1 Density of asperities

There are several definitions of a summit. When measuring a profile, a summit is generally defined as a point which is higher than its two neighbours. For a surface measurement there are basically two possibilities:

- A point is higher than its four direct neighbours. This is generally called a peak.

- A point is higher than its eight neighbours, i.e. its four direct neighbours and the neighbours on the diagonals (see Fig. A.1). This is generally called a summit.

The analyses in Sections 5.4.1.2 and 5.4.2 are based on the latter definition. Since the demand on a point to be a summit is stricter than the demand for a point to be a peak, there are always more peaks than summits on a surface.

The position of the summits being known, the number of asperities can be counted. By dividing this number through the surface area of the measured surface, the density of summits $n$ becomes known.
A.2 Radius of asperities

The radius of the asperities $\beta$ can be calculated in several ways. Greenwood (1984) introduced the 3-point definition:

$$
\beta_x^{-1} = -\frac{z_{x-\Delta x,y} - 2z_{x,y} + z_{x+\Delta x,y}}{\Delta x^2},
$$

$$
\beta_y^{-1} = -\frac{z_{x,y-\Delta y} - 2z_{x,y} + z_{x,y+\Delta y}}{\Delta y^2},
$$

(A.1)

with $\beta_x$ and $\beta_y$ the radii in, respectively, the $x$- and the $y$-direction, $z_{x,y}$ the local surface height at location $(x, y)$ and $\Delta x$ and $\Delta y$ the pixel size or sample interval in, respectively, the $x$- and the $y$-direction. The radius of the summit $\beta$ can be found by combining the radii in the two perpendicular directions by applying:

$$
\beta = \frac{\beta_x + \beta_y}{2}.
$$

(A.2)

The value of $\beta$ used in the calculations is actually the average summit radius. This is simply calculated by adding all radii of the asperities and dividing the sum by the number of asperities on the measured surface. This definition has been used in Sections 5.4.1.2 and 5.4.2.

It should be noted that for practical situations this value might be a little too high, since it are the higher summits that are in contact with the opposite surface. In general, it can be said that with increasing height of the summit, the summit tends to be sharper, i.e. the radius of the summit is lower for high summits.

A.3 Standard deviation of the summits

The standard deviation of the summit heights $\sigma_s$ can now be calculated from the measured summit distribution:

$$
\sigma_s^2 = \frac{1}{i_{\text{max}}} \sum_{i} (s_i - d_d)^2,
$$

(A.3)
with \( s_i \) the height of summit \( i \), \( i_{\text{max}} \) the number of asperities on the surface and \( d_d \) the distance between the mean line of the summits and the mean line through all points of the surface.

### A.4 Combining roughness of two surfaces

The calculations in this thesis were done, with one rough surface in contact with a smooth surface. In practical cases a completely smooth surface does not exist. However, when one of the two contacting surfaces is much rougher compared to the smooth surface, the roughness on the smooth surface can be neglected.

However, when the roughnesses of two surfaces are of the same order of magnitude, such an assumption is no longer valid. Greenwood and Tripp (1970-71) described the contact between two rough surfaces, using the model of Greenwood and Williamson (1966). Greenwood and Tripp combined the roughness parameters of both surfaces in such a way that the model of one rough surface in contact with a smooth surface can be applied again.

The standard deviations of the summits are combined as follows:

\[
\sigma_s = \sqrt{\sigma_{s,1}^2 + \sigma_{s,2}^2},
\]  

(A.4)

with \( \sigma_{s,1} \) and \( \sigma_{s,2} \) the standard deviations of the summits of, respectively, surface 1 and surface 2, and \( \sigma_s \) the equivalent standard deviation of the summits. From this equation it can be concluded that, when the surfaces have equal roughness, \( \sigma_s \) is approximately 1.4 times the \( \sigma_s \) of a single surface. In case one of the surfaces is much rougher than the other surface, \( \sigma_s \) will be approximately the \( \sigma_s \) of the rough surface.

The radii of the summits of the two surfaces are combined by:

\[
\frac{1}{\bar{\beta}} = \frac{1}{\bar{\beta}_1} + \frac{1}{\bar{\beta}_2},
\]

(A.5)

with \( \beta_1 \) and \( \beta_2 \) the average radii of the summits of, respectively, surface 1 and surface 2. For surfaces of equal roughness, the equivalent radius \( \beta \) will thus be a factor of 2 lower than the radius of the asperities of a single surface. This is in line with the fact that for a rougher surface (the equivalent \( \sigma_s \) is higher than the single \( \sigma_s \)) the summits are sharper. When one of the surfaces is again much rougher, its summits will be sharper and thus the equivalent radius will be equal to radius of that surface.

In their analysis Greenwood and Tripp (1970-71) dealt with two surfaces with an equal density of asperities and used the value of a single rough surface as the value
for the equivalent density. It is quite difficult to give a fair value for the equivalent density. For surfaces of equal roughness the value of a single surface can be taken, as in the analysis of Greenwood and Tripp. When one surface is much rougher, then the density of that surface is taken. In case the differences are relatively small the best approximation is to take the density of the roughest surface. In that case both limiting conditions are satisfied.

Another value that has to be combined when two rough surface are in contact instead of one is the distance $d_d$ between the mean value of the summits and the mean value of the surface. In Fig. A.2 the two surfaces have been drawn as well as the distances that are important in this case.

![Diagram showing mean height of summits and mean level of surfaces](image)

**FIGURE A.2:** Distances in the contact between two rough surfaces.

From Fig. A.2 it can be concluded that the equivalent distance between the mean plane of the summits and the mean plane through the surface is simply the sum for the two planes:

$$d_d = d_{d,1} + d_{d,2},$$

(A.6)

with $d_{d,1}$ and $d_{d,2}$ the distance between the planes of, respectively, surface 1 and surface 2.
Appendix B

Hertzian theory

In Chapter 3 the deformation of rough line contacts is described. In this chapter the quantities are made dimensionless using the Hertzian parameters. In 1881 Hertz formulated his theory on the deformation of elastic bodies. It is based on smooth surfaces. In this appendix the most important quantities for line contacts are given.

Half Hertzian width:

\[ b = 2\sqrt{\frac{2F_NR}{\pi BE'}} \]  \hspace{1cm} (B.1)

Maximum Hertzian pressure:

\[ p_h = \frac{2F_N}{\pi b} = \sqrt{\frac{F_N E'}{2\pi BR}}. \]  \hspace{1cm} (B.2)

Average Hertzian pressure:

\[ p_{av} = \frac{\pi}{4} p_h = \frac{F_N}{2b} = \frac{1}{4} \sqrt{\frac{\pi F_N E'}{2BR}}. \]  \hspace{1cm} (B.3)

The pressure distribution for a Hertzian contact is given by:

\[ p(x) = p_h \sqrt{1 - \left(\frac{x}{b}\right)^2}. \]  \hspace{1cm} (B.4)

To be complete, the values of \( E' \) and \( R \) are given here as well. The value of the reduced elastic modulus is given by:

\[ \frac{2}{E'} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}, \]  \hspace{1cm} (B.5)

with \( E_1 \) and \( E_2 \) the elastic moduli and \( \nu_1 \) and \( \nu_2 \) Poisson’s ratios of surfaces 1 and 2 respectively. The reduced radius is defined by:

\[ \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \]  \hspace{1cm} (B.6)

with \( R_1 \) and \( R_2 \) the radii of cylinders 1 and 2, respectively.
Appendix C

Distributions

In this appendix two different distribution are described. In Section C.1 the summit distribution, corresponding to a Gaussian heights distribution, is described. Section C.2 describes the $\chi^2$ distribution, which can be used for non-Gaussian surfaces.

C.1 Summit distributions according to Nayak

In Chapter 3 the deformation has been calculated using the model of Greenwood and Williamson (1966), based on a Gaussian asperity distribution. This distribution has been used, since much surfaces have a Gaussian height distribution. However, when a surface has a Gaussian height distribution this does not automatically mean that the asperity distribution is Gaussian as well. In particular Nayak (1971) pointed out that the description of rough surfaces requires the study of several random variables. The work of Nayak is based on Longuet-Higgins (1957a, 1957b). The work of Longuet-Higgins originated from a theoretical study of ocean surfaces. Longuet-Higgins described the ocean surfaces with the theory of statistical geometry. The main difference between the work of Nayak and Longuet-Higgins is that the surfaces of the latter are time dependent.

The power of statistical geometry applies to cases in which the surface height, gradient and curvature are random. In his analysis of random surfaces Nayak therefore assumed that the surface heights, slopes and curvatures possess a multi-Gaussian probability density. Nayak used the result from the random process theory that a linear combination of Gaussian processes is in itself a Gaussian process. Thus, when the surface has a Gaussian height distribution, the surface slopes will have a Gaussian distribution, because the slopes are the difference between two heights. Similarly the second derivatives (the curvatures) will also be Gaussian. Therefore, in essence Nayak’s model only applies to surfaces with a Gaussian height distribution.

In random surface theory the moments of the profile and surface power spectral density function (PSDF) are of special importance. The moments $m$ of the PSDF
are defined as:
\[ m_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\omega_x, \omega_y) \omega_x^p \omega_y^q \, d\omega_x \, d\omega_y, \]  
(C.1)

with:
\( \omega \): wave-vector
\( \Phi \): power spectral density.

In the case of isotropic surfaces, the surface roughness is equal in all directions and therefore the double integral in eq. (C.1) simplifies to a single integral:
\[ m_n = \int_{-\infty}^{\infty} \Phi(\omega) \omega^n \, d\omega. \]  
(C.2)

From the definition of the standard deviation it follows that:
\[ m_0 = \sigma^2. \]  
(C.3)

According to Nayak (1971), for isotropic surfaces the uneven moments are zero.

Nayak defines a bandwidth parameter\(^1\) \( \alpha_N \):
\[ \alpha_N = \frac{m_0 m_4}{m_2^2}. \]  
(C.4)

This parameter \( \alpha_N \) is related to the width of the PSDF. For \( \alpha_N \rightarrow 1.5 \) the spectrum gets narrower; it has waves of approximately equal wavelength; as for \( \alpha_N \rightarrow \infty \) the spectrum gets wider, it has a large range of wavelengths. Longuet-Higgins (1957a) has shown that \( \alpha_N \geq 1.5 \) for a random, isotropic surface.

From the joint probability distribution for \( s(= h/\sigma) \) and its first and second derivatives in the \( x \) and \( y \) direction, Nayak (1971) derives a probability distribution for the summits. Integration of this gives the density of summits, \( n \):
\[ n = \frac{1}{6\pi \sqrt{3}} \frac{m_4}{m_2}. \]  
(C.5)

This quantity can be used to normalize the probability function of the summits\(^2\):
\[ p_{\text{sum}}(s) = \frac{\sqrt{3}}{2\pi} \left[ \sqrt{\frac{3}{2\alpha_N} - 3} \, s \, e^{-\frac{s^2}{\alpha_N}} + \frac{3\sqrt{2\pi}}{2\alpha_N} \, e^{-\frac{s^2}{\alpha_N}} \, (1 + \text{erf}(\beta_N))(s^2 - 1) \right. \]
\[ + \left. \sqrt{\frac{2\pi}{3\cdot(\alpha_N - 1)}} \, e^{-\frac{\alpha_N s^2}{2\pi}} \, (1 + \text{erf}(\gamma_N)) \right], \]  
(C.6)

---

\(^1\)In the article of Nayak this parameter is called \( \alpha \).

\(^2\)In both Nayak (1971) and Thomas (1982) the distribution function is not printed properly, however, the figures, resulting from the function, are correct.
C.1. Summit distributions according to Nayak

this function contains three auxiliary variables: \( C_1, \beta_N \) and \( \gamma_N \):

\[
C_1 = \frac{\alpha_N}{2\alpha_N - 3} \quad \beta_N = \sqrt{\frac{3}{2 \cdot (2\alpha_N - 3)}} \cdot s \quad \gamma_N = \sqrt{\frac{C_1}{2 \cdot (\alpha_N - 1)}} \cdot s.
\]

It is important to note that Nayak defined the dimensionless parameter \( s \) as the height divided by the standard deviation of the plane, thus \( s = h/\sigma \). This is in contrast with many other papers where instead of \( \sigma \) the standard deviation of the summits \( \sigma_s \) is used.

In the probability function of the summits two limiting cases can be observed:

1.

\[
\lim_{\alpha_N \to 1.5} p_{\text{sum}}(s) = \begin{cases} 
\sqrt{\frac{6}{\pi}} e^{-\frac{s^2}{2}} \left( s^2 - 1 + e^{-s^2} \right) & \text{if } s \geq 0; \\
0 & \text{if } s < 0.
\end{cases}
\]

(C.7)

2.

\[
\lim_{\alpha_N \to \infty} p_{\text{sum}}(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}.
\]

(C.8)

In eq. (C.8) the Gaussian distribution can again be recognized. In this case the distribution of the summits is equal to the distribution of all heights. The spectrum is so wide that the number of asperities has become very high and the surface is completely covered with summits.

The distribution function of the summits in eq. (C.6) has been plotted in Figure C.1. In this figure the two limiting case are plotted as well as four intermediate values of \( \alpha_N \). It is striking that for \( \alpha_N \to 1.5 \) all summits are on the positive site of the distribution. Obviously, for a roughness with a very narrow width, all the summits are above the mean plane through the summits.

Based on eq. (C.6) the distance between the mean plane through the summits and the mean plane through the surface can also be calculated using:

\[
d_d = \int_{-\infty}^{\infty} s p_{\text{sum}}(s) \, ds.
\]

(C.9)

This results for Nayak’s distribution in:

\[
d_d = \frac{4 \sigma}{\sqrt{\pi} \alpha_N}.
\]

(C.10)
FIGURE C.1: The summit distribution according to Nayak (1971) for different values of $\alpha_N$. 1: $\alpha_N \rightarrow 1.5$, 2: $\alpha_N = 2$, 3: $\alpha_N = 3$, 4: $\alpha_N = 5$, 5: $\alpha_N = 20$, 6: $\alpha_N \rightarrow \infty$.

Also the standard deviation through the summits can be calculating by:

$$\sigma_s^2 = \int_{-\infty}^{\infty} (s - d_d)^2 p_{\text{sum}}(s) \, ds,$$

and fitted:

$$\sigma_s = \sqrt{1 - \frac{16 + \pi - 3 \pi \sqrt{3}}{\pi \alpha_N} \cdot \sigma}.$$  

For small values of $\alpha_N$ the distributions have a slight skewness, with a kurtosis, slightly larger than three. The values of the different parameters are given in Table C.1. In this table the difference between the mean planes ($d_d$), the standard deviation of the summits ($\sigma_s$), the skewness of the summits ($Sk_s$) as well as the kurtosis of the summits ($Ku_s$) are given as function of $\alpha_N$. From Table C.1 it can be seen that the distribution of the summits have only deviates only slightly deviation from the Gaussian distribution. This fact can be explained by pointing out that the distribution of the surface heights itself is Gaussian. Therefore, especially for high $\alpha_N$, the parameters of the summits will take after the parameters of the surface (and therefore the parameters of a Gaussian roughness distribution).
TABLE C.1: Roughness parameters for the summit distribution according to Nayak (1971)

<table>
<thead>
<tr>
<th>$\alpha_N$</th>
<th>$\frac{d_d}{\sigma}$</th>
<th>$\frac{\sigma_s}{\sigma}$</th>
<th>$Sk_s$</th>
<th>$Ku_s$</th>
<th>$n\beta\sigma_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>1.843</td>
<td>0.643</td>
<td>0.469</td>
<td>3.187</td>
<td>0.016</td>
</tr>
<tr>
<td>2</td>
<td>1.596</td>
<td>0.743</td>
<td>0.190</td>
<td>3.056</td>
<td>0.021</td>
</tr>
<tr>
<td>3</td>
<td>1.303</td>
<td>0.837</td>
<td>0.072</td>
<td>3.015</td>
<td>0.030</td>
</tr>
<tr>
<td>5</td>
<td>1.009</td>
<td>0.906</td>
<td>0.026</td>
<td>3.004</td>
<td>0.041</td>
</tr>
<tr>
<td>20</td>
<td>0.505</td>
<td>0.977</td>
<td>0.003</td>
<td>3.000</td>
<td>0.089</td>
</tr>
<tr>
<td>100</td>
<td>0.226</td>
<td>0.996</td>
<td>$2 \cdot 10^{-4}$</td>
<td>3.000</td>
<td>0.204</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Nayak (1971) has made an estimate of the mean summit curvature as well. From his statistical approach he found that, in general, the highest summits (i.e. the summits which are found at the upper part of the surface distribution) have the largest curvatures. Intuitively this can be understood, since high summits are sharper and those summits will therefore have a low radius (which can be taken as being inverse to the curvature).

The result that higher summits have larger curvatures especially applies for low values of $\alpha_N$, although for higher values of $\alpha_N$ this effect can still be quite large. For $\alpha_N \to \infty$ the mean summit curvature is constant. It does not depend on the height of the summit:

$$\kappa = \frac{8}{3} \sqrt{\frac{m_4}{\pi}} = \beta^{-1}. \quad (C.13)$$

Using this relation it is possible to relate the product $n\beta\sigma_s$ of Greenwood and Williamson (1966) to $\alpha_N$. For this purpose eqns. (C.3), (C.4), (C.5), (C.12) and (C.13) are combined:

$$n\beta\sigma_s = \frac{\sqrt{\pi\alpha_N - 16 - \pi + 3\pi\sqrt{3}}}{16\pi \sqrt{3}}. \quad (C.14)$$

The values of $n\beta\sigma_s$ are given in Table C.1 also. Johnson et al. (1972) proved that this product is constant for many surfaces, i.e. it is relatively insensitive to the sampling frequency of a roughness measurement apparatus. For most surfaces, the value of $n\beta\sigma_s$ falls between 0.03 and 0.10. These values correspond to $\alpha_N$ values of approximately 3 and 25. It can thus be concluded that $\alpha_N$ values larger than 25 are unrealistic.
Contact

The summit distribution of eq. (C.6) has been scaled on the standard deviation of the surface heights. In the work of e.g. Greenwood and Williamson (1966) the summit distribution is scaled on the standard deviation of the summits. This fact is important when the summit distribution according to Nayak (1971) is implemented in a contact model. Using the theory of Nayak it is possible to use the joint probability distribution of both summit heights and curvatures to set up a contact model. For simplicity, however, it also possible to implement Nayak’s summits distribution in the model of Greenwood and Williamson:

\[ F_N = \frac{2}{3} n \beta \sigma \sqrt{\frac{\sigma}{\beta}} E' A_{\text{nom}} \int_{\frac{h}{\sigma}}^{\infty} \left( s - \frac{h}{\sigma} \right)^{\frac{3}{2}} p_{\text{sum}}(s) \, ds. \]  

(C.15)

Notice that the subscript on \( \sigma \), from eq. (3.8) has been dropped. The real area of contact can, of course, also be calculated (see also eq. (3.7)):

\[ A_r = \pi n \beta \sigma A_{\text{nom}} \int_{\frac{h}{\sigma}}^{\infty} \left( s - \frac{h}{\sigma} \right) p_{\text{sum}}(s) \, ds. \]  

(C.16)

Since \( n \beta \sigma \) depends only on \( \alpha_N \):

\[ n \beta \sigma = \frac{1}{16} \sqrt{\frac{\alpha_N}{3 \pi}}, \]  

(C.17)

the percentage of real contact \( A_r/A_{\text{nom}} \) depends only on \( \alpha_N \) and the dimensionless separation \( (\lambda = h/\sigma) \). For calculating the load carried by the asperities \( F_C \), the elasticity modulus \( E' \), the value of \( \sigma/\beta \) are important as well. From eq. (C.3) and (C.13) the latter quotient can be calculated as:

\[ \frac{\sigma}{\beta} = \frac{3 \sqrt{\pi}}{8} \sqrt{\frac{m_0}{m_4}}. \]  

(C.18)

Therefore, in order to calculate \( F_C \) not only \( \alpha_N \) is needed, but either the ratio of \( \sigma \) and \( \beta \), (or \( m_0 \) and \( \sigma \)) and \( m_4 \) (or \( \beta \)) as well (or at least a ratio of \( m_0 \) and \( m_4 \)). In Fig. C.2 the load carried by the asperities for \( \sigma/\beta = 10^{-3} \) and for different values of \( \alpha_N \) is presented as function of \( \lambda \). On the vertical axis of this figure the total load \( F_N \) is given, but for an asperity contact model this is equal to \( F_C \). From this figure it can be concluded that for positive values of \( \lambda \) the load carried by the asperities hardly depends on the bandwidth parameter \( \alpha_N \). Only for higher values of \( \alpha_N \) larger differences can be seen when \( \lambda \) is small or negative.

The same effect can be seen in Fig. C.3. In this figure the real contact area \( A_r \) is plotted as function of \( \lambda \) for different values of \( \alpha_N \). For positive values of \( \lambda \)
FIGURE C.2: The dimensionless load $W$ carried by the summit distribution according to Nayak (1971) as function of $\lambda$ for $\sigma/\beta = 10^{-3}$ and different values of $\alpha_N$. 1: $\alpha_N \to 1.5$, 2: $\alpha_N = 2$, 3: $\alpha_N = 3$, 4: $\alpha_N = 5$, 5: $\alpha_N = 20$, 6: $\alpha_N = 100$.

FIGURE C.3: The real contact area of contact $A_r$ as function of $\lambda$ using the summit distribution according to Nayak (1971) for different values of $\alpha_N (\sigma/\beta = 10^{-3})$. 1: $\alpha_N \to 1.5$, 2: $\alpha_N = 2$, 3: $\alpha_N = 3$, 4: $\alpha_N = 5$, 5: $\alpha_N = 20$, 6: $\alpha_N = 100$. 
there is not much difference between the different curves. This is caused by the fact that although the summit distribution curves are quite different, the load carried by the asperities differs not much. For low values of $\alpha_N$ the summit distributions are shifted to higher separations. However, for low values of $\alpha_N$ there are less summits available and therefore less summits can carry the load.

From Fig. C.3 it can also be concluded that the value of $A_r/A_{\text{nom}}$ is less than unity for all positive $\lambda$-values. The nominal contact area is set equal to the Hertzian contact area. For $\alpha_N = 100$ the ratio of real contact to apparent contact is 0.33. This ratio is acceptable for the Greenwood and Williamson contact model.

Figures C.2 and C.3 have a strong resemblance. This feature can also be recognized in Fig. C.4. In this figure the real contact area has been plotted as function of

$$W = \frac{F_N}{B E R}$$

**Figure C.4:** The real contact area of contact $A_r$ as function of the dimensionless load $W$ using the summit distribution according to Nayak (1971) for different values of $\alpha_N$ ($\sigma/\beta = 10^{-3}$). 1: $\alpha_N \to 1.5$, 2: $\alpha_N = 2$, 3: $\alpha_N = 3$, 4: $\alpha_N = 5$, 5: $\alpha_N = 20$, 6: $\alpha_N = 100$.

the applied load. For low loads the ratio $A_r/A_{\text{nom}}$ is the same for all values of $\alpha_N$. For larger loads the curves begin to diverge and the real contact area is relatively small for low values of $\alpha_N$. 

C.2. The $\chi^2$-height distribution

For low loads the slope of the curves in Fig. C.4 is approximately 0.5. Thus since the nominal contact area $A_{\text{nom}}$ is set equal to the Hertzian contact area $A_H$ and the Hertzian contact area is proportional to $F_N^{0.5}$, for low loads the real contact area is approximately proportional to the load.

The $\chi^2$-height distribution

The summit distribution, proposed by Nayak (1971) applies to Gaussian height distributions only. Some surface finishing operations such as lapping or polishing modify the summit distribution, flattening the high summits but leaving the low summits undisturbed. The same applies to (mild) wear. Therefore, Adler and Firman (1981) have proposed a non-gaussian model for rough surfaces, in which the surface heights follow a scaled $\chi^2$ distribution (in full: the M-inverted $\chi^2$-distribution). The model of Adler and Firman is intended for height distributions, not summit distributions.

The probability distribution of the scaled $\chi^2$ distribution is:

$$
\phi_{\chi^2}(s) = \begin{cases} 
\frac{\exp\left(-\frac{M-s}{2N}\right)}{2^{\frac{n_x}{2}} N^{\frac{n_x}{2}} \Gamma\left(\frac{n_x}{2}\right)} (M - s)^{\frac{n_x}{2} - 1} & \text{if } M - s \geq 0; \\
0 & \text{if } M - s < 0,
\end{cases}
$$

(C.19)

with

$$
M = \sqrt{\frac{n_x}{2}}
$$

and

$$
N = \frac{1}{\sqrt{2n_x}}.
$$

In eq. (C.19) $n_x$ stands for a parameter, representing the degrees of freedom (McCool, 1992); it is an integer. In McCool this parameter is called $n$, the subscript $\chi$ has been added to distinguish it from the density of summits. The parameter $M$ is used to shift the mean of the distribution to zero, thus the mean value of this distribution is zero. The skewness of the $\chi^2$ model is (McCool, 1992):

$$
Sk_{\chi^2} = -\sqrt{\frac{8}{n_x}},
$$

(C.20)

and the kurtosis can be proven to be:

$$
Ku_{\chi^2} = 3 + \frac{12}{n_x}.
$$

(C.21)
FIGURE C.5: Different height distributions according to the M-inverted $\chi^2_n$ distribution for different values of the $n_\chi$-parameter, Gaussian distribution (G).

In Fig. C.5 the $\chi^2$ distribution of eq. (C.19) has been plotted. The influence of the $n_\chi$-parameter on the distribution is clearly shown. For larger values of $n_\chi$ the Gaussian heights distribution is approached. For low values of $n_\chi$ the skewed character of the $\chi^2$-distribution becomes clear. The slope of the distribution function can be steep for low values of $n_\chi$. The Gaussian distribution has an infinite tail of heights in both directions (positive and negative side of the distribution). From eq. (C.19) it is clear that for larger positive values of $s$ the distribution shows no tail.

Although the $\chi^2$ distribution has been proposed for surface heights distribution, it can also be used for summit heights distributions. It was shown in Section C.1 that for larger values of $\alpha_N$ the summit distribution resembles the surface heights distribution. Adler and Firman (1981) have proven that this also applied for the $\chi^2$-distributions.
**Appendix D**

**The Ocvirk solution**

Often, in journal bearing theory use is made of the Ocvirk solution (Ocvirk, 1952). This solution of the Ocvirk equation describes the short-bearing asymptote in the Hydrodynamic Lubrication theory. For short bearings the Poiseuille term in the tangential direction can be neglected, whereas in the axial direction the term must be taken into account. Equation (2.1) can thus be written as:

\[
\frac{\partial}{\partial y} \left( h^3 \frac{\partial p}{\partial y} \right) = 6 \eta v^+ \frac{\partial (\psi h)}{\partial x},
\]

assuming the fluid incompressible and including the liquid fraction \( \psi \). This is a measure for the amount of cavitation in the fluid. If \( \psi = 1 \) the fluid at that particular place does not cavitate. When \( \psi < 1 \) the fluid cavitates.

Integration of eq. (D.1) gives:

\[
h^3 \frac{\partial p}{\partial y} = 6 \eta v^+ \frac{\partial (\psi h)}{\partial x} \left( y - \frac{1}{2} B \right),
\]

where the fact that the pressure distribution must be symmetric is used:

\[
\left. \frac{dp}{dy} \right|_{y=\frac{1}{2}B} = 0.
\]

Integrating for the second time gives, with the fact the pressure is zero at the edges:

\[
p = 3 \eta v^+ \frac{1}{h^3} \frac{d(\psi h)}{dx} y (y - B).
\]

At this point a distinction must be made between the area with cavitation and that without cavitation:

\[
p = 3 \eta v^+ \frac{1}{h^3} \frac{dh}{dx} y (y - B), \quad \psi = 1 \quad (-\infty \leq x \leq x_c), \tag{D.5}
\]

\[
p = 0, \quad \psi = \frac{h(x_c)}{h} \quad (x_c \leq x \leq \infty), \tag{D.6}
\]
where the cavitated area begins at \( x = x_c \). The value of \( x_c \) can be found from (Moes, 1997):

\[
\left. \frac{dh}{dx} \right|_{x=x_c} = 0 \quad \left. \frac{d^2h}{dx^2} \right|_{x=x_c} > 0. \tag{D.7}
\]

Since the film thickness between a cylinder and a plane can be approximated by:

\[
h = h_0 + \frac{x^2}{2R}, \tag{D.8}
\]

it can easily be seen that the cavitation area starts at \( x_c=0 \).

Now the pressure distribution is known, thus integrating in both tangential and axial direction gives:

\[
F = \frac{1}{4} \frac{\eta_0 v^+ B^3}{h_0^2}. \tag{D.9}
\]

The viscosity is assumed to be isoviscous.
References


References


Moës, H. (1997), Lubrication and Beyond, lecture notes 115531, University of Twente, Enschede, The Netherlands.


Ocvirk, F. W. (1952), Short Bearing Approximation for Full Journal Bearings, Technical note 2808, NACA.


References


