Multibody Syst Dyn (2009) 21: 99–122
DOI 10.1007/s11044-008-9130-6

A linearized input–output representation of flexible multibody systems for control synthesis

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Received: 19 December 2007 / Accepted: 18 September 2008 / Published online: 25 October 2008

Abstract In this paper, a linearized input–output representation of flexible multibody systems is proposed in which an arbitrary combination of positions, velocities, accelerations, and forces can be taken as input variables and as output variables. The formulation is based on a nonlinear finite element approach in which a multibody system is modeled as an assembly of rigid body elements interconnected by joint elements such as flexible hinges and beams. The proposed formulation is general in nature and can be applied for prototype modeling and control system analysis of mechatronic systems. Application of the theory is illustrated through a detailed model development of an active vibration isolation system for a metrology frame of a lithography machine.

Keywords Flexible multibody systems · Input–output equations · State-space equations · Linearized equations · Mechatronics

1 Introduction

For design of mechatronic systems, it is essential to make use of simple prototype models with a few degrees of freedom that capture only the relevant system dynamics. The multibody system approach is a well-suited method to model the dynamic behavior of the mechanical part of such systems. In this approach, the mechanical components are considered as rigid bodies that interact with each other through a variety of connections such as flexure hinges and flexible beams, also called flexures; see Fig. 1. Flexures are commonly used in

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mechatronic devices to guide motion because they provide accurate linear and rotational motion and do not suffer from clearance, friction, and wear. A mathematical description of the models is represented by the equations of motion derived from the multibody system approach.

For control synthesis, a control engineering based system representation is required. State-space equations are well suited to deal with Multiple Input Multiple Output (MiMo) systems. Linearization of the equations of motion is a well-known technique to obtain the state-space matrices of a linearized state-space formulation for flexible multibody systems. The derivation of the state-space matrices as described in control literature usually deals with forces and torques as inputs, and positions and velocities as outputs. Accelerations can be defined as output variables as well, although in that case a nonzero direct feed-through matrix is found. However, this representation is insufficient for modeling of, e.g., vibration isolation control systems in which sometimes (floor) accelerations are inputs and (internal) forces are considered as outputs. Kübler and Schiehlen [11] presented a general nonlinear state-space formulation of multibody systems with corresponding input and output quantities. The input and output vectors consist of (applied or constraint) forces and torques, and prescribed motion of bodies and coupling elements, consisting of positions, velocities, and accelerations. The determination of output quantities is discussed and the feed-through property is investigated in view of modular simulation by simulator coupling.

In this paper, a linearized state-space formulation for flexible multibody systems is proposed in which an arbitrary combination of positions, velocities, and accelerations of rheonomic degrees of freedom as well as forces and torques can be taken both as input variables and as output variables, according to the control problem being solved. The dynamic degrees of freedom and their time derivatives are the state variables. The input–output equations are derived from the linearized equations of kinematics and the linearized equations of reaction. The formulation is based on a nonlinear finite element description which was originally developed by Besseling [2] for stability and post-buckling analysis of frame structures. A key aspect of this finite element approach is the specification of independent deformation modes in the description of strain, stress, and associated stiffness of the elements. The deformation modes are defined by nonlinear functions of the nodal coordinates, valid for arbitrary large displacements and rotations, and include the specification of rigid body motions as displacements for which the element deformations are zero. Deformable elements are described by allowing nonzero deformations and specifying constitutive equations relating the deformation parameters and dual stress resultant parameters. They may express simple linear elastic behavior, but with these constitutive equations also active elements such as actuators can be modeled. Van der Werff [21] generalized this particular finite element approach to flexible mechanisms and multibody systems by introducing the concept of geometric transfer functions. The geometric transfer function formalism provides a systematic approach for generating nonlinear equations of motion and linearized multibody models in terms of generalized coordinates which are suitable for control system analysis. Furthermore, this finite element formulation accounts for geometric nonlinear effects such as geometric stiffening. The ease with which the deformability can be handled, leads to interesting possibilities of modeling flexible joint elements like beams and hinges [7]. Geometric nonlinear effects of flexible beam elements due to interaction of axial and bending deformations [3], and torsional and bending deformations [13] are naturally introduced in the finite element model. This considerably reduces the number of elements necessary to obtain a sufficiently accurate model which makes the analysis fast and reliable, even for large deflections. The latter is important because flexures often exhibit substantial deflections as these are used to accomplish large relative displacements and rotations. The method is applicable for flexible multibody systems as well as for flexible structures in which the system members experience only small

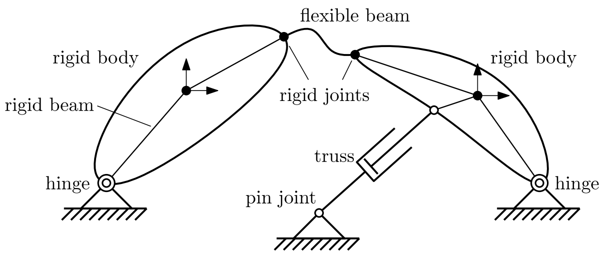


Fig. 1 Physical description of a flexible multibody system

displacement motions and elastic deformations with respect to an equilibrium position or a state of stationary motion.

The implementation of the derivation of the state-space matrices has been added to the program SPACAR [8] which has an interface to MATLAB. The state-space matrices derived in this paper can specify exactly how, e.g., an input position, an input velocity and an input acceleration in one nodal point affect any output. Yet the state-space equations do not take intrinsically into account the obvious derivative and integral relations between position, velocity and acceleration. In this respect, the input-output relation is expressed more conveniently by means of a transfer function which will be shown in an example. Subsequently, all MATLAB tools for the analysis of (linear) systems are available including graphical means like Bode plots and s-plane representations.

This paper is organized as follows. Section 2 defines some finite element notions and Sect. 3 briefly presents the concept of geometric transfer functions. In Sect. 4, the equations of motion are presented in terms of independent generalized coordinates and in Sect. 5 the nonlinear state equations are outlined. In Sect. 6, the equations of reaction are given and a solution method is presented. In Sect. 7, the linearized equations are derived analytically. The matrix coefficients of the linearized equations are identified and the functional dependencies of the coefficients on the nominal positions, velocities, accelerations, and forces are outlined. In Sect. 8, the linearized state-space formulation with corresponding input and output quantities is established. In Sect. 9, a method is described for computation of stationary and equilibrium solutions. In this way, the method will be applicable for flexible structures as well. In Sect. 10, the system's transfer function matrix is derived from the state space equations. Finally, in Sect. 11, two illustrative examples are discussed to demonstrate the applicability of the method for generation of linearized state-space equations with an arbitrary combination of inputs and outputs. First, an example is presented in which the linearized state-space equations and transfer functions of a simple linear mass-spring-damper system are calculated for the case when rheonomic degrees of freedom are chosen as inputs. In a second example, a detailed model development of an active vibration isolation system of a metrology frame suspension for a lithography machine is presented including a modal decoupling controller design.

2 Finite element representation of flexible multibody systems

In the presented finite element method, a flexible multibody system is modeled as an assembly of rigid body structures interconnected by joint elements such as flexible hinges and beams; see Fig. 1. A rigid body structure consists of a system of rigid beam elements that link the body's center of gravity with the interconnection points. The location of each element is described relative to a fixed inertial coordinate system by a set of nodal coordinates $\boldsymbol{x}^{(k)}$, valid for large displacements and rotations. Some coordinates may be Cartesian coordinates of the end nodes, while others describe the orientation of orthogonal base vectors or triads, rigidly attached to the element nodes. The superscript k is added to show that a specific element k is considered. With respect to some reference configuration of the element, the instantaneous values of the nodal coordinates represent a fixed number of deformation modes of the element. The deformation modes are specified by a vector of deformation parameters $\boldsymbol{e}^{(k)}$. The number of deformation parameters is equal to the number of nodal coordinates minus the number of degrees of freedom of the element as a rigid body. In the example of a spatial beam element, there are twelve independent nodal coordinates and six rigid body degrees of freedom, so that six independent deformation modes can be defined which describe the elongation, torsion and bending deformations of the element. Rigid body motions of the elements are characterized by displacements and rotations of the nodal points for which the deformations are zero. Since the deformation, modes are invariant for rigid body motions of the element they are described by nonlinear functions of the nodal coordinates. In this way, we can define for each element k a vector function $\boldsymbol{e}^{(k)} = \boldsymbol{\mathcal{D}}^{(k)}(\boldsymbol{x}^{(k)})$. For a detailed description of the deformation functions of the finite elements, the reader is referred to references [7, 13, 18].

3 Kinematic analysis

A multibody system can be built up with finite elements by letting them have nodal points in common. The assembly of finite elements is realized by defining a global vector $\boldsymbol{x}$ of nodal coordinates for the entire multibody system. The deformation functions of the elements constituting the multibody system can then be described in terms of the components of vector $\boldsymbol{x}$ yielding the nonlinear vector function

$$\boldsymbol{e} = \boldsymbol{\mathcal{D}}(\boldsymbol{x}), \tag{1}$$

which represents the basic equations for the kinematic analysis. Kinematic constraints can be introduced by putting conditions on the nodal coordinates $\boldsymbol{x}$ as well as by imposing conditions on the deformation parameters $\boldsymbol{e}$ which are all assumed to be holonomic. For instance, the condition for rigidity of the elements is characterized by displacements and rotations of the nodal points for which the deformations, denoted $\boldsymbol{e}^{(o)} = \boldsymbol{\mathcal{D}}^{(o)}(\boldsymbol{x})$, are zero. Since the positions and orientations of nodal points are described with respect to a global coordinate system, constraint conditions arising from support coordinates can be accounted for directly by prescribing the associated nodal coordinates, denoted $\boldsymbol{x}^{(o)}$, by a fixed value. An important notion in the kinematic and dynamic analysis of mechanical systems is that of degrees of freedom. The number of kinematic degrees of freedom is the smallest number of coordinates, denoted *ndof*, that describe, together with the fixed, time-independent kinematic constraints, the configuration of the multibody system. We call them independent or generalized coordinates which can be either absolute generalized coordinates, denoted $\boldsymbol{x}^{(m)}$, as well as relative generalized coordinates, denoted $\boldsymbol{e}^{(m)}$. Some of the relative generalized coordinates are associated with large relative displacements and rotations between element nodes, while others describe small elastic deformations of the element. In accordance with the above specified constraints and the choice of generalized coordinates, the vectors $\boldsymbol{x}$ and $\boldsymbol{e}$ can now be partitioned as

$$\boldsymbol{x} = \left[\boldsymbol{x}^{(o)T}, \boldsymbol{x}^{(c)T}, \boldsymbol{x}^{(m)T}\right]^T, \qquad \boldsymbol{e} = \left[\boldsymbol{e}^{(o)T}, \boldsymbol{e}^{(m)T}, \boldsymbol{e}^{(c)T}\right]^T, \tag{2}$$

where the superscript o denotes invariant nodal coordinates or deformations having a fixed prescribed value, the superscript c denotes dependent nodal coordinates or deformations and the superscript m denotes independent (or generalized) nodal coordinates or deformations. If the constraints are independent, the nodal coordinates and deformation parameters can be expressed as functions of the generalized coordinates $\boldsymbol{q} = (\boldsymbol{x}^{(m)}, \boldsymbol{e}^{(m)})$. The solution is expressed by means of the geometric transfer functions $\boldsymbol{\mathcal{F}}^{(x)}$ and $\boldsymbol{\mathcal{F}}^{(e)}$:

$$\boldsymbol{x} = \boldsymbol{\mathcal{F}}^{(x)}(\boldsymbol{q}), \qquad \boldsymbol{e} = \boldsymbol{\mathcal{F}}^{(e)}(\boldsymbol{q}), \qquad \boldsymbol{q} = (q_1, \ldots, q_{\mathrm{ndof}}), \tag{3}$$

where ndof is the total number of kinematic degrees of freedom. The velocity vectors $\dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{e}}$ can be calculated from (3) as[1]

$$\dot{\boldsymbol{x}} = \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(x)} \dot{\boldsymbol{q}}, \tag{4a}$$

$$\dot{\boldsymbol{e}} = \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(e)} \dot{\boldsymbol{q}}, \tag{4b}$$

or in partitioned form:

$$\dot{\boldsymbol{x}}^{(o)} = \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(x,o)} \dot{\boldsymbol{q}}, \qquad \dot{\boldsymbol{e}}^{(o)} = \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(e,o)} \dot{\boldsymbol{q}}, \tag{5a}$$

$$\dot{\boldsymbol{x}}^{(c)} = \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(x,c)} \dot{\boldsymbol{q}}, \qquad \dot{\boldsymbol{e}}^{(m)} = \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(e,m)} \dot{\boldsymbol{q}}, \tag{5b}$$

$$\dot{\boldsymbol{x}}^{(m)} = \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(x,m)} \dot{\boldsymbol{q}}, \qquad \dot{\boldsymbol{e}}^{(c)} = \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(e,c)} \dot{\boldsymbol{q}}, \tag{5c}$$

where the derivative functions $\mathbf{D}_q \boldsymbol{\mathcal{F}}^{(x)}$ and $\mathbf{D}_q \boldsymbol{\mathcal{F}}^{(e)}$ are called the first-order geometric transfer functions. By definition we have:

$$\mathbf{D}_q \boldsymbol{\mathcal{F}}^{(x,o)} = \left[\boldsymbol{O}^{(x^o \times x^m)}, \boldsymbol{O}^{(x^o \times e^m)}\right], \qquad \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(e,o)} = \left[\boldsymbol{O}^{(e^o \times x^m)}, \boldsymbol{O}^{(e^o \times e^m)}\right], \tag{6a}$$

$$\mathbf{D}_q \boldsymbol{\mathcal{F}}^{(x,m)} = \left[\boldsymbol{I}^{(x^m \times x^m)}, \boldsymbol{O}^{(x^m \times e^m)}\right], \qquad \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(e,m)} = \left[\boldsymbol{O}^{(e^m \times x^m)}, \boldsymbol{I}^{(e^m \times e^m)}\right]. \tag{6b}$$

The acceleration vectors $\ddot{\boldsymbol{x}}^{(c)}$ and $\ddot{\boldsymbol{e}}^{(c)}$ are obtained by differentiating (4) again with respect to time

$$\ddot{\boldsymbol{x}}^{(c)} = \mathbf{D} \boldsymbol{\mathcal{F}}^{(x,c)} \ddot{\boldsymbol{q}} + \mathbf{D}^2 \boldsymbol{\mathcal{F}}^{(x,c)} \dot{\boldsymbol{q}} \dot{\boldsymbol{q}}, \tag{7a}$$

$$\ddot{\boldsymbol{e}}^{(c)} = \mathbf{D} \boldsymbol{\mathcal{F}}^{(e,c)} \ddot{\boldsymbol{q}} + \mathbf{D}^2 \boldsymbol{\mathcal{F}}^{(e,c)} \dot{\boldsymbol{q}} \dot{\boldsymbol{q}}, \tag{7b}$$

where $\mathbf{D}^2 \boldsymbol{\mathcal{F}}^{(x,c)}$ and $\mathbf{D}^2 \boldsymbol{\mathcal{F}}^{(e,c)}$ are the nonzero parts of the second order geometric transfer functions $\mathbf{D}^2 \boldsymbol{\mathcal{F}}^{(x)}$ and $\mathbf{D}^2 \boldsymbol{\mathcal{F}}^{(e)}$. Generally, the geometric transfer functions cannot be calculated explicitly, but have to be determined by solving (1) numerically in an iterative way. For a method for computing the unknown parts $\boldsymbol{\mathcal{F}}^{(x,c)}$, $\boldsymbol{\mathcal{F}}^{(e,c)}$ and their derivatives $\mathbf{D}_q \boldsymbol{\mathcal{F}}^{(x,c)}$, $\mathbf{D}^2 \boldsymbol{\mathcal{F}}^{(x,c)}$, $\mathbf{D}_q \boldsymbol{\mathcal{F}}^{(e,c)}$, we refer to the literature [7].

4 Equations of motion

By means of the first and second order geometric transfer functions, the equations of motion are expressed in terms of the kinematic degrees of freedom $\boldsymbol{q}$:

$$\bar{\boldsymbol{M}}(\boldsymbol{q}) \ddot{\boldsymbol{q}} = \mathbf{D} \boldsymbol{\mathcal{F}}^{(x)T} \left(\boldsymbol{f} - \boldsymbol{M} \mathbf{D}^2 \boldsymbol{\mathcal{F}}^{(x,c)} \dot{\boldsymbol{q}} \dot{\boldsymbol{q}}\right) - \mathbf{D} \boldsymbol{\mathcal{F}}^{(e)T} \boldsymbol{\sigma}, \tag{8}$$

[1] The subscript q will be omitted if there is no possibility for confusion.

where

$$\bar{\boldsymbol{M}} = \mathbf{D}\mathcal{F}^{(x)T}\boldsymbol{M}^{(x)}, \quad \text{with } \boldsymbol{M}^{(x)} = \boldsymbol{M}\mathbf{D}\mathcal{F}^{(x)}, \tag{9}$$

is the system mass matrix and $\boldsymbol{M}(\boldsymbol{x})$ is the global mass matrix, obtained by assembling the lumped and consistent element mass matrices [12, 17]. The vector $\boldsymbol{f}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$ of nodal forces also includes the velocity dependent inertia forces $\boldsymbol{f}_{\text{in}}(\boldsymbol{x}, \dot{\boldsymbol{x}})$. Furthermore, the loading state of each element is described by a vector of generalized stress resultants. These vectors are assembled in the global vector $\boldsymbol{\sigma}$ which is dual to $\dot{\boldsymbol{e}}$. The terms including the vectors $\boldsymbol{f}$ and $\boldsymbol{\sigma}$ can be expanded as

$$\mathbf{D}\mathcal{F}^{(x)T}\boldsymbol{f} = \mathbf{D}\mathcal{F}^{(x,c)T}\boldsymbol{f}^{(c)} + \mathbf{D}\mathcal{F}^{(x,m)T}\boldsymbol{f}^{(m)}, \tag{10a}$$

$$\mathbf{D}\mathcal{F}^{(e)T}\boldsymbol{\sigma} = \mathbf{D}\mathcal{F}^{(e,m)T}\boldsymbol{\sigma}^{(m)} + \mathbf{D}\mathcal{F}^{(e,c)T}\boldsymbol{\sigma}^{(c)}, \tag{10b}$$

where $\boldsymbol{f}^{(c)}$ and $\boldsymbol{f}^{(m)}$ represent externally applied nodal forces and driving forces dual to $\dot{\boldsymbol{x}}^{(c)}$ and $\dot{\boldsymbol{x}}^{(m)}$, respectively. Generalized stress resultants $\boldsymbol{\sigma}^{(m)}, \boldsymbol{\sigma}^{(c)}$ of elastic elements are determined from the constitutive equation

$$\begin{bmatrix} \boldsymbol{\sigma}^{(m)} \\ \boldsymbol{\sigma}^{(c)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}_a^{(m)} \\ \boldsymbol{\sigma}_a^{(c)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{S}^{(m,m)} & \boldsymbol{S}^{(m,c)} \\ \boldsymbol{S}^{(c,m)} & \boldsymbol{S}^{(c,c)} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}^{(m)} \\ \boldsymbol{e}^{(c)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{S}_d^{(m,m)} & \boldsymbol{S}_d^{(m,c)} \\ \boldsymbol{S}_d^{(c,m)} & \boldsymbol{S}_d^{(c,c)} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{e}}^{(m)} \\ \dot{\boldsymbol{e}}^{(c)} \end{bmatrix}, \tag{11}$$

where $\boldsymbol{S}^{(m,m)}$, $\boldsymbol{S}^{(m,c)}$ and $\boldsymbol{S}^{(c,c)}$ are symmetric matrices containing the elastic coefficients and $\boldsymbol{S}_d^{(m,m)}$, $\boldsymbol{S}_d^{(m,c)}$ and $\boldsymbol{D}_d^{(c,c)}$ are symmetric matrices containing the viscous damping coefficients. Driving forces and torques of built-in driving actuators are represented by the vectors $\boldsymbol{\sigma}_a^{(m)}$ and $\boldsymbol{\sigma}_a^{(c)}$. These are characterized by special constitutive equations describing the behavior of the actuators. In this way it is possible to study the dynamics of active multibody systems. The reaction forces $\boldsymbol{f}^{(o)}$ and generalized stress resultants $\boldsymbol{\sigma}^{(o)}$ associated with rigid elements are eliminated due to the orthogonality $\mathbf{D}\mathcal{F}^{(x,o)}\boldsymbol{f}^{(o)} = \mathbf{0}$ and $\mathbf{D}\mathcal{F}^{(e,o)T}\boldsymbol{\sigma}^{(o)} = \mathbf{0}$. Consequently, the equations of motion (8) represent *ndof* ordinary differential equations.

5 State equations

In order to transform the equations of motion to the more general state variable form the vector of generalized coordinates $\boldsymbol{q}$ is partitioned as

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}^d \\ \boldsymbol{q}^r \end{bmatrix}, \tag{12}$$

where $\boldsymbol{q}^d$ is the vector of dynamic degrees of freedom and $\boldsymbol{q}^r$ is the vector of rheonomic degrees of freedom which are known explicit functions of time representing the rheonomic constraints [5]. Substitution of (12) into (8) yields the reduced equations of motion

$$\bar{\boldsymbol{M}}_{dd}(\boldsymbol{q})\ddot{\boldsymbol{q}}^d = \bar{\boldsymbol{f}}_d - \bar{\boldsymbol{M}}_{dr}\ddot{\boldsymbol{q}}^r, \tag{13}$$

where

$$\bar{\boldsymbol{M}}_{dd} = \mathbf{D}_{q^d}\mathcal{F}^{(x)T}\boldsymbol{M}\mathbf{D}_{q^d}\mathcal{F}^{(x)}, \tag{14a}$$

$$\bar{\boldsymbol{M}}_{dr} = \mathbf{D}_{q^d}\mathcal{F}^{(x)T}\boldsymbol{M}\mathbf{D}_{q^r}\mathcal{F}^{(x)}, \tag{14b}$$

are reduced system mass matrices and

$$\bar{\boldsymbol{f}}_d = \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)T}\big(\boldsymbol{f} - \boldsymbol{M}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x,c)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}\big) - \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e)T}\boldsymbol{\sigma}, \tag{15}$$

and again

$$\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)T}\boldsymbol{f} = \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x,c)T}\boldsymbol{f}^{(c)} + \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x,m)T}\boldsymbol{f}^{(m)}, \tag{16a}$$

$$\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e)T}\boldsymbol{\sigma} = \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e,m)T}\boldsymbol{\sigma}^{(m)} + \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e,c)T}\boldsymbol{\sigma}^{(c)}, \tag{16b}$$

where $\boldsymbol{\sigma}^{(m)}$ and $\boldsymbol{\sigma}^{(c)}$ are defined in (11). Since matrix $\bar{\boldsymbol{M}}_{dd}$ is symmetric and positive definite, (13) can be written in a nonlinear state variable form as

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{q}^d\\ \dot{\boldsymbol{q}}^d\end{bmatrix} = \begin{bmatrix}\dot{\boldsymbol{q}}^d\\ \bar{\boldsymbol{M}}_{dd}^{-1}(\bar{\boldsymbol{f}}_d - \bar{\boldsymbol{M}}_{dr}\ddot{\boldsymbol{q}}^r)\end{bmatrix}. \tag{17}$$

The vector $[\boldsymbol{q}^{dT}, \dot{\boldsymbol{q}}^{dT}]^T$, hereafter denoted $\boldsymbol{z}$, is called the state vector, consisting of the vector of dynamic degrees of freedom $\boldsymbol{q}^d$ and its time derivative $\dot{\boldsymbol{q}}^d$.

6 Equations of reaction

The unknown stress resultants and reaction forces are computed from the equations of reaction

$$(\mathbf{D}_x\boldsymbol{\mathcal{D}})^T\boldsymbol{\sigma} = \boldsymbol{f} - \boldsymbol{M}\ddot{\boldsymbol{x}}, \tag{18}$$

where differentiation operator $\mathbf{D}_x$ represents partial differentiation with respect to the nodal coordinate $\boldsymbol{x}$. In order to solve these equations, the nodal force vector $\boldsymbol{f}$ and the vector of generalized stress resultants $\boldsymbol{\sigma}$ will be partitioned in accordance with (2) as

$$\boldsymbol{f} = \big[\boldsymbol{f}^{(o)T},\ \boldsymbol{f}^{(c)T},\ \boldsymbol{f}^{(m)T}\big]^T \quad \text{and} \quad \boldsymbol{\sigma} = \big[\boldsymbol{\sigma}^{(o)T},\ \boldsymbol{\sigma}^{(m)T},\ \boldsymbol{\sigma}^{(c)T}\big]^T. \tag{19}$$

With these partitions, (18) can be written as:

$$\begin{bmatrix}(\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(m)})^T & (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(c)})^T\\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T & (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)})^T\\ (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(m)})^T & (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(c)})^T\end{bmatrix}\begin{bmatrix}\boldsymbol{\sigma}^{(o)}\\ \boldsymbol{\sigma}^{(m)}\\ \boldsymbol{\sigma}^{(c)}\end{bmatrix} = \begin{bmatrix}\boldsymbol{f}^{(o)} & -\boldsymbol{M}^{(o,c)}\ddot{\boldsymbol{x}}^{(c)} & -\boldsymbol{M}^{(o,m)}\ddot{\boldsymbol{x}}^{(m)}\\ \boldsymbol{f}^{(c)} & -\boldsymbol{M}^{(c,c)}\ddot{\boldsymbol{x}}^{(c)} & -\boldsymbol{M}^{(c,m)}\ddot{\boldsymbol{x}}^{(m)}\\ \boldsymbol{f}^{(m)} & -\boldsymbol{M}^{(m,c)}\ddot{\boldsymbol{x}}^{(c)} & -\boldsymbol{M}^{(m,m)}\ddot{\boldsymbol{x}}^{(m)}\end{bmatrix}, \tag{20}$$

where, the superscripts o, c and m combined with the operator $\mathbf{D}$ represent partial differentiation with respect to the corresponding nodal coordinates $\boldsymbol{x}^{(o)}$, $\boldsymbol{x}^{(c)}$, and $\boldsymbol{x}^{(m)}$.

The partitioned matrix $[(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T,\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T]$ is a square matrix and if in addition the multibody system is not in a singular configuration, it is nonsingular and $\boldsymbol{\sigma}^{(o)}$, $\boldsymbol{\sigma}^{(m)}$ can be computed by

$$\begin{bmatrix}\boldsymbol{\sigma}^{(o)}\\ \boldsymbol{\sigma}^{(m)}\end{bmatrix} = \tilde{\boldsymbol{D}}_1\big[\boldsymbol{f}^{(c)} - \boldsymbol{M}^{(c,c)}\ddot{\boldsymbol{x}}^{(c)} - \boldsymbol{M}^{(c,m)}\ddot{\boldsymbol{x}}^{(m)} - (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)})^T\boldsymbol{\sigma}^{(c)}\big], \tag{21}$$

where

$$\tilde{\boldsymbol{D}}_1 = \left[(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T,\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T\right]^{-1}. \tag{22}$$

Vector $\boldsymbol{\sigma}^{(c)}$ is defined by (11). Finally, from (20), the vector of reaction forces $\boldsymbol{f}^{(o)}$ and the vector of driving forces $\boldsymbol{f}^{(m)}$ are determined.

7 Linearized equations

The linearized equations of motion are of interest from both analysis and control point of view. For analysis, they enable us to study the natural frequencies and corresponding eigenmodes as well as stability of motion and static stability (buckling) of flexible multibody systems. For control system design, the linearized equations provide a basis for developing the linearized state-space equations suitable for control system design. The output quantities are derived from the linearized equations of kinematics, the linearized equations of motion and the linearized equations of reaction.

7.1 Linearized equations of kinematics

If variations of quantities are denoted by the prefix δ, the linear approximations of (3), (4), and (7a) are

$$\delta\boldsymbol{x} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\delta\boldsymbol{q}, \tag{23a}$$

$$\delta\dot{\boldsymbol{x}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\delta\dot{\boldsymbol{q}} + \left(\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\right)\delta\boldsymbol{q}, \tag{23b}$$

$$\delta\ddot{\boldsymbol{x}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\delta\ddot{\boldsymbol{q}} + 2\left(\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\right)\delta\dot{\boldsymbol{q}} + \left(\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\ddot{\boldsymbol{q}} + \mathbf{D}^3\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}\right)\delta\boldsymbol{q}, \tag{23c}$$

and

$$\delta\boldsymbol{e} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(e)}\delta\boldsymbol{q}, \tag{24a}$$

$$\delta\dot{\boldsymbol{e}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(e)}\delta\dot{\boldsymbol{q}} + \left(\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(e)}\dot{\boldsymbol{q}}\right)\delta\boldsymbol{q}, \tag{24b}$$

where $\mathbf{D}^3\boldsymbol{\mathcal{F}}^{(x)}$ is the third-order geometric transfer function [9]. Equations (23) and (24) are used later on in Sect. 8.2 to derive the kinematic part of the output equations (58).

7.2 Linearized equations of motion

Expanding the equations of motion (see (8)) in their Taylor series expansion and disregarding second and higher order terms yields with (23) and (24), the linearized equations of motion

$$\bar{\boldsymbol{M}}\delta\ddot{\boldsymbol{q}} + (\bar{\boldsymbol{C}} + \bar{\boldsymbol{D}})\delta\dot{\boldsymbol{q}} + (\bar{\boldsymbol{K}} + \bar{\boldsymbol{N}} + \bar{\boldsymbol{G}})\delta\boldsymbol{q} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}\delta\boldsymbol{f} - \mathbf{D}\boldsymbol{\mathcal{F}}^{(e)T}\delta\boldsymbol{\sigma}_a, \tag{25}$$

where

$$\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}\delta\boldsymbol{f} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x,c)T}\delta\boldsymbol{f}^{(c)} + \mathbf{D}\boldsymbol{\mathcal{F}}^{(x,m)T}\delta\boldsymbol{f}^{(m)}, \tag{26a}$$

$$\mathbf{D}\boldsymbol{\mathcal{F}}^{(e)T}\delta\boldsymbol{\sigma}_a = \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,m)T}\delta\boldsymbol{\sigma}_a^{(m)} + \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,c)T}\delta\boldsymbol{\sigma}_a^{(c)}, \tag{26b}$$

and $\bar{\boldsymbol{M}}$ is the system mass matrix from (9), $\bar{\boldsymbol{C}}$ is the velocity sensitive matrix, $\bar{\boldsymbol{D}}$ denotes the damping matrix, $\bar{\boldsymbol{K}}$ denotes the structural stiffness matrix and $\bar{\boldsymbol{N}}$, $\bar{\boldsymbol{G}}$ are the dynamic stiffening matrix and the geometric stiffening matrix, respectively. These matrices are calculated by [1]:

$$\bar{\boldsymbol{K}} = \begin{bmatrix} \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,m)T} & \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,c)T} \end{bmatrix} \begin{bmatrix} \boldsymbol{S}^{(m,m)} & \boldsymbol{S}^{(m,c)} \\ \boldsymbol{S}^{(c,m)} & \boldsymbol{S}^{(c,c)} \end{bmatrix} \begin{bmatrix} \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,m)} \\ \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,c)} \end{bmatrix}, \tag{27}$$

$$\bar{\boldsymbol{D}} = \begin{bmatrix} \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,m)T} & \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,c)T} \end{bmatrix} \begin{bmatrix} \boldsymbol{S}_d^{(m,m)} & \boldsymbol{S}_d^{(m,c)} \\ \boldsymbol{S}_d^{(c,m)} & \boldsymbol{S}_d^{(c,c)} \end{bmatrix} \begin{bmatrix} \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,m)} \\ \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,c)} \end{bmatrix}, \tag{28}$$

$$\bar{\boldsymbol{C}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}\boldsymbol{C}^{(x)}, \tag{29}$$

where

$$\boldsymbol{C}^{(x)} = (\mathbf{D}_{\dot{x}}\boldsymbol{f}_{\text{in}})\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)} + 2\boldsymbol{M}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x,c)}\dot{\boldsymbol{q}}, \tag{30}$$

and

$$\bar{\boldsymbol{N}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}\boldsymbol{N}^{(x)} + \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,m)T}\boldsymbol{N}^{(e,m)} + \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,c)T}\boldsymbol{N}^{(e,c)}, \tag{31}$$

with

$$\boldsymbol{N}^{(x)} = \mathbf{D}_x(\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{f}_{\text{in}})\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)} + (\mathbf{D}_{\dot{x}}\boldsymbol{f}_{\text{in}})\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x,c)}\dot{\boldsymbol{q}} + \boldsymbol{M}(\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x,c)}\ddot{\boldsymbol{q}} + \mathbf{D}^3\boldsymbol{\mathcal{F}}^{(x,c)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}), \tag{32}$$

$$\boldsymbol{N}^{(e,m)} = \boldsymbol{S}_d^{(m,c)}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(e,c)}\dot{\boldsymbol{q}}, \tag{33}$$

$$\boldsymbol{N}^{(e,c)} = \boldsymbol{S}_d^{(c,c)}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(e,c)}\dot{\boldsymbol{q}}, \tag{34}$$

where $\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(e,c)}$ is defined in (7b). Finally, we have,

$$\bar{\boldsymbol{G}} = \mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x,c)T}(\boldsymbol{M}\ddot{\boldsymbol{x}} - \boldsymbol{f}) - \mathbf{D}^2\boldsymbol{\mathcal{F}}^{(e,c)T}\boldsymbol{\sigma}^{(c)}. \tag{35}$$

The matrix coefficients in (25) to (35) are functions of time since they depend on the nominal positions $\boldsymbol{q}$, velocities $\dot{\boldsymbol{q}}$, and accelerations $\ddot{\boldsymbol{q}}$ of the system. They are derived analytically and are evaluated numerically. $\bar{\boldsymbol{M}}$, $\bar{\boldsymbol{D}}$, $\bar{\boldsymbol{K}}$, and $\bar{\boldsymbol{G}}$ are symmetric matrices, but $\bar{\boldsymbol{C}}$ and $\bar{\boldsymbol{N}}$ need not. The vectors $\delta\boldsymbol{f}^{(c)}(t)$, $\delta\boldsymbol{f}^{(m)}(t)$ and $\delta\boldsymbol{\sigma}_a^{(m)}(t)$, $\delta\boldsymbol{\sigma}_a^{(c)}(t)$ at the right-hand sides of (26) represent time-varying perturbations of nodal forces and torques and internal driving forces and torques applied to the multibody system. Note that the conservative parts of $\delta\boldsymbol{f}$ and $\delta\boldsymbol{\sigma}$ are included in the matrices $\bar{\boldsymbol{C}}$, $\bar{\boldsymbol{N}}$, $\bar{\boldsymbol{G}}$, and $\bar{\boldsymbol{D}}$, $\bar{\boldsymbol{K}}$ [14].

7.3 Linearized equations of reaction

Expanding the equations of reaction (18) in their Taylor series expansion and disregarding second and higher order terms yields

$$(\mathbf{D}_x\boldsymbol{\mathcal{D}})^T\delta\boldsymbol{\sigma} + \big((\mathbf{D}_x^2\boldsymbol{\mathcal{D}})^T\boldsymbol{\sigma}\big)\delta\boldsymbol{x} = \delta\boldsymbol{f} + (\mathbf{D}_x\boldsymbol{f}_{\text{in}})\delta\boldsymbol{x} + (\mathbf{D}_{\dot{x}}\boldsymbol{f}_{\text{in}})\delta\dot{\boldsymbol{x}} - \mathbf{D}_x(\boldsymbol{M}\ddot{\boldsymbol{x}})\delta\boldsymbol{x} - \boldsymbol{M}\delta\ddot{\boldsymbol{x}}. \tag{36}$$

Substitution of (23) yields the linearized equations of reaction

$$(\mathbf{D}_x\boldsymbol{\mathcal{D}})^T\delta\boldsymbol{\sigma} = \delta\boldsymbol{f} - \boldsymbol{M}^{(x)}\delta\ddot{\boldsymbol{q}} - \boldsymbol{C}^{(x)}\delta\dot{\boldsymbol{q}} - \big(\boldsymbol{N}^{(x)} + \boldsymbol{G}^{(x)}\big)\delta\boldsymbol{q}, \tag{37}$$

where

$$\boldsymbol{G}^{(x)} = \boldsymbol{G}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}, \tag{38}$$

in which

$$\boldsymbol{G} = \left(\mathbf{D}_x^2\boldsymbol{\mathcal{D}}\right)^T\boldsymbol{\sigma}, \tag{39}$$

is the geometrical stiffness matrix due to the reference stresses $\boldsymbol{\sigma}$ [2, 3]. The coefficient matrices $\boldsymbol{M}^{(x)}$, $\boldsymbol{C}^{(x)}$ and $\boldsymbol{N}^{(x)}$ are defined by (9), (30), and (32), respectively. With the vector partitions of (19), (37) can be written as

$$\begin{bmatrix} (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(m)})^T \\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T \\ (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(m)})^T \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\sigma}^{(o)} \\ \delta\boldsymbol{\sigma}^{(m)} \end{bmatrix}$$

$$= \begin{bmatrix} \delta\boldsymbol{f}^{(o)} \\ \delta\boldsymbol{f}^{(c)} \\ \delta\boldsymbol{f}^{(m)} \end{bmatrix} - \begin{bmatrix} (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(c)})^T \\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)})^T \\ (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(c)})^T \end{bmatrix} \left[\delta\boldsymbol{\sigma}^{(c)}\right]$$

$$- \begin{bmatrix} \boldsymbol{M}^{(x,o)} & \boldsymbol{C}^{(x,o)} & (\boldsymbol{N}^{(x,o)} + \boldsymbol{G}^{(x,o)}) \\ \boldsymbol{M}^{(x,c)} & \boldsymbol{C}^{(x,c)} & (\boldsymbol{N}^{(x,c)} + \boldsymbol{G}^{(x,c)}) \\ \boldsymbol{M}^{(x,m)} & \boldsymbol{C}^{(x,m)} & (\boldsymbol{N}^{(x,m)} + \boldsymbol{G}^{(x,m)}) \end{bmatrix} \begin{bmatrix} \delta\ddot{\boldsymbol{q}} \\ \delta\dot{\boldsymbol{q}} \\ \delta\boldsymbol{q} \end{bmatrix}, \tag{40}$$

where stress resultants $\delta\boldsymbol{\sigma}^{(c)}$ of redundant elastic elements are determined by

$$\delta\boldsymbol{\sigma}^{(c)} = \delta\boldsymbol{\sigma}_a^{(c)} + \left(\boldsymbol{K}^{(e,c)} + \boldsymbol{N}^{(e,c)}\right)\delta\boldsymbol{q} + \boldsymbol{D}^{(e,c)}\delta\dot{\boldsymbol{q}}, \tag{41}$$

where

$$\boldsymbol{K}^{(e,c)} = \boldsymbol{S}^{(c,m)}\mathbf{D}\boldsymbol{\mathcal{F}}^{(e,m)} + \boldsymbol{S}^{(c,c)}\mathbf{D}\boldsymbol{\mathcal{F}}^{(e,c)}, \tag{42a}$$

$$\boldsymbol{D}^{(e,c)} = \boldsymbol{S}_d^{(c,m)}\mathbf{D}\boldsymbol{\mathcal{F}}^{(e,m)} + \boldsymbol{S}_d^{(c,c)}\mathbf{D}\boldsymbol{\mathcal{F}}^{(e,c)}. \tag{42b}$$

Matrix $\boldsymbol{N}^{(e,c)}$ is defined by (34). The partitioned matrix $[(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T,\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T]$ is a square matrix and if in addition the multibody system is not in a singular configuration, it is nonsingular and the generalized stress resultant components of $\delta\boldsymbol{\sigma}^{(o)}$ and $\delta\boldsymbol{\sigma}^{(m)}$ can be computed by

$$\begin{bmatrix} \delta\boldsymbol{\sigma}^{(o)} \\ \delta\boldsymbol{\sigma}^{(m)} \end{bmatrix} = \tilde{\boldsymbol{D}}_1\left[\delta\boldsymbol{f}^{(c)} - \boldsymbol{M}^{(x,c)}\delta\ddot{\boldsymbol{q}} - \boldsymbol{C}^{(x,c)}\delta\dot{\boldsymbol{q}} - \left(\boldsymbol{N}^{(x,c)} + \boldsymbol{G}^{(x,c)}\right)\delta\boldsymbol{q}\right] - \tilde{\boldsymbol{D}}_2\delta\boldsymbol{\sigma}^{(c)}, \tag{43}$$

where matrix $\boldsymbol{D}_1$ is defined in (22) and

$$\tilde{\boldsymbol{D}}_2 = \tilde{\boldsymbol{D}}_1\left(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)}\right)^T. \tag{44}$$

Substitution of (43) into the upper part of (40) yields the expression for the reaction forces $\delta\boldsymbol{f}^{(o)}$:

$$\begin{aligned} \delta\boldsymbol{f}^{(o)} = {} & \tilde{\boldsymbol{D}}_3\delta\boldsymbol{f}^{(c)} + \left(\boldsymbol{M}^{(x,o)} - \tilde{\boldsymbol{D}}_3\boldsymbol{M}^{(x,c)}\right)\delta\ddot{\boldsymbol{q}} + \left(\boldsymbol{C}^{(x,o)} - \tilde{\boldsymbol{D}}_3\boldsymbol{C}^{(x,c)}\right)\delta\dot{\boldsymbol{q}} \\ & + \left[\boldsymbol{N}^{(x,o)} + \boldsymbol{G}^{(x,o)} - \tilde{\boldsymbol{D}}_3\left(\boldsymbol{N}^{(x,c)} + \boldsymbol{G}^{(x,c)}\right)\right]\delta\boldsymbol{q} - \tilde{\boldsymbol{D}}_4\delta\boldsymbol{\sigma}^{(c)}, \end{aligned} \tag{45}$$

where

$$\tilde{\boldsymbol{D}}_3 = \big[(\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(o)})^T,\ (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(m)})^T\big]\tilde{\boldsymbol{D}}_1, \tag{46}$$

$$\tilde{\boldsymbol{D}}_4 = \tilde{\boldsymbol{D}}_3\big(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)}\big)^T - \big(\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(c)}\big)^T. \tag{47}$$

Substitution of (43) into the lower part of (40) yields the expression for the external driving forces $\delta\boldsymbol{f}^{(m)}$:

$$\begin{aligned}\delta\boldsymbol{f}^{(m)} &= \tilde{\boldsymbol{D}}_5\delta\boldsymbol{f}^{(c)} + \big(\boldsymbol{M}^{(x,m)} - \tilde{\boldsymbol{D}}_5\boldsymbol{M}^{(x,c)}\big)\delta\ddot{\boldsymbol{q}} + \big(\boldsymbol{C}^{(x,m)} - \tilde{\boldsymbol{D}}_5\boldsymbol{C}^{(x,c)}\big)\delta\dot{\boldsymbol{q}} \\ &\quad + \big[\boldsymbol{N}^{(x,m)} + \boldsymbol{G}^{(x,m)} - \tilde{\boldsymbol{D}}_5\big(\boldsymbol{N}^{(x,c)} + \boldsymbol{G}^{(x,c)}\big)\big]\delta\boldsymbol{q} - \tilde{\boldsymbol{D}}_6\delta\boldsymbol{\sigma}^{(c)},\end{aligned} \tag{48}$$

where

$$\tilde{\boldsymbol{D}}_5 = \big[(\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(o)})^T,\ (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(m)})^T\big]\tilde{\boldsymbol{D}}_1, \tag{49}$$

$$\tilde{\boldsymbol{D}}_6 = \tilde{\boldsymbol{D}}_5\big(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)}\big)^T - \big(\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(c)}\big)^T. \tag{50}$$

Equations (43)–(48) are used later on in Sect. 8.2 to derive the dynamic part of the output equations (58).

8 Linearized state-space equations

The state-space formulation is the most common description of dynamic systems [15, 19], which consists of the state equations to describe the internal dynamics and the output equations to determine output quantities of the system.

8.1 State equations

By selecting the state of the multibody system to be the vector $\boldsymbol{z} = [\boldsymbol{q}^{dT}, \dot{\boldsymbol{q}}^{dT}]^T$ as in (17), the linearized equations of motion of (25) can be transformed to the linearized state equations

$$\delta\dot{\boldsymbol{z}} = \boldsymbol{A}\delta\boldsymbol{z} + \boldsymbol{B}\delta\boldsymbol{u}, \tag{51}$$

where the input vector $\delta\boldsymbol{u}$ consists of time-varying applied nodal forces and torques $\delta\boldsymbol{f}^{(c)}(t)$, $\delta\boldsymbol{f}^{(m)}(t)$, generalized stress resultants, i.e., internal driving forces and torques $\delta\boldsymbol{\sigma}_a^{(m)}(t)$, $\delta\boldsymbol{\sigma}_a^{(c)}(t)$ and prescribed motions in terms of rheonomic degrees of freedom $\delta\ddot{\boldsymbol{q}}^r(t)$, $\delta\dot{\boldsymbol{q}}^r(t)$, $\delta\boldsymbol{q}^r(t)$, respectively:

$$\delta\boldsymbol{u} = \big[\delta\boldsymbol{f}^{(c)T},\ \delta\boldsymbol{f}^{(m)T},\ \delta\boldsymbol{\sigma}_a^{(m)T},\ \delta\boldsymbol{\sigma}_a^{(c)T},\ \delta\ddot{\boldsymbol{q}}^{rT},\ \delta\dot{\boldsymbol{q}}^{rT},\ \delta\boldsymbol{q}^{rT}\big]^T. \tag{52}$$

The state matrix and input matrix are written as

$$\boldsymbol{A} = \begin{bmatrix}\boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix}\boldsymbol{O} \\ \boldsymbol{B}_2\end{bmatrix}, \tag{53}$$

with

$$[\boldsymbol{A}_{21}|\boldsymbol{A}_{22}] = \boldsymbol{A}_2 = \big[-\bar{\boldsymbol{M}}_{dd}^{-1}(\bar{\boldsymbol{K}}_{dd} + \bar{\boldsymbol{N}}_{dd} + \bar{\boldsymbol{G}}_{dd})| - \bar{\boldsymbol{M}}_{dd}^{-1}(\bar{\boldsymbol{C}}_{dd} + \bar{\boldsymbol{D}}_{dd})\big], \tag{54}$$

and

$$\boldsymbol{B}_2 = \big[\bar{\boldsymbol{M}}_{dd}^{-1}\big]\big[\overset{(\delta \boldsymbol{f}^{(c)})}{\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x,c)T}}\ \big|\overset{(\delta \boldsymbol{f}^{(m)})}{\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x,m)T}}\ \big| \overset{(\delta \boldsymbol{\sigma}_a^{(m)})}{-\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e,m)T}}\ \big| \overset{(\delta \boldsymbol{\sigma}_a^{(c)})}{-\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e,c)T}}\big|$$
$$\big| \overset{(\delta \ddot{\boldsymbol{q}}^r)}{-\bar{\boldsymbol{M}}_{dr}}\ \big| \overset{(\delta \dot{\boldsymbol{q}}^r)}{-(\bar{\boldsymbol{C}}_{dr} + \bar{\boldsymbol{D}}_{dr})}\big| \overset{(\delta \boldsymbol{q}^r)}{-(\bar{\boldsymbol{K}}_{dr} + \bar{\boldsymbol{N}}_{dr} + \bar{\boldsymbol{G}}_{dr})}\big], \tag{55}$$

where the subscripts dd and dr stand for partitioned matrices analogous to (14). Matrices $\boldsymbol{A}_{21}$, $\boldsymbol{A}_{22}$, and $\boldsymbol{B}_2$ are derived from the linearized equations of motion (25).

8.2 Output equations

In addition to the linearized state equations we can define the linearized output equations

$$\delta \boldsymbol{y} = \boldsymbol{C}\delta \boldsymbol{z} + \boldsymbol{D}\delta \boldsymbol{u}, \tag{56}$$

where $\boldsymbol{C}$ is called the output matrix and $\boldsymbol{D}$ the feed-through matrix. The matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ depend on the chosen input vector $\delta \boldsymbol{u}$ and the output vector $\delta \boldsymbol{y}$. Once the state variables have been found as known functions of time, the system outputs can be found as a linear combination of states and inputs of the system. For readability, the output vector $\delta \boldsymbol{y}$ is partitioned into a kinematic part and a dynamic part as

$$\delta \boldsymbol{y} = \begin{bmatrix} \delta \boldsymbol{y}^{(\text{kin})} \\ \delta \boldsymbol{y}^{(\text{dyn})} \end{bmatrix}, \quad \text{with } \delta \boldsymbol{y}^{(\text{kin})} = \begin{bmatrix} \delta \boldsymbol{x} \\ \delta \dot{\boldsymbol{x}} \\ \delta \ddot{\boldsymbol{x}} \end{bmatrix}, \ \delta \boldsymbol{y}^{(\text{dyn})} = \begin{bmatrix} \delta \boldsymbol{\sigma}^{(o)} \\ \delta \boldsymbol{\sigma}^{(m)} \\ \delta \boldsymbol{f}^{(o)} \\ \delta \boldsymbol{f}^{(m)} \end{bmatrix}. \tag{57}$$

Then the output equations are defined by

$$\begin{bmatrix} \delta \boldsymbol{y}^{(\text{kin})} \\ \delta \boldsymbol{y}^{(\text{dyn})} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}^{(\text{kin})} \\ \boldsymbol{C}^{(\text{dyn})} \end{bmatrix} \delta \boldsymbol{z} + \begin{bmatrix} \boldsymbol{D}^{(\text{kin})} \\ \boldsymbol{D}^{(\text{dyn})} \end{bmatrix} \delta \boldsymbol{u}, \tag{58}$$

where $\boldsymbol{C}^{(\text{kin})}$, $\boldsymbol{C}^{(\text{dyn})}$ are kinematic and dynamic output matrices and $\boldsymbol{D}^{(\text{kin})}$, $\boldsymbol{D}^{(\text{dyn})}$ are the kinematic and dynamic feed-through matrices.

8.3 Kinematic output matrices

The components of the output vector $\delta \boldsymbol{y}^{(\text{kin})}$ are available as the nodal positions $\delta \boldsymbol{x}(t)$, velocities $\delta \dot{\boldsymbol{x}}(t)$ and accelerations $\delta \ddot{\boldsymbol{x}}(t)$ of (23) as functions of the generalized dynamic and rheonomic degrees of freedom $\delta \boldsymbol{q}^d(t)$, $\delta \boldsymbol{q}^r(t)$, $\delta \dot{\boldsymbol{q}}^d(t)$, $\delta \dot{\boldsymbol{q}}^r(t)$, and $\delta \ddot{\boldsymbol{q}}^d(t)$, $\delta \ddot{\boldsymbol{q}}^r(t)$, where $\delta \ddot{\boldsymbol{q}}^d(t)$ is determined from the linearized state (51). Substitution of (23) into (58) yields with the linearized state-space equations (51)–(55) the following expressions of the kinematic output matrix $\boldsymbol{C}^{(\text{kin})}$ and feed-through matrix $\boldsymbol{D}^{(\text{kin})}$:

$$\boldsymbol{C}^{(\text{kin})} = \left[\begin{array}{c|c} \overset{(\delta \boldsymbol{q}^d)}{\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}} & \overset{(\delta \dot{\boldsymbol{q}}^d)}{\boldsymbol{O}} \\ \mathbf{D}_{q^d}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} & \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)} \\ \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\mathbf{A}_{21} + \mathbf{D}_{q^d}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\ddot{\boldsymbol{q}} + \mathbf{D}_{q^d}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}} & \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\mathbf{A}_{22} + 2\mathbf{D}_{q^d}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} \end{array}\right] \tag{59}$$

and

$$\boldsymbol{D}^{(\mathrm{kin})} = \begin{bmatrix} \overset{(\delta\boldsymbol{u})}{\boldsymbol{O}} \\ \boldsymbol{O} \\ \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\boldsymbol{B}_2 \end{bmatrix} + \left[\begin{array}{c|c|c} \overset{(\delta\ddot{\boldsymbol{q}}^r)}{\boldsymbol{O}} & \overset{(\delta\dot{\boldsymbol{q}}^r)}{\boldsymbol{O}} & \overset{(\delta\boldsymbol{q}^r)}{\mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)}} \\ \boldsymbol{O} & \mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)} & \mathbf{D}_{q^r}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} \\ \mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)} & 2\mathbf{D}_{q^r}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} & \mathbf{D}_{q^r}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\ddot{\boldsymbol{q}} + \mathbf{D}_{q^r}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}} \end{array}\right] \tag{60}$$

The zero-components of the matrix $\boldsymbol{D}^{(\mathrm{kin})}$ associated with the applied nodal forces $\delta\boldsymbol{f}^{(c)}$, $\delta\boldsymbol{f}^{(m)}$ and the generalized stress resultants $\delta\boldsymbol{\sigma}^{(m)}$ are omitted. These force quantities have no feed-through on the nodal positions $\delta\boldsymbol{x}$ and velocities $\delta\dot{\boldsymbol{x}}$.

8.4 Dynamic output matrices

The dynamic output matrices are derived from the linearized equations of reaction (37). The components of the output vector $\delta\boldsymbol{y}^{(\mathrm{dyn})}$ are available as the vectors of generalized stress resultants $\delta\boldsymbol{\sigma}^{(o)}(t)$, $\delta\boldsymbol{\sigma}^{(m)}(t)$ calculated by (43) and the vector of reaction forces $\delta\boldsymbol{f}^{(o)}(t)$ and external driving forces $\delta\boldsymbol{f}^{(m)}(t)$ calculated by (45) and (48), respectively. Substitution of (41)–(48) into (58) yields with the linearized state-space equations (51)–(55) the following expressions for the dynamic output matrix $\boldsymbol{C}^{(\mathrm{dyn})}$ and feed-through matrix $\boldsymbol{D}^{(\mathrm{dyn})}$:

$$\boldsymbol{C}^{(\mathrm{dyn})} = \left[\tilde{\boldsymbol{M}}_d^{(x)}\boldsymbol{A}_2\right] + \left[\overset{(\delta\boldsymbol{q}^d)}{\tilde{\boldsymbol{K}}_d^{(x)} + \tilde{\boldsymbol{N}}_d^{(x)} + \tilde{\boldsymbol{G}}_d^{(x)}} \mid \overset{(\delta\dot{\boldsymbol{q}}^d)}{\tilde{\boldsymbol{C}}_d^{(x)} + \tilde{\boldsymbol{D}}_d^{(x)}}\right], \tag{61}$$

and

$$\boldsymbol{D}^{(\mathrm{dyn})} = \left[\tilde{\boldsymbol{M}}_d^{(x)}\boldsymbol{B}_2\right] + \Big[\overset{(\delta\boldsymbol{f}^{(c)})}{\tilde{\boldsymbol{D}}_{\boldsymbol{f}^{(c)}}} \mid \overset{(\delta\boldsymbol{f}^{(m)})}{\boldsymbol{O}} \mid \overset{(\delta\boldsymbol{\sigma}_a^{(m)})}{\boldsymbol{O}} \mid \overset{(\delta\boldsymbol{\sigma}_a^{(c)})}{\tilde{\boldsymbol{D}}_{\boldsymbol{\sigma}^{(c)}}} \mid$$

$$\mid \overset{(\delta\ddot{\boldsymbol{q}}^r)}{\tilde{\boldsymbol{M}}_r^{(x)}} \mid \overset{(\delta\dot{\boldsymbol{q}}^r)}{\tilde{\boldsymbol{C}}_r^{(x)} + \tilde{\boldsymbol{D}}_r^{(x)}} \mid \overset{(\delta\boldsymbol{q}^r)}{\tilde{\boldsymbol{K}}_r^{(x)} + \tilde{\boldsymbol{N}}_r^{(x)} + \tilde{\boldsymbol{G}}_r^{(x)}}\Big], \tag{62}$$

where

$$\tilde{\boldsymbol{M}}^{(x)} = \tilde{\boldsymbol{D}}\boldsymbol{M}^{(x)}, \qquad \tilde{\boldsymbol{C}}^{(x)} = \tilde{\boldsymbol{D}}\boldsymbol{C}^{(x)}, \qquad \tilde{\boldsymbol{D}}^{(x)} = \tilde{\boldsymbol{D}}_{\boldsymbol{\sigma}^{(c)}}\boldsymbol{D}^{(e,c)}, \tag{63a}$$

$$\tilde{\boldsymbol{K}}^{(x)} = \tilde{\boldsymbol{D}}_{\boldsymbol{\sigma}^{(c)}}\boldsymbol{K}^{(e,c)}, \qquad \tilde{\boldsymbol{N}}^{(x)} = \tilde{\boldsymbol{D}}\boldsymbol{N}^{(x)} + \tilde{\boldsymbol{D}}_{\boldsymbol{\sigma}^{(c)}}\boldsymbol{N}^{(e,c)}, \qquad \tilde{\boldsymbol{G}}^{(x)} = \tilde{\boldsymbol{D}}\boldsymbol{G}^{(x)}, \tag{63b}$$

in which

$$\tilde{\boldsymbol{D}} = \overset{(\boldsymbol{x}^{(o)},\ \boldsymbol{x}^{(c)},\ \boldsymbol{x}^{(m)})}{\begin{bmatrix} \boldsymbol{O} & -\tilde{\boldsymbol{D}}_1 & \boldsymbol{O} \\ \boldsymbol{I} & -\tilde{\boldsymbol{D}}_3 & \boldsymbol{O} \\ \boldsymbol{O} & -\tilde{\boldsymbol{D}}_5 & \boldsymbol{I} \end{bmatrix}}, \qquad \tilde{\boldsymbol{D}}_{\boldsymbol{f}^{(c)}} = \begin{bmatrix} \tilde{\boldsymbol{D}}_1 \\ \tilde{\boldsymbol{D}}_3 \\ \tilde{\boldsymbol{D}}_5 \end{bmatrix}, \qquad \tilde{\boldsymbol{D}}_{\boldsymbol{\sigma}^{(c)}} = \begin{bmatrix} -\tilde{\boldsymbol{D}}_2 \\ -\tilde{\boldsymbol{D}}_4 \\ -\tilde{\boldsymbol{D}}_6 \end{bmatrix}. \tag{63c}$$

The subscripts d and r associated with the matrices $\tilde{\boldsymbol{M}}^{(x)}$, $\tilde{\boldsymbol{C}}^{(x)}$, $\tilde{\boldsymbol{N}}^{(x)}$ and $\tilde{\boldsymbol{G}}^{(x)}$ represent partitioned matrices corresponding to $\boldsymbol{q}^d$ and $\boldsymbol{q}^r$; see (12). The matrices $\boldsymbol{N}^{(e,c)}$, $\boldsymbol{K}^{(e,c)}$ and $\boldsymbol{D}^{(e,c)}$ are defined by (34) and (42a). Note that the subvectors $\delta\boldsymbol{f}^{(m)}$ may appear both as input quantity in the vector $\delta\boldsymbol{u}$ and as output quantity in the vector $\delta\boldsymbol{y}^{(\mathrm{dyn})}$. Matrices $\tilde{\boldsymbol{D}}_1 - \tilde{\boldsymbol{D}}_6$ are defined in (22), (44), (46), (47), (49), and (50).

9 Stationary and equilibrium solutions

Stationary solutions are solutions for which the vector of dynamic degrees of freedom has a constant value, i.e., $\dot{\boldsymbol{q}}^d = \mathbf{0}$, $\ddot{\boldsymbol{q}}^d = \mathbf{0}$. This can represent a state of stationary motion or an equilibrium position. In the latter case, the multibody system behaves as a (flexible) structure, in which the system members experience only small displacement motions and elastic deformations with respect to the equilibrium position. Equilibrium positions are also important as initial values for a dynamic analysis. According to (17), stationary solutions can be obtained for $\ddot{\boldsymbol{q}}^r = \mathbf{0}$, by solving the algebraic equations

$$\begin{bmatrix} \dot{\boldsymbol{q}}^d \\ \bar{\boldsymbol{f}}_d(\boldsymbol{q}^d, \dot{\boldsymbol{q}}^d) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{64}$$

This equation can be solved with the Newton–Raphson method, which converges quadratically if the initial guess of a solution is sufficiently close to a true solution and the state matrix $\boldsymbol{A}$ (see (53)) is regular at this solution [12]. In a typical case stability of a stationary solution is determined by the eigenvalues of the state matrix $\boldsymbol{A}$: if all eigenvalues have negative real parts, the solution is stable and if some eigenvalue has a positive real part, the solution is unstable [14]. If some eigenvalue is purely imaginary or zero, we are in a bifurcation point. The associated frequency equation for the undamped system is given by

$$\det\big(-\omega_i^2 \bar{\boldsymbol{M}}_{dd} + \bar{\boldsymbol{K}}_{dd} + \bar{\boldsymbol{N}}_{dd} + \bar{\boldsymbol{G}}_{dd}\big) = 0, \tag{65}$$

where the quantities ω_i are the natural frequencies of the system under dynamic and static loading conditions.

In case of an equilibrium configuration, the kinematic output matrix $\boldsymbol{C}^{(\mathrm{kin})}$ and feed-through matrix $\boldsymbol{D}^{(\mathrm{kin})}$ defined in (59) and (60) become

$$\boldsymbol{C}^{(\mathrm{kin})} = \left[\begin{array}{c|c} \mathrm{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)} & \boldsymbol{O} \\ \boldsymbol{O} & \mathrm{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)} \\ \mathrm{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\boldsymbol{A}_{21} & \mathrm{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\boldsymbol{A}_{22} \end{array}\right], \tag{66}$$

and

$$\boldsymbol{D}^{(\mathrm{kin})} = \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{O} \\ \mathrm{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\boldsymbol{B}_2 \end{bmatrix} + \left[\begin{array}{c|c|c} \boldsymbol{O} & \boldsymbol{O} & \mathrm{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)} \\ \boldsymbol{O} & \mathrm{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)} & \boldsymbol{O} \\ \mathrm{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)} & \boldsymbol{O} & \boldsymbol{O} \end{array}\right]. \tag{67}$$

Since for an equilibrium configuration the matrices $\bar{\boldsymbol{N}}$ and $\tilde{\boldsymbol{N}}^{(x)}$ are identically zero, the dynamic output matrix $\boldsymbol{C}^{(\mathrm{dyn})}$ and feed-through matrix $\boldsymbol{D}^{(\mathrm{dyn})}$ defined in (61) and (62) become

$$\boldsymbol{C}^{(\mathrm{dyn})} = \tilde{\boldsymbol{M}}_d^{(x)}\boldsymbol{A}_2 + \big[\tilde{\boldsymbol{K}}_d^{(x)} + \tilde{\boldsymbol{G}}_d^{(x)} \mid \tilde{\boldsymbol{D}}_d^{(x)}\big], \tag{68}$$

$$\boldsymbol{D}^{(\mathrm{dyn})} = \tilde{\boldsymbol{M}}_d^{(x)}\boldsymbol{B}_2 + \big[\tilde{\boldsymbol{D}}_{\boldsymbol{f}^{(c)}} \mid \boldsymbol{O} \mid \boldsymbol{O} \mid \tilde{\boldsymbol{D}}_{\boldsymbol{\sigma}^{(c)}} \mid \tilde{\boldsymbol{M}}_r^{(x)} \mid \tilde{\boldsymbol{D}}_r^{(x)} \mid \tilde{\boldsymbol{K}}_r^{(x)} + \tilde{\boldsymbol{G}}_r^{(x)}\big], \tag{69}$$

where

$$\boldsymbol{A}_2 = \big[-\bar{\boldsymbol{M}}_{dd}^{-1}(\bar{\boldsymbol{K}}_{dd} + \bar{\boldsymbol{G}}_{dd})\big| -\bar{\boldsymbol{M}}_{dd}^{-1}\bar{\boldsymbol{D}}_{dd}\big], \tag{70}$$

and

$$\begin{aligned} \boldsymbol{B}_2 = \big[\bar{\boldsymbol{M}}_{dd}^{-1}\big]\big[&\mathrm{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x,c)T}\big|\mathrm{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x,m)T}\big|-\mathrm{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e,m)T}\big|-\mathrm{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e,c)T}\big| \\ &\big|-\bar{\boldsymbol{M}}_{dr}\big|-\bar{\boldsymbol{D}}_{dr}\big|-(\bar{\boldsymbol{K}}_{dr} + \bar{\boldsymbol{G}}_{dr})\big]. \end{aligned} \tag{71}$$

Stability of an equilibrium configuration is determined by the eigenvalues of matrix $(\bar{\boldsymbol{K}}_{dd} + \bar{\boldsymbol{G}}_{dd})$. In a linear buckling problem, critical load multipliers λ_i are determined by solving the eigenvalue problem

$$\det(\bar{\boldsymbol{K}}_{dd} + \lambda_i \bar{\boldsymbol{G}}_{dd}) = 0, \tag{72}$$

where $\lambda_i = \boldsymbol{f}_i / \boldsymbol{f}_0$. Here, $\bar{\boldsymbol{K}}_{dd}$ is the structural stiffness matrix, $\bar{\boldsymbol{G}}_{dd}$ the geometric stiffness matrix due to the reference load $\boldsymbol{f}_0$ and $\boldsymbol{f}_i$ is the buckling load.

10 From state space equations to transfer function matrix

The linearized state equations and linearized output equations of (51) and (56) have been derived in Sect. 8 for the state vector $\boldsymbol{z} = [\boldsymbol{q}^{dT}, \dot{\boldsymbol{q}}^{dT}]^T$, the general input vector $\delta\boldsymbol{u}$ from (52) and the general output vector $\delta\boldsymbol{y}$ from (57). Next, it will be outlined how this state space representation can be transformed into a transfer function matrix representation that clearly expresses the relations between all inputs and outputs. The standard expression [15]

$$\widetilde{\boldsymbol{G}}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}, \tag{73}$$

clearly relates the state space matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ with the transfer function matrix $\widetilde{\boldsymbol{G}}(s)$ where s is the Laplace variable. However, this expression is only correct for the input parts $\delta\boldsymbol{f}^{(c)}$, $\delta\boldsymbol{f}^{(m)}$, $\delta\boldsymbol{\sigma}_a^{(m)}$, and $\delta\boldsymbol{\sigma}_a^{(c)}$, and it will fail due to the occurrence of $\delta\boldsymbol{q}^r$ combined with its time derivatives $\delta\dot{\boldsymbol{q}}^r$ and $\delta\ddot{\boldsymbol{q}}^r$ in the general input vector $\delta\boldsymbol{u}$.

This can be understood by recognizing that the transfer function matrix $\widetilde{\boldsymbol{G}}(s)$ in general relates the Laplace transforms of the system's input $\delta\boldsymbol{u}$ and output $\delta\boldsymbol{y}$

$$\mathcal{L}\{\delta\boldsymbol{y}(t)\} = \delta\boldsymbol{y}(s) = \widetilde{\boldsymbol{G}}(s)\delta\boldsymbol{u}(s) = \widetilde{\boldsymbol{G}}(s)\mathcal{L}\{\delta\boldsymbol{u}(t)\}, \tag{74}$$

where $\mathcal{L}\{\ldots\}$ denotes the Laplace transform. In the general input vector $\delta\boldsymbol{u}$ from (52), the Laplace transforms of $\delta\boldsymbol{q}^r$, $\delta\dot{\boldsymbol{q}}^r$, and $\delta\ddot{\boldsymbol{q}}^r$ are not independent as for zero initial conditions

$$\mathcal{L}\{\delta\dot{\boldsymbol{q}}^r(t)\} = s\mathcal{L}\{\delta\boldsymbol{q}^r(t)\} \quad \text{and} \quad \mathcal{L}\{\delta\ddot{\boldsymbol{q}}^r(t)\} = s^2\mathcal{L}\{\delta\boldsymbol{q}^r(t)\}. \tag{75}$$

This dependency has to be accounted for in the parts of the transfer function matrix relating any of the inputs $\delta\boldsymbol{q}^r$, $\delta\dot{\boldsymbol{q}}^r$, and $\delta\ddot{\boldsymbol{q}}^r$ with the output $\delta\boldsymbol{y}(t)$. In the remainder of this section the transfer function matrix will be derived for these inputs only. To simplify the notation, the contribution of the input parts $\delta\boldsymbol{f}^{(c)}$, $\delta\boldsymbol{f}^{(m)}$, $\delta\boldsymbol{\sigma}_a^{(m)}$, and $\delta\boldsymbol{\sigma}_a^{(c)}$ is not included as the accompanying transfer function matrix can be obtained directly with (73). Hence, the input vector is limited to

$$\delta\boldsymbol{u}_r = \left[\delta\ddot{\boldsymbol{q}}^{rT},\ \delta\dot{\boldsymbol{q}}^{rT},\ \delta\boldsymbol{q}^{rT}\right]^T \tag{76}$$

and only the parts of the input and feedthrough matrices $\boldsymbol{B}$ and $\boldsymbol{D}$ for these inputs are discussed.

For the first example, we consider an input vector including only the displacement input $\delta\boldsymbol{q}^r$ such as

$$\delta\boldsymbol{u}_1 = \delta\boldsymbol{q}^r \tag{77}$$

and we determine the transfer function matrix in

$$\delta\boldsymbol{y}(s) = \boldsymbol{G}_1(s)\delta\boldsymbol{u}_1(s). \tag{78}$$

Using (75), the input vector in the Laplace domain $\delta\boldsymbol{u}_r(s)$ can be written as

$$\delta\boldsymbol{u}_r(s) = \begin{bmatrix} s^2\boldsymbol{I} \\ s\boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix} \delta\boldsymbol{u}_1(s) \tag{79}$$

and consequently the transfer function matrix from $\delta\boldsymbol{u}_1$ to $\delta\boldsymbol{y}$ is

$$\boldsymbol{G}_1(s) = \left\{\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right\} \begin{bmatrix} s^2\boldsymbol{I} \\ s\boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix}. \tag{80}$$

Alternatively, the input vector can be defined to include the velocity $\delta\dot{\boldsymbol{q}}^r$

$$\delta\boldsymbol{u}_2 = \delta\dot{\boldsymbol{q}}^r \tag{81}$$

such that the input vector is

$$\delta\boldsymbol{u}_r(s) = \begin{bmatrix} s\boldsymbol{I} \\ \boldsymbol{I} \\ \frac{1}{s}\boldsymbol{I} \end{bmatrix} \delta\boldsymbol{u}_2(s). \tag{82}$$

The transfer function from $\delta\boldsymbol{u}_2$ to $\delta\boldsymbol{y}$ is then

$$\boldsymbol{G}_2(s) = \left\{\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right\} \begin{bmatrix} s\boldsymbol{I} \\ \boldsymbol{I} \\ \frac{1}{s}\boldsymbol{I} \end{bmatrix}. \tag{83}$$

An acceleration input can be treated analogously. Obviously, these cases can be combined, e.g., to define an input $\delta\boldsymbol{u}$ with the position of one rheonomic degree of freedom and the acceleration of another rheonomic degree of freedom. Note further that in the case only accelerations $\delta\ddot{\boldsymbol{q}}^r$ and no velocities $\delta\dot{\boldsymbol{q}}^r$ and positions $\delta\boldsymbol{q}^r$ appear in the input, the transfer function matrix can also be obtained by adding the velocities and positions to the state vector $\delta\boldsymbol{x}$.

11 Illustrative examples

11.1 Spring-mass-damper system mounted on a cart

Consider the spring-mass-damper system mounted on a massless cart as shown in Fig. 2. In this system, δx^1 is the rheonomic displacement of the cart and is the input to the system. The displacement δx^2 of the mass m is the output, i.e., $\delta\boldsymbol{y} = \delta x^2$. The connection between this mass and the massless cart is described by the viscous damping coefficient d and a spring constant k. The relative position of the mass on the massless cart

$$\delta q = \delta x^2 - \delta x^1, \tag{84}$$

is chosen as dynamic degree of freedom of the system. For the definition of the input vector, it is important to recognize that as outlined in Sect. 10, next to the input position, δx^1 also

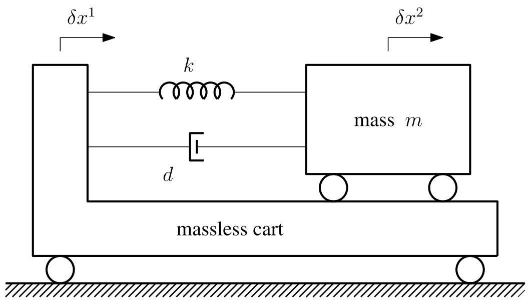


Fig. 2 Spring-mass-damper system mounted on a cart

the velocity $\delta\dot{x}^1$, and acceleration $\delta\ddot{x}^1$ have to be included in the input in order to obtain state space matrices that can be used in (80). By defining the input vector as

$$\delta\boldsymbol{u} = \left[\delta\ddot{x}^1,\ \delta\dot{x}^1,\ \delta x^1\right]^T, \tag{85}$$

the state space matrices are

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -m^{-1}k & -m^{-1}d \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \\ \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \boldsymbol{D} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \tag{86}$$

By substituting the state-space matrices of (86) into (73), we obtain the transfer function matrix

$$\widetilde{\boldsymbol{G}}(s) = \begin{bmatrix} \dfrac{-1}{s^2 + \frac{d}{m}s + \frac{k}{m}} & 0 & 1 \end{bmatrix}, \tag{87}$$

where the three components of the matrix specify how the Laplace transforms of all three components in the input vector are combined to obtain the Laplace transform of the output

$$\delta x^2(s) = \widetilde{\boldsymbol{G}}(s)\delta\boldsymbol{u}(s). \tag{88}$$

As was outlined in Sect. 10, the three components of this matrix do not directly specify the transfer functions from each of the components in the input vector to the output $\delta x^2(s)$ as (75) have to be accounted for. To obtain the transfer function from a single input, e.g., $\delta x^1(s)$, to the output $\delta x^2(s)$, (79) is rewritten for this particular example as

$$\delta\boldsymbol{u}(s) = \begin{bmatrix} s^2 & s & 1 \end{bmatrix}^T \delta x^1(s), \tag{89}$$

so the Single-input Single-output (SiSo) transfer function from input $\delta x^1(s)$ to output $\delta x^2(s)$ is according to (80)

$$G_1(s) = \frac{\delta x^2(s)}{\delta x^1(s)} = \widetilde{\boldsymbol{G}}(s)\begin{bmatrix} s^2 \\ s \\ 1 \end{bmatrix} = \frac{\frac{d}{m}s + \frac{k}{m}}{s^2 + \frac{d}{m}s + \frac{k}{m}}. \tag{90}$$

The same approach can be followed in the case the velocity $\delta\dot{x}^1 = \delta v^1$ is the input. Then (82) is rewritten as

$$\delta\boldsymbol{u}(s) = \begin{bmatrix} s & 1 & 1/s \end{bmatrix}^T \delta v^1(s), \tag{91}$$

so the SiSo transfer function from input $\delta v^1(s)$ to $\delta x^2(s)$ is according to (83)

$$G_2(s) = \frac{\delta x^2(s)}{\delta v^1(s)} = \widetilde{\boldsymbol{G}}(s) \begin{bmatrix} s \\ 1 \\ 1/s \end{bmatrix} = \frac{\frac{d}{m}s + \frac{k}{m}}{s(s^2 + \frac{d}{m}s + \frac{k}{m})}. \tag{92}$$

Taking the acceleration $\delta \ddot{x}^1$ as input can be analyzed analogously. Clearly, the transfer function for any of these inputs to the output can be obtained from the state space matrices, provided the input vector $\delta \boldsymbol{u}$ is defined according to (85) to include the position, velocity, and acceleration of the rheonomic degree of freedom.

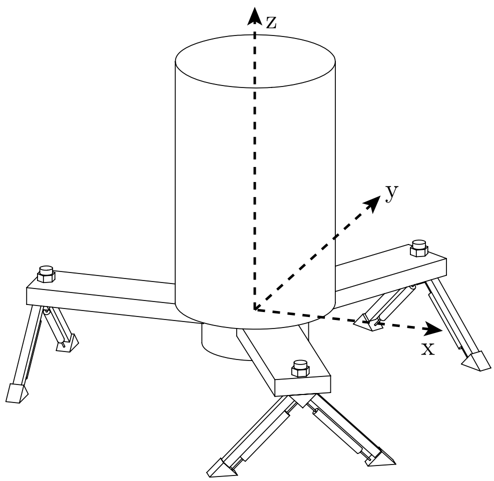
11.2 Active vibration isolation of a metrology frame

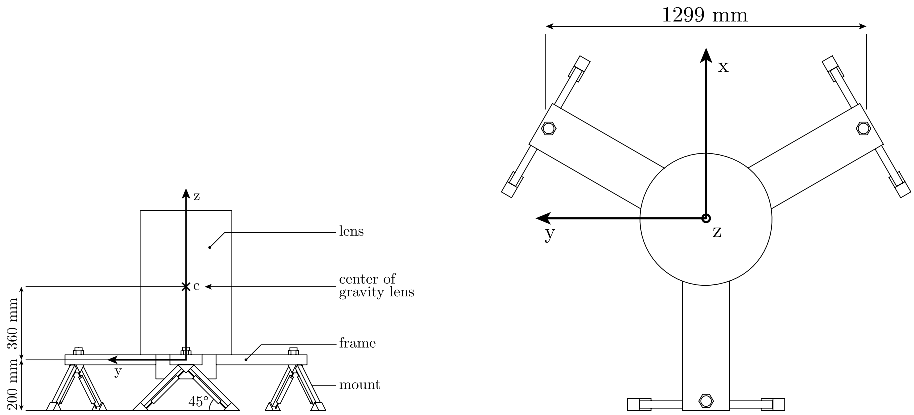
As a second example, a six degree of freedom model of a lens suspension frame of a wafer stepper/scanner, a so-called metrology frame, is presented; see Figs. 3 and 4. The model is used for a conceptual design phase of active mounts for a vibration isolation system of the metrology frame, i.e., the combined frame and lens. The idea is to design a hybrid-elastic mount with a high stiffness (typically 100–200× higher than for pneumatic isolators). The transmissibility of floor vibrations is actively reduced, using force sensors, built-in piezo actuators, and a control system. Based on the linearized input-output representation a Multi-input Multi-output (MiMo) transfer-matrix adequate for analysis and active vibration control design is obtained.

The metrology frame and lens do not show natural frequencies in the frequency range of interest, and hence they are considered rigid, i.e., no internal deformation is taken into account. Table 1 gives an overview of the inertia properties of the frame and lens. The moments of inertia I_{xx}, I_{yy}, and I_{zz} are defined with respect to the center of gravity of the frame and lens, respectively.

The mounts are modeled by a beam-like structure as shown in Fig. 5. Each mount consists of two legs which are modeled by an elastic beam element with both ends clamped. The beam element is modeled as an active element which provides for the passive elastic properties of the leg and the longitudinal force of the piezo actuator. The constitutive equation for the longitudinal stress resultant (normal force) $\sigma_1^{(k)}$ of beam element (k) is defined

Fig. 3 3D view of lens suspension frame of a wafer stepper/scanner





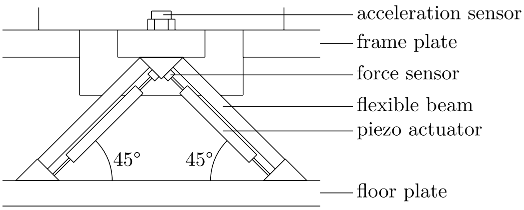
(a) Front view.
(b) Top view.

Fig. 4 Front and top view of the metrology frame

Table 1 Inertia properties of the metrology frame

	Mass [kg]	z_c [m]	I_{xx} [kg/m^2]	I_{yy} [kg/m^2]	I_{zz} [kg/m^2]
Frame	742	0	52.25	52.25	104.5
Lens	853	0.36	118.32	118.32	44.79

Fig. 5 Detailed view of a mount



by

$$\sigma_1^{(k)} = \sigma_a^{(k)} + s^{(k)} e_1^{(k)}, \tag{93}$$

where $\sigma_a^{(k)}$ represents the longitudinal force of the piezo actuator, $s^{(k)} = E^{(k)} A^{(k)} / l^{(k)}$ is the longitudinal stiffness coefficient and $e_1^{(k)}$ is the longitudinal deformation of the beam element; see Fig. 6. Table 2 shows the equivalent stiffness properties of the elastic beam elements.

The metrology frame is modeled using 11 spatial beam elements numbered (1) to (11), and hereafter simply called beams; see Fig. 7. The beams (1), (2), and (3) represent the frame. The beams (10) and (11) represent the lens. Beam-elements (1), (2), (3), (10), and (11) are rigid. The inertia properties of the rigid beams match the inertia properties of frame and lens as in Table 1. Beams (4), (5), (6), (7), (8), and (9) represent the active-elastic beams. They are considered mass-less with respect to the heavy frame and lens.

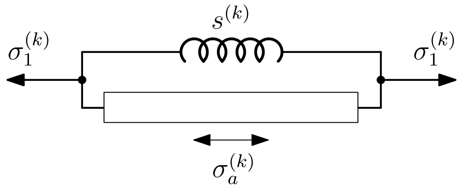


Fig. 6 Piezo actuator (force $\sigma_a^{(k)}$) with parallel spring (stiffness $s^{(k)}$)

Table 2 Stiffness properties of elastic beams

Length	long. stiffn. $\frac{EA}{l}$	bend. stiffn. $\frac{EI}{l}$	torsion stiffn. $\frac{S_T}{l}$
0.265 m	$8.398 \cdot 10^6$ N/m	7.431 Nm	5.8 Nm

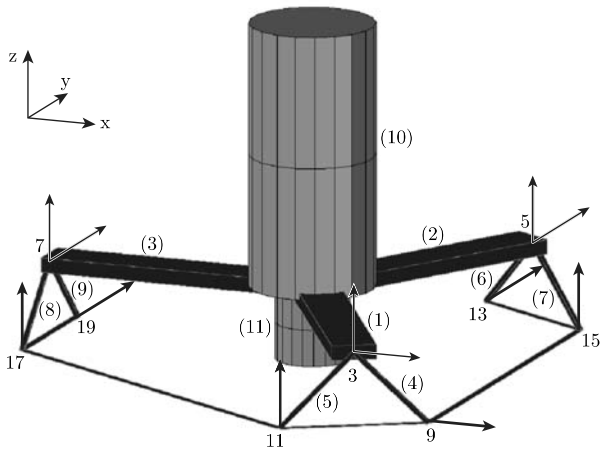


Fig. 7 Finite element model of metrology frame and floor using beams

As dynamic degrees of freedom we choose the longitudinal deformations of the suspension beams constituting the legs, i.e.,

$$\boldsymbol{q}^{(d)} = \left[e_1^{(4)}, e_1^{(5)}, e_1^{(6)}, e_1^{(7)}, e_1^{(8)}, e_1^{(9)}\right]^T, \tag{94}$$

where the numbers between the brackets denote the element numbers.

The floor is modeled as a rigid body configuration built-up by means of rigid beam elements. Because we are interested in the open loop and later on also in the closed loop transfer functions between floor vibration and frame vibrations, the floor excitations are defined as rheonomic accelerations applied at the nodal points between legs and floor as shown in Fig. 7. They are defined by the input vector

$$\boldsymbol{u}^{(\text{floor})} = \left[\ddot{x}^9, \ddot{z}^{11}, \ddot{y}^{13}, \ddot{z}^{15}, \ddot{z}^{17}, \ddot{y}^{19}\right]^T, \tag{95}$$

where the superscript numbers represent the associated node numbers; see Fig. 7.

Next to it, we define the input vector of actuator forces, associated with the active beam elements numbered (4)–(9) defined by the input vector

$$\boldsymbol{u}^{(\text{actuator})} = \left[\sigma_a^{(4)}, \sigma_a^{(5)}, \sigma_a^{(6)}, \sigma_a^{(7)}, \sigma_a^{(8)}, \sigma_a^{(9)}\right]^T. \tag{96}$$

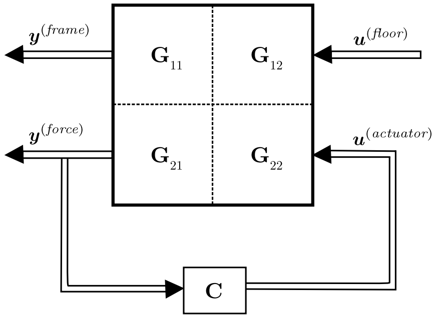


Fig. 8 Generalized plant $\boldsymbol{G}$ with 12 inputs and 12 outputs and controller $\boldsymbol{C}$ with 6 inputs and 6 outputs

The outputs are defined in two parts as well. The first part contains the output-signals of so-called performance acceleration sensors which are attached at nodal points 3, 5, and 7. These accelerations are included in the output vector

$$\boldsymbol{y}^{(\text{frame})} = \left[\ddot{x}^3, \ddot{z}^3, \ddot{y}^5, \ddot{z}^5, \ddot{z}^7, \ddot{y}^7\right]^T. \tag{97}$$

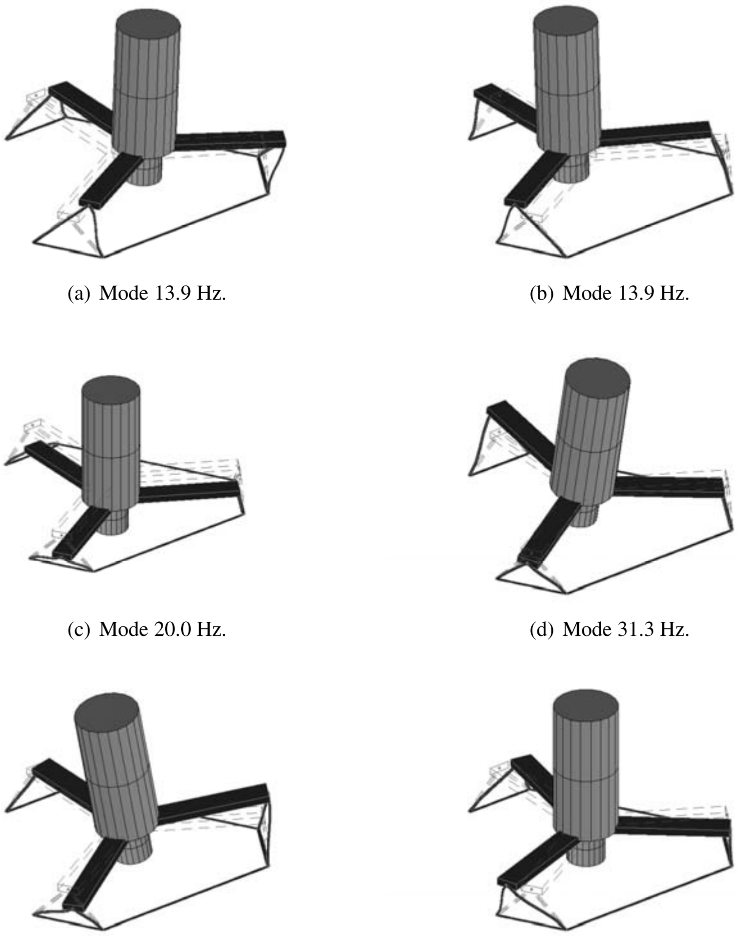
The second part contains the outputs of force sensors which measure the longitudinal stress resultant $\sigma_1^{(k)}$ of the elastic beams, i.e., the actuator forces diminished by the normal forces due to the elongation of the elastic beams. In the controlled case, they serve as the feedback sensors (error-sensors). The force sensor signals are included in the output vector

$$\boldsymbol{y}^{(\text{force})} = \left[\sigma_1^{(4)}, \sigma_1^{(5)}, \sigma_1^{(6)}, \sigma_1^{(7)}, \sigma_1^{(8)}, \sigma_1^{(9)}\right]^T. \tag{98}$$

Figure 8 shows the 12×12 generalized plant $\boldsymbol{G}$ with the input vectors $\boldsymbol{u}^{(\text{floor})}$ and $\boldsymbol{u}^{(\text{actuator})}$ and the output vectors $\boldsymbol{y}^{(\text{frame})}$ and $\boldsymbol{y}^{(\text{force})}$ defined by (95) to (98). Matrix $\boldsymbol{G}$ is partitioned in four transfer matrices $\boldsymbol{G}_{11}, \boldsymbol{G}_{12}, \boldsymbol{G}_{21,}$ and $\boldsymbol{G}_{22}$. Of interest are the singular values of the transfer matrix between floor accelerations and frame accelerations which are in the open loop case the singular values of $\boldsymbol{G}_{11}$. The singular values represent the principal gains of the transfer matrix, especially the largest singular value is important because it shows the worst-case gain frequency relation ship between an input and an output vector of the given input and output set. Therefore, this largest singular value gives an impression of the passive vibration isolation in the uncontrolled (open loop) case. Figure 9 shows the mode shapes and corresponding natural frequencies of the passive system. As outlined before, the lens and frame behave as a rigid body. Figure 10 shows the singular values (solid lines) of the open loop transfer function $\boldsymbol{G}_{11}$. From both figures, we can conclude that passive vibration isolation is obtained for the frequency region beyond 50 Hz.

In order to provide isolation of floor vibrations from 1 Hz and beyond and to provide sufficient artificial damping of the suspension modes, additional control forces $\boldsymbol{u}^{(\text{actuator})}$ are applied. These forces are computed on the basis of six force output signals $\boldsymbol{y}^{(\text{force})}$. This implies a co-located sensor-actuator control principle [16]. The control strategy is to combine proportional and integral force feedback. This is equivalent with adding virtual mass and artificial damping. Using a modal decoupling approach [6], the control forces are computed by the following force feedback control equations

$$\boldsymbol{u}(s)^{(\text{actuator})} = \boldsymbol{C}(s)\,\boldsymbol{y}(s)^{(\text{force})}, \tag{99}$$



(a) Mode 13.9 Hz. (b) Mode 13.9 Hz.

(c) Mode 20.0 Hz. (d) Mode 31.3 Hz.

(e) Mode 31.3 Hz. (f) Mode 49.0 Hz.

Fig. 9 Mode shapes and natural frequencies of the passive system

where

$$\boldsymbol{C}(s) = -\left(\boldsymbol{K}^{(P)} + \frac{\boldsymbol{K}^{(I)}}{s}\right), \tag{100}$$

with

$$\boldsymbol{K}^{(P)} = \left(\omega_d^2 \boldsymbol{I} \bar{\boldsymbol{M}}_{dd}\right)^{-1} \bar{\boldsymbol{K}}_{dd} - \boldsymbol{I}, \tag{101}$$

$$\boldsymbol{K}^{(I)} = 2\zeta\omega_d\left(\boldsymbol{I} + \boldsymbol{K}^{(P)}\right), \tag{102}$$

i.e., a Laplace domain representation (with s the Laplace variable) of a PI-controller. Herein, $\boldsymbol{K}^{(P)}$ and $\boldsymbol{K}^{(I)}$ represent the proportional and integral controller gains, ω_d the desired corner frequency, ζ the desired relative damping, $\bar{\boldsymbol{M}}_{dd}$ and $\bar{\boldsymbol{K}}_{dd}$ are the reduced system mass matrix and stiffness matrix, defined in (14a) and (54) and $\boldsymbol{I}$ represents a 6×6 identity matrix. For a more detailed discussion, the reader is referred to [20].

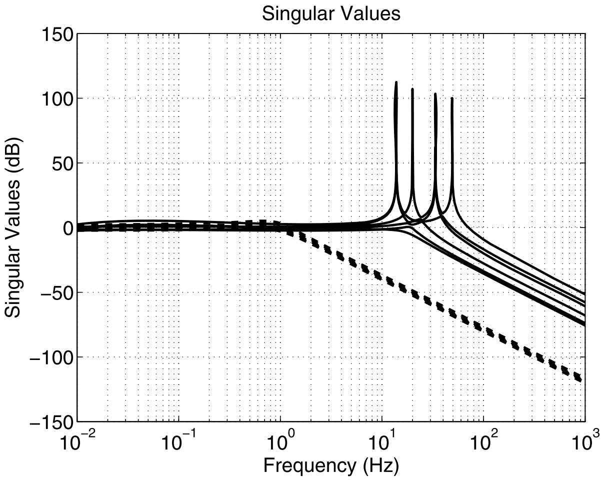


Fig. 10 Singular values; solid line is open loop, dashed line is closed loop

The controller $\boldsymbol{C}$ in the closed loop configuration as shown in Fig. 8 changes the open loop transfer $\boldsymbol{G}_{11}$ to $\boldsymbol{T}$:

$$\boldsymbol{T} = \boldsymbol{G}_{11} + \boldsymbol{G}_{12} \cdot \boldsymbol{C} \cdot (\boldsymbol{I} - \boldsymbol{G}_{22} \cdot \boldsymbol{C})^{-1} \cdot \boldsymbol{G}_{21}. \tag{103}$$

Figure 10 shows the singular value plot of the closed loop transfer function $\boldsymbol{T}$ in Bode representation (dashed line). It can be observed that the natural frequencies of all modes are brought back to 1 Hz by active means and that all modes are well damped.

12 Conclusions

A linearized state-space formulation for flexible multibody systems is developed with applied forces and prescribed motions as input and resulting absolute motions, generalized stress resultants, and reaction forces as output. Its feasibility has been demonstrated with a detailed model development of an active vibration isolation system for a metrology frame suspension modeled as a multibody system. The formulation is based on a nonlinear finite element description for flexible multibody systems. System components which show no vibrational behavior in the frequency range of interest are assumed rigid and are modeled by rigid beam elements. On the other hand, flexible joints like flexure hinges and leaf springs can be modeled adequately using only a few number of flexible beam elements as these elements account for geometric nonlinear effects such as geometric stiffening and interaction between deformation modes. In this way, a low dimensional description of prototype models can be obtained which is suitable for mechatronic design, i.e., the mechanical design as well as control system synthesis. It allows a designer to perform iterations quickly to optimize parameters.

The geometric transfer function formalism enables the derivation of linearized state space models and provides a clear check for the consistency of the constraints and the choice of degrees of freedom. The latter is important as complex interactions between rigid and elastic components makes it difficult to define a correct set of constraints and degrees of freedom. From a design point of view, the geometric transfer function concept can be employed for designing prototype models, while avoiding overconstraint design, e.g., in line with so-called Exact Constraint Design principles [4, 10].

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