Efficient dynamic closed-loop simulations of flexible manipulators

- Linearised equations of motion (for e.g. closed-loop simulations)
- Superposition of rigid link motion and small elastic deformations
- Perturbation method
- Mode-Acceleration Method / Adaptive Modal Integration
- Examples:
  One-link manipulator with constrained motion
  Spatial two-link flexible manipulator with PID control
- Conclusions

Equations of motion

Flexible manipulator with
$\underline{e}^m$: large relative displacements and rotations,
$\underline{\varepsilon}^m$: flexible deformation parameters.

Equations of motion, adapted from slide DvM/93:

$$\begin{bmatrix} \bar{M}^{ee} & \bar{M}^{e\varepsilon} \\ \bar{M}^{\varepsilon e} & \bar{M}^{\varepsilon\varepsilon} \end{bmatrix} \begin{bmatrix} \ddot{\underline{e}}^m \\ \ddot{\underline{\varepsilon}}^m \end{bmatrix} + \begin{bmatrix} \mathrm{D}_{e^m}\underline{\mathcal{F}}^T \\ \mathrm{D}_{\varepsilon^m}\underline{\mathcal{F}}^T \end{bmatrix} \left[M(\mathrm{D}^2\underline{\mathcal{F}}\cdot(\dot{\underline{e}}^m, \dot{\underline{\varepsilon}}^m))\cdot(\dot{\underline{e}}^m, \dot{\underline{\varepsilon}}^m) - \underline{f} \right] = - \begin{bmatrix} \underline{\sigma}^{em} \\ \underline{\sigma}^{\varepsilon m} \end{bmatrix}$$

Components of the reduced mass matrix

$$\bar{M}^{ee} = \mathrm{D}_{e^m}\underline{\mathcal{F}}^T M \mathrm{D}_{e^m}\underline{\mathcal{F}}, \qquad \bar{M}^{e\varepsilon} = \mathrm{D}_{e^m}\underline{\mathcal{F}}^T M \mathrm{D}_{\varepsilon^m}\underline{\mathcal{F}},$$
$$\bar{M}^{\varepsilon e} = \mathrm{D}_{\varepsilon^m}\underline{\mathcal{F}}^T M \mathrm{D}_{e^m}\underline{\mathcal{F}}, \qquad \bar{M}^{\varepsilon\varepsilon} = \mathrm{D}_{\varepsilon^m}\underline{\mathcal{F}}^T M \mathrm{D}_{\varepsilon^m}\underline{\mathcal{F}}.$$

Equations of motion (2)

With generalised coordinates (D.O.F.) $\underline{q} = \begin{bmatrix} \underline{e}^m \\ \underline{\varepsilon}^m \end{bmatrix}$

$$\bar{M}(\underline{q})\ddot{\underline{q}} + C(\underline{q}, \dot{\underline{q}})\dot{\underline{q}} + \underline{g}(\underline{q}) + \bar{K}\underline{q} = B\underline{u},$$

- $\bar{M}(\underline{q})$ is the reduced mass matrix $\bar{M} = \mathrm{D}\underline{\mathcal{F}}^T M \mathrm{D}\underline{\mathcal{F}}$
- $C(\underline{q}, \dot{\underline{q}})\dot{\underline{q}}$ represents the Coriolis and centrifugal forces
- $\underline{g}(\underline{q})$ is the vector of external nodal forces, including gravity,
- $\underline{\sigma}^{em}$ are the driving forces and torques, i.e. control input vector $-\underline{u}$ (note the sign), and $B = \begin{bmatrix} I \\ 0 \end{bmatrix}$
- $\underline{\sigma}^{\varepsilon m}$ is the stress resultant vector of flexible elements, characterised by Hooke's law: Symmetric stiffness matrix $K^{\varepsilon\varepsilon}$ with the elastic constants and $\bar{K} = \begin{bmatrix} 0 & 0 \\ 0 & K^{\varepsilon\varepsilon} \end{bmatrix}$

The direct solution of the non-linear equations of motion is rather time consuming (both low and high frequent behaviour).

Perturbation method

Model the vibrational motion of the manipulator as a first-order perturbation $\delta\underline{q}$ of the nominal rigid link motion $\underline{q}_0$

$$\underline{q} = \underline{q}_0 + \delta\underline{q} \text{ or } \begin{bmatrix} \underline{e}^m \\ \underline{\varepsilon}^m \end{bmatrix} = \begin{bmatrix} \underline{e}_0^m \\ \underline{0} \end{bmatrix} + \begin{bmatrix} \delta\underline{e}^m \\ \delta\underline{\varepsilon}^m \end{bmatrix} ; \quad \underline{u} = \underline{u}_0 + \delta\underline{u}$$

The perturbation method involves two steps:

1. Compute nominal rigid link motion $\underline{q}_0$ from the non-linear equations of motion with the rigidified model, i.e. all $\underline{\varepsilon}^m \equiv \underline{0}$.

$$\bar{M}_0^{ee}\ddot{\underline{e}}_0^m + \mathrm{D}_{e^m}\underline{\mathcal{F}}_0^T \left[M_0(\mathrm{D}^2\underline{\mathcal{F}}_0\cdot(\dot{\underline{e}}_0^m, \underline{0}))\cdot(\dot{\underline{e}}_0^m, \underline{0}) - \underline{f} \right] = -\underline{\sigma}_0^{em} = \underline{u}_0,$$
$$\bar{M}_0^{\varepsilon e}\ddot{\underline{e}}_0^m + \mathrm{D}_{\varepsilon^m}\underline{\mathcal{F}}_0^T \left[M_0(\mathrm{D}^2\underline{\mathcal{F}}_0\cdot(\dot{\underline{e}}_0^m, \underline{0}))\cdot(\dot{\underline{e}}_0^m, \underline{0}) - \underline{f} \right] = -\underline{\sigma}_0^{\varepsilon m}.$$

For a known nominal trajectory $\underline{e}_0^m$, $\dot{\underline{e}}_0^m$, $\ddot{\underline{e}}_0^m$ the generalised stress resultants $\underline{\sigma}_0^{em} = -\underline{u}_0$ and $\underline{\sigma}_0^{\varepsilon m}$ are obtained.

2. Compute the (small) vibrational motion $\delta\underline{q}$ from linearised equations of motion:

$$\bar{M}_0\delta\ddot{\underline{q}} + C_0\delta\dot{\underline{q}} + \bar{K}_0\delta\underline{q} = \begin{bmatrix}\delta\underline{u}\\ \underline{\sigma}_0^{\varepsilon m}\end{bmatrix}.$$

$\underline{\sigma}_0^{\varepsilon m}$ are the generalised stress resultants applied as internal excitation forces.

$\bar{M}_0$ is the system mass matrix,

C_0 is the velocity sensitivity matrix,

$\bar{K}_0$ is the combined stiffness matrix defined as

$$\bar{K}_0 = \begin{bmatrix}0 & 0\\ 0 & K_0^{\varepsilon\varepsilon}\end{bmatrix} + G_0 + N_0,$$

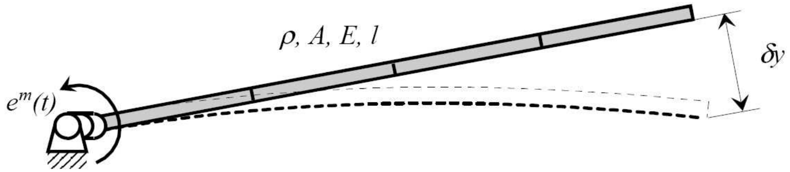
including the structural stiffness matrix $K_0^{\varepsilon\varepsilon}$, the geometric stiffening matrix G_0 and the dynamic stiffening matrix N_0.

Perturbation method: Applications

$$\bar{M}_0\begin{bmatrix}\delta\ddot{\underline{e}}^m\\ \ddot{\underline{\varepsilon}}^m\end{bmatrix} + C_0\begin{bmatrix}\delta\dot{\underline{e}}^m\\ \dot{\underline{\varepsilon}}^m\end{bmatrix} + \bar{K}_0\begin{bmatrix}\delta\underline{e}^m\\ \underline{\varepsilon}^m\end{bmatrix} = \begin{bmatrix}-\delta\underline{\sigma}_0^{em}\\ \underline{\sigma}_0^{\varepsilon m}\end{bmatrix}.$$

1. Constrained motion (§ 12.6 or paper LP-1): $\underline{e}^m = \underline{e}_0^m$, so $\delta\underline{e}^m \equiv 0$.
   → Solve differential equation for $\underline{\varepsilon}^m$ and compute $\delta\underline{\sigma}_0^{em}$.
   → Example of one-link manipulator.
2. Prescribed forces and torques (§ 12.6): $\underline{\sigma}^{em} = \underline{\sigma}_0^{em}$, so $\delta\underline{\sigma}_0^{em} \equiv 0$.
   → Solve differential equation for $\underline{e}^m$ and $\underline{\varepsilon}^m$.
3. Controlled trajectory motion (paper LP-1): $\delta\underline{\sigma}_0^{em} = -\delta\underline{u}$ from control system.
   → Solve differential equation for $\underline{e}^m$ and $\underline{\varepsilon}^m$.
   → Example of two-link manipulator.

Constrained motion: One-link flexible manipulator



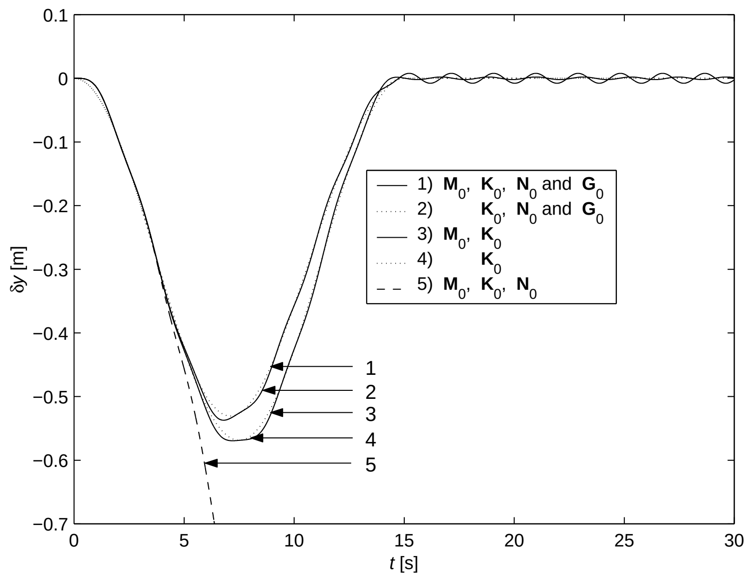
$$\begin{bmatrix}\bar{m}_0^{ee} & \bar{M}_0^{e\varepsilon}\\ \bar{M}_0^{\varepsilon e} & \bar{M}_0^{\varepsilon\varepsilon}\end{bmatrix}\begin{bmatrix}\delta\ddot{e}^m\\ \ddot{\underline{\varepsilon}}^m\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \bar{K}_0^{\varepsilon\varepsilon} + \bar{N}_0^{\varepsilon\varepsilon} + \bar{G}_0^{\varepsilon\varepsilon}\end{bmatrix}\begin{bmatrix}\delta e^m\\ \underline{\varepsilon}^m\end{bmatrix} = \begin{bmatrix}-\delta\sigma^{em}\\ \underline{\sigma}_0^{\varepsilon m}\end{bmatrix}$$

Constrained motion: $e^m = \underline{e}^m(t)$; $\delta\underline{e}^m(t) = 0$.

$$e^m(t) = \begin{cases} 0 & t < 0,\\ \frac{\Omega}{T}\left[\frac{1}{2}t^2 + \frac{T^2}{4\pi^2}(\cos\frac{2\pi t}{T} - 1)\right] & 0 \le t \le T,\\ \Omega(t - \frac{1}{2}T) & t > T.\end{cases}$$

Flexible motion: $\bar{M}_0^{\varepsilon\varepsilon}\ddot{\underline{\varepsilon}}^m + \left[\bar{K}_0^{\varepsilon\varepsilon} + \bar{N}_0^{\varepsilon\varepsilon} + \bar{G}_0^{\varepsilon\varepsilon}\right]\underline{\varepsilon}^m = \underline{\sigma}_0^{\varepsilon m}$

Tip deflection δy using different superposition approximations



Comparison of the maximum tip deflection

Model	number of elements	Max. deflection [m]
SPACAR, non-linear	4	0.536
Wu & Haug	4 substructures	0.556
idem	6 substructures	0.543
superposition, case 1	4	0.537
idem, case 2	4	0.531
idem, case 3	4	0.569
idem, case 4	4	0.569
idem, case 5	4	∞

case 1 $\quad \bar{M}_0^{\varepsilon\varepsilon}\ddot{\underline{\varepsilon}}^m + \left[\bar{K}_0^{\varepsilon\varepsilon} + \bar{N}_0^{\varepsilon\varepsilon} + \bar{G}_0^{\varepsilon\varepsilon}\right]\underline{\varepsilon}^m = \underline{\sigma}_0^{\varepsilon m}$

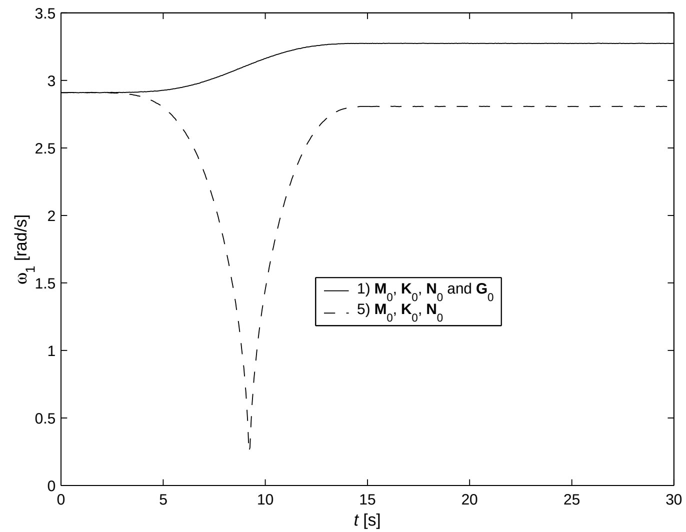
case 2 $\quad \left[\bar{K}_0^{\varepsilon\varepsilon} + \bar{N}_0^{\varepsilon\varepsilon} + \bar{G}_0^{\varepsilon\varepsilon}\right]\underline{\varepsilon}^m = \underline{\sigma}_0^{\varepsilon m}$

case 3 $\quad \bar{M}_0^{\varepsilon\varepsilon}\ddot{\underline{\varepsilon}}^m + \left[\bar{K}_0^{\varepsilon\varepsilon}\right]\underline{\varepsilon}^m = \underline{\sigma}_0^{\varepsilon m}$

case 4 $\quad \left[\bar{K}_0^{\varepsilon\varepsilon}\right]\underline{\varepsilon}^m = \underline{\sigma}_0^{\varepsilon m}$

case 5 $\quad \bar{M}_0^{\varepsilon\varepsilon}\ddot{\underline{\varepsilon}}^m + \left[\bar{K}_0^{\varepsilon\varepsilon} + \bar{N}_0^{\varepsilon\varepsilon}\right]\underline{\varepsilon}^m = \underline{\sigma}_0^{\varepsilon m}$

Natural frequency of the first constrained bending mode



Frequency equation: $\det(-\omega_i^2\bar{M}_0^{\varepsilon\varepsilon} + \bar{K}_0^{\varepsilon\varepsilon} + \bar{N}_0^{\varepsilon\varepsilon} + \bar{G}_0^{\varepsilon\varepsilon}) = 0$

Perturbation method for controlled trajectory motion

As before, model the vibrational motion of the manipulator as a first-order perturbation $\delta\underline{q}$ of the nominal rigid link motion $\underline{q}_0$

$$\underline{q} = \underline{q}_0 + \delta\underline{q},$$

Apply the two steps of the perturbation method:

1. Compute nominal rigid link motion $\underline{q}_0$ from the non-linear equations of motion with all $\underline{\varepsilon}^m \equiv \underline{0}$, i.e. all vibrational modes set to zero.
2. Compute the vibrational motion $\delta\underline{q}$ from linearised equations of motion:

$$\bar{M}_0\delta\ddot{\underline{q}} + C_0\delta\dot{\underline{q}} + \bar{K}_0\delta\underline{q} = \underline{\sigma}_0, \qquad \underline{\sigma}_0 = \begin{bmatrix}\delta\underline{u}_\mathrm{d}\\ \underline{\sigma}_0^{\varepsilon m}\end{bmatrix}.$$

$\underline{\sigma}_0^{\varepsilon m}$ are the generalized stress resultants applied as internal excitation forces.
$\delta\underline{u}_\mathrm{d}$ is the control input vector (minus $\underline{u}_0$) and ...

$\delta\underline{u}_\mathrm{d}$ is the control input vector $\delta\underline{u} = \underline{u} - \underline{u}_0$, *possibly* minus the proportional action of the controller represented by a matrix K_p

$$\delta\underline{u} = -K_p\delta\underline{e}^m + \delta\underline{u}_\mathrm{d}.$$

$\bar{M}_0$ is the system mass matrix,

C_0 is the velocity sensitivity matrix,

$\bar{K}_0$ is the combined stiffness matrix defined as

$$\bar{K}_0 = \begin{bmatrix}0 & 0\\ 0 & K_0^{\varepsilon\varepsilon}\end{bmatrix} + G_0 + N_0 + \begin{bmatrix}K_p & 0\\ 0 & 0\end{bmatrix}.$$

It includes the structural stiffness matrix $K_0^{\varepsilon\varepsilon}$, the geometric stiffening matrix G_0, the dynamic stiffening matrix N_0 and the matrix K_p of the proportional control action.

Note that one can also take $\delta\underline{u}_\mathrm{d} = \delta\underline{u}$ in which case $K_p = 0$ and the proportional control action is *not* included in the linearised equation of motion. Using a realistic K_p is particular beneficial for the modal analysis to be discussed next.

Mode-superposition method

Equations of motion in n principal coordinates η

$$\delta\underline{q} = \Phi\underline{\eta} \qquad \Phi = \left[\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_n\right]$$

$$\delta\underline{\dot{q}} = \Phi\underline{\dot{\eta}} + \dot{\Phi}\underline{\eta} \qquad (\bar{K}_0^S - \omega_i^2 \bar{M}_0)\underline{\phi}_i = \underline{0}, \quad i = 1, 2, ..., n,$$

$$\delta\underline{\ddot{q}} = \Phi\underline{\ddot{\eta}} + 2\dot{\Phi}\underline{\dot{\eta}} + \ddot{\Phi}\underline{\eta} \qquad \text{Symmetric } \bar{K}_0^S = \tfrac{1}{2}(\bar{K}_0 + \bar{K}_0^T)$$

$\bar{K}_0$ includes K_p.

$$\hat{M}\underline{\ddot{\eta}} + \hat{C}\underline{\dot{\eta}} + \hat{K}\underline{\eta} = \underline{\hat{\sigma}},$$

where

$\hat{M} = \Phi^T \bar{M}_0 \Phi$	modal mass matrix,
$\hat{C} = \Phi^T C_0 \Phi + 2\Phi^T \bar{M}_0 \dot{\Phi}$	modal damping matrix,
$\hat{K} = \Phi^T \bar{K}_0 \Phi + \Phi^T C_0 \dot{\Phi} + \Phi^T \bar{M}_0 \ddot{\Phi}$	modal stiffness matrix,
$\underline{\hat{\sigma}} = \Phi^T \underline{\sigma}_0$	modal force vector.

"Adaptive Modal Integration" (AMI): Time-varying nature of modal matrix Φ is taken into account.

Mode-Displacement Method (MDM)

Solution $\delta\underline{\hat{q}}$ using only $\hat{n} < n$ modes

$$\delta\underline{\hat{q}} = \hat{\Phi}\underline{\hat{\eta}}, \qquad \hat{\Phi} = \left[\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_{\hat{n}}\right].$$

Mode-Acceleration Method (MAM)

Improved convergence after rewriting the equations of motion

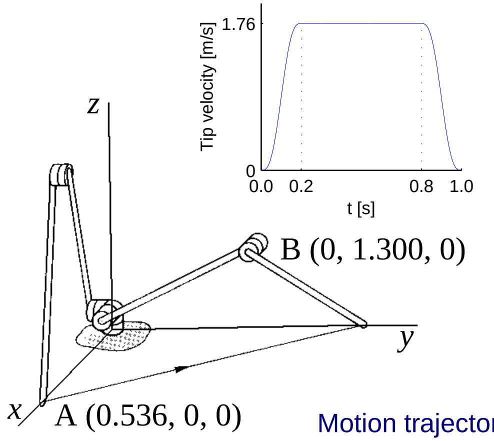
$$\delta\underline{q} = \bar{K}_0^{-1}(\underline{\sigma}_0 - \bar{M}_0\delta\underline{\ddot{q}} - C_0\delta\underline{\dot{q}}),$$

and substitution the MDM solution $\delta\underline{\hat{q}}$ in the right hand side

$$\delta\underline{\tilde{q}} = \bar{K}_0^{-1}\underline{\sigma}_0 - \bar{K}_0^{-1}(\bar{M}_0\delta\underline{\ddot{\hat{q}}} + C_0\delta\underline{\dot{\hat{q}}}).$$

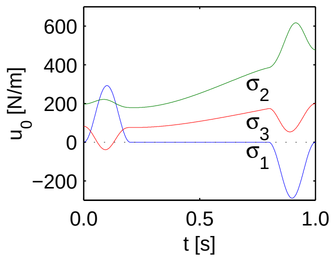
First term in the expression of the MAM solution $\delta\underline{\tilde{q}}$ represents a pseudo static response of the system.

Controlled trajectory motion



Velocity profile of the manipulator tip

Motion trajectory

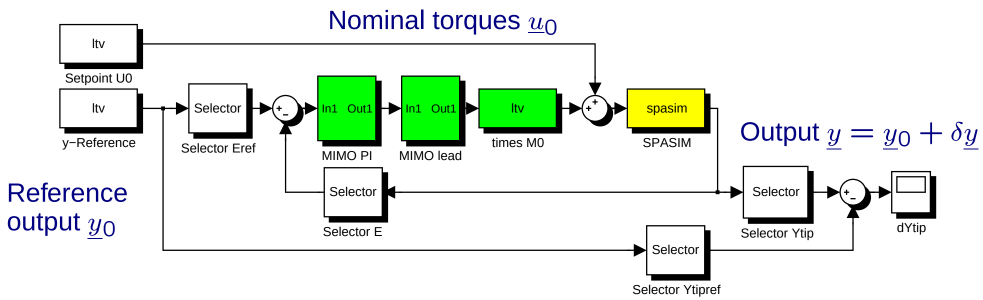


Nominal torques $\underline{u}_0$ for the three actuators

MIMO PID feedback control: $\delta\underline{u} = -\bar{M}_0^{ee}\, H(s)\, \delta\underline{e}^m$.
$\bar{M}_0^{ee}$: mass matrix for decoupling between the actuators.
$H(s)$: controller with three SISO PID controllers on the diagonal.

Block diagram for simulations in Simulink

Non-linear simulation:

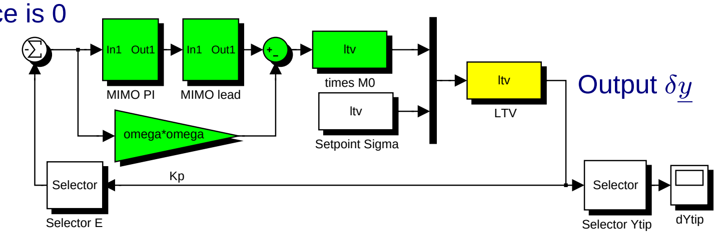


SPASIM block for (non-linear) mechanism simulation.

Nominal torques $\underline{u}_0$ (applied as feedforward) and reference output $\underline{y}_0$ are read from files.

Perturbation method with modal analysis (“AMI”):

Reference is 0



Output δy

LTV-block for simulation of Linear Time-Varying state space system

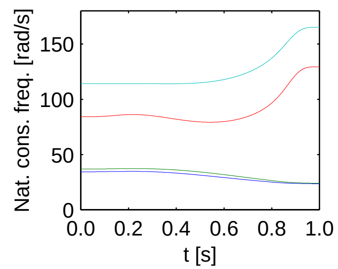
$$\dot{\underline{x}}_{\text{SS}} = A_{\text{SS}}\,\underline{x}_{\text{SS}} + B_{\text{SS}}\,\underline{u}_{\text{SS}} \qquad \underline{x}_{\text{SS}} = \begin{bmatrix}\delta\underline{q}\\ \delta\dot{\underline{q}}\end{bmatrix} \text{ or } \begin{bmatrix}\underline{\eta}\\ \dot{\underline{\eta}}\end{bmatrix}, \qquad \underline{u}_{\text{SS}} = \begin{bmatrix}\delta\underline{u}_{\text{d}}\\ \underline{\sigma}_0^{\varepsilon m}\end{bmatrix},$$

$$\underline{y}_{\text{SS}} = C_{\text{SS}}\,\underline{x}_{\text{SS}}$$

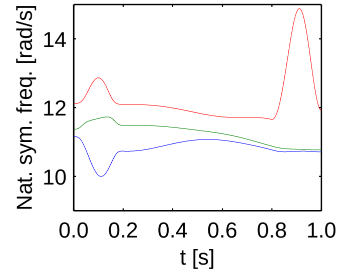
$\underline{y}_{\text{SS}}$: User defined outputs.

Proportional controller part K_p is included in the LTV block and has to be excluded in the controller.

Analysis 1: Natural frequencies along the trajectory

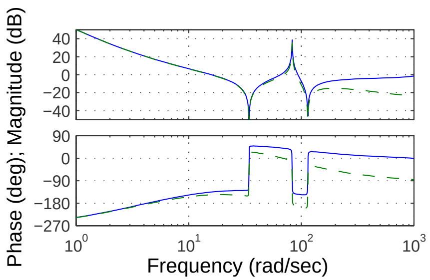


Four lowest natural frequencies $\omega_{c,i}$ for a constrained manipulator. The bandwidth of the PID controllers is set to $\omega_b = 12$ rad/s.

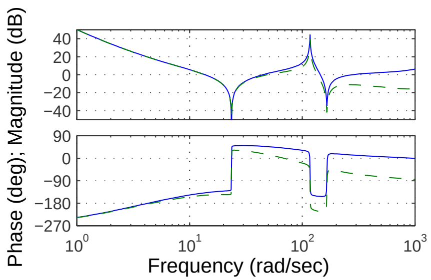


Three lowest closed-loop natural frequencies ω_i during the controlled trajectory motion of the manipulator computed with a symmetric $\bar{K}_0^S$.

Analysis 2: Open loop Bode plots for actuator 1 + controller (SISO)



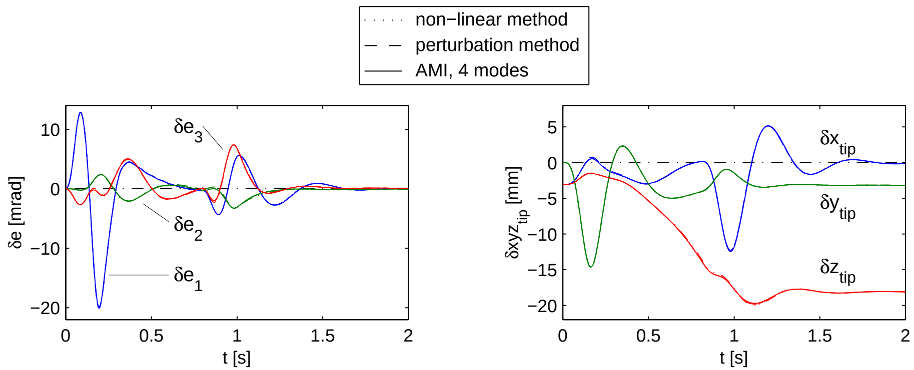
Initial configuration



Final configuration

The solid lines are with the PID controller.
The dashed lines are for a controller with an additional pole: Unstable behaviour is expected from the graph near 100 rad/s, which can be confirmed by simulations.

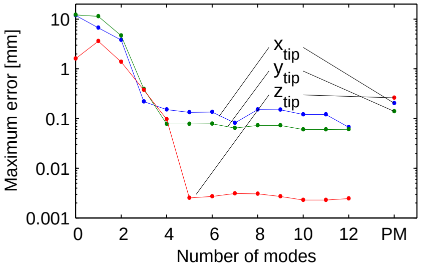
Simulations 1a: Deviations from the nominal trajectory



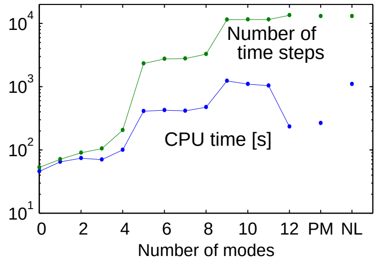
Actuator rotations Tip co-ordinates

Deviations from the nominal trajectory according to three simulation methods: No differences, so the AMI method performs well with only 4 modes (3 modified rigid link modes + 1 additional mode).

Simulations 2a: Errors and CPU time



Maximum errors tip co-ordinates

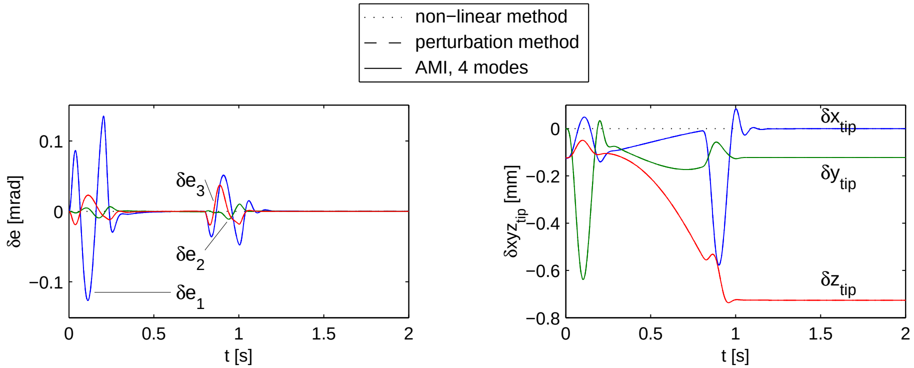


CPU time

- The maximum error in the tip co-ordinates found with the AMI method with only 3 modes is comparable to the difference between the perturbation method and the non-linear simulation (indicated with "PM").
- With more modes the accuracy hardly improves at the expense of slower simulations. A significant reduction in CPU time is obtained in comparison the perturbation method ("PM") and the non-linear simulation ("NL").

Simulations 1b: Deviations from the nominal trajectory

Second case: A stiffer manipulator with a controller bandwidth of approximately 60 rad/s = 9.5 Hz.

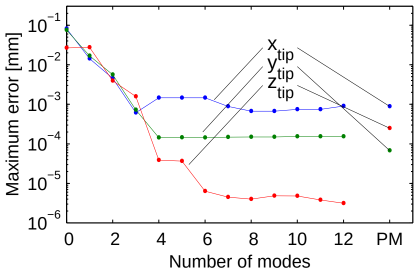


Actuator rotations

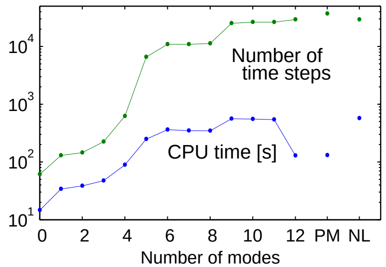
Tip co-ordinates

- Conclusions as before.

Simulations 2b: Errors and CPU time



Maximum errors tip co-ordinates



CPU time (faster PC)

- For the stiffer manipulator the approach works as well. As before the AMI method with only 3 modes gives accurate results for e.g. the tip position.

Conclusions

- The presented perturbation method allows an efficient numerical simulation of the controlled trajectory motion of a flexible manipulator as well as a straightforward vibration control formulation.
- A further reduction of the simulation time was obtained by applying a modal reduction technique, which we refer to as the Adaptive Modal Integration (AMI) method.
- For the spatial flexible two-link manipulator, results of both the perturbation method and the AMI method agree well with the results obtained from a full non-linear analysis. In the AMI method only three (modified rigid link) or four degrees of freedom are needed to reach a satisfying accuracy.
- Crucial elements in the AMI method are the availability of accurately linearized equations and a careful modal analysis in which the time-varying nature of the mode shape functions and the proportional feedback gains are taken into account.