A Linearised Input-Output Representation for Control Synthesis in Flexible Multibody System Dynamics

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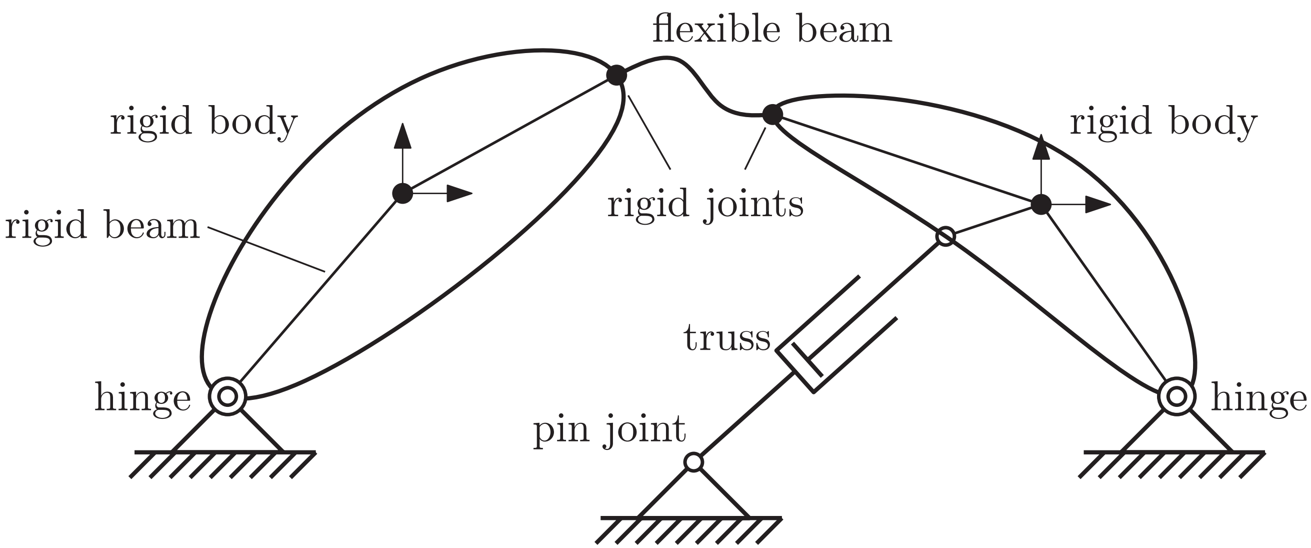
Layout

- Finite element representation of flexible multibody systems
- Equations of motion and reaction
- Linearised equations of motion and reaction
- Linearised state-space equations
- Stationary and equilibrium solutions
- From state-space equations to transfer function(s)
- Illustrative examples
- Conclusions

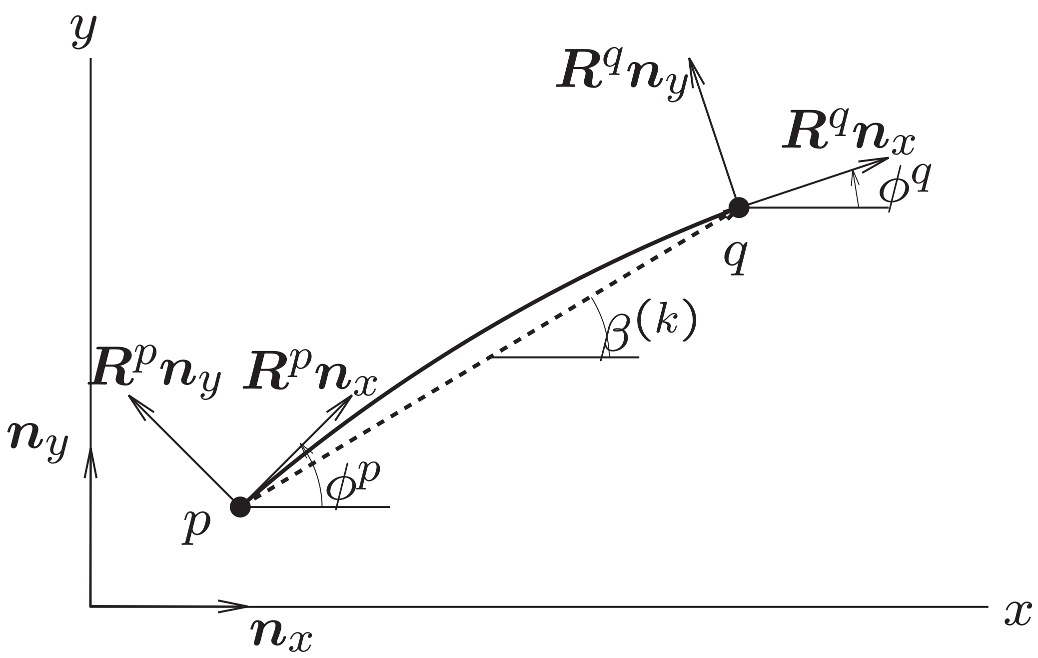
Finite element representation of multibody systems



Physical description of a flexible multibody system

Element k with set of nodal coordinates $\boldsymbol{x}^{(k)}$ (Cartesian and rotational) in a fixed inertial coordinate system and deformation modes specified by a vector of deformation parameters $\boldsymbol{e}^{(k)}$.

Planar flexible beam element



$$\boldsymbol{R}^p \equiv \begin{bmatrix} \cos\phi^p & -\sin\phi^p \\ \sin\phi^p & \cos\phi^p \end{bmatrix}$$

$$\boldsymbol{R}^q \equiv \begin{bmatrix} \cos\phi^q & -\sin\phi^q \\ \sin\phi^q & \cos\phi^q \end{bmatrix}$$

$$\boldsymbol{l}^{(k)} \equiv \boldsymbol{x}^q - \boldsymbol{x}^p = [x^q - x^p,\ y^q - y^p]^T$$

Elongation: $$\varepsilon_1^{(k)} = \mathcal{D}_1^{(k)}(\boldsymbol{x}^{(k)}) = \left((x^q - x^p)^2 + (y^q - y^p)^2\right)^{1/2} - l_0^{(k)}$$

Bending: $$\varepsilon_2^{(k)} = \mathcal{D}_2^{(k)}(\boldsymbol{x}^{(k)}) = -(\boldsymbol{R}^p\boldsymbol{n}_y, \boldsymbol{l}^{(k)})$$

$$\varepsilon_3^{(k)} = \mathcal{D}_3^{(k)}(\boldsymbol{x}^{(k)}) = (\boldsymbol{R}^q\boldsymbol{n}_y, \boldsymbol{l}^{(k)})$$

Kinematic analysis

Deformation equations

$$\boldsymbol{e} = \boldsymbol{\mathcal{D}}(\boldsymbol{x})$$

$$\dot{\boldsymbol{e}} = \frac{\partial \boldsymbol{\mathcal{D}}}{\partial \dot{\boldsymbol{x}}} = \mathbf{D}_x \boldsymbol{\mathcal{D}} \dot{\boldsymbol{x}}$$

$\boldsymbol{x}$: nodal coordinates

$\boldsymbol{e}$: deformation mode coordinates

Partitioning:

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^{(o)} \\ \boldsymbol{x}^{(c)} \\ \boldsymbol{x}^{(m)} \end{bmatrix}$$

fixed coordinates
dependent nodal coordinates
absolute generalized / independent coordinates

$$\boldsymbol{e} = \begin{bmatrix} \boldsymbol{e}^{(o)} \\ \boldsymbol{e}^{(m)} \\ \boldsymbol{e}^{(c)} \end{bmatrix}$$

rigid / zero deformations
relative generalized /independent coordinates
dependent deformations

Generalised coordinates $\boldsymbol{x}^{(m)}, \boldsymbol{e}^{(m)}$ collected in vector $\boldsymbol{q}$ with $ndof$ kinematic degrees of freedom.

Geometric transfer functions

$$\boldsymbol{x} = \boldsymbol{\mathcal{F}}^{(x)}(\boldsymbol{q})$$
$$\boldsymbol{e} = \boldsymbol{\mathcal{F}}^{(e)}(\boldsymbol{q})$$

$\boldsymbol{q}$: generalised coordinates $\boldsymbol{x}^{(m)}$ and $\boldsymbol{e}^{(m)}$

Velocities

$$\dot{\boldsymbol{x}} = \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(x)} \dot{\boldsymbol{q}}$$
$$\dot{\boldsymbol{e}} = \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(e)} \dot{\boldsymbol{q}}$$

$\mathbf{D}_q \boldsymbol{\mathcal{F}}$: first-order geometric transfer function

$\mathbf{D}_q^2 \boldsymbol{\mathcal{F}}$: second-order geometric transfer function

Accelerations

$$\ddot{\boldsymbol{x}} = \mathbf{D}_q^2 \boldsymbol{\mathcal{F}}^{(x)} \dot{\boldsymbol{q}} \dot{\boldsymbol{q}} + \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(x)} \ddot{\boldsymbol{q}}$$
$$\ddot{\boldsymbol{e}} = \mathbf{D}_q^2 \boldsymbol{\mathcal{F}}^{(e)} \dot{\boldsymbol{q}} \dot{\boldsymbol{q}} + \mathbf{D}_q \boldsymbol{\mathcal{F}}^{(e)} \ddot{\boldsymbol{q}}$$

Equations of motion expressed the kinematic degrees of freedom $\boldsymbol{q}$:

$$\bar{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}(\boldsymbol{f} - M\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x,c)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}) - \mathbf{D}\boldsymbol{\mathcal{F}}^{(e)T}\boldsymbol{\sigma}$$

$$\bar{M} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}M\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}$$ system mass matrix

$$\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}\boldsymbol{f} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x,c)T}\boldsymbol{f}^{(c)} + \mathbf{D}\boldsymbol{\mathcal{F}}^{(x,m)T}\boldsymbol{f}^{(m)}$$ nodal forces

$$\mathbf{D}\boldsymbol{\mathcal{F}}^{(e)T}\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,m)T}\boldsymbol{\sigma}^{(m)} + \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,c)T}\boldsymbol{\sigma}^{(c)}$$ stress resultants

$$\begin{bmatrix}\boldsymbol{\sigma}^{(m)}\\ \boldsymbol{\sigma}^{(c)}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\sigma}_a^{(m)}\\ \boldsymbol{\sigma}_a^{(c)}\end{bmatrix} + \begin{bmatrix}\boldsymbol{S}^{(m,m)} & \boldsymbol{S}^{(m,c)}\\ \boldsymbol{S}^{(c,m)} & \boldsymbol{S}^{(c,c)}\end{bmatrix}\begin{bmatrix}\boldsymbol{e}^{(m)}\\ \boldsymbol{e}^{(c)}\end{bmatrix} + \begin{bmatrix}\boldsymbol{S}_d^{(m,m)} & \boldsymbol{S}_d^{(m,c)}\\ \boldsymbol{S}_d^{(c,m)} & \boldsymbol{S}_d^{(c,c)}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{e}}^{(m)}\\ \dot{\boldsymbol{e}}^{(c)}\end{bmatrix}$$

Elastic coefficients $\boldsymbol{S}^{(m,m)}$, $\boldsymbol{S}^{(m,c)}$ and $\boldsymbol{S}^{(c,c)}$ (symmetric matrices)

Viscous damping coefficients $\boldsymbol{S}_d^{(m,m)}$, $\boldsymbol{S}_d^{(m,c)}$ and $\boldsymbol{D}_d^{(c,c)}$ (symmetric matrices)

Driving forces and torques $\boldsymbol{\sigma}_a^{(m)}$ and $\boldsymbol{\sigma}_a^{(c)}$.

Equations of reaction for unknown stress resultants and reaction forces

$$(\mathbf{D}_x\boldsymbol{\mathcal{D}})^T\boldsymbol{\sigma} = \boldsymbol{f} - \boldsymbol{M}\ddot{\boldsymbol{x}} \quad \text{with partitioning } \boldsymbol{f} = \begin{bmatrix} \boldsymbol{f}^{(o)} \\ \boldsymbol{f}^{(c)} \\ \boldsymbol{f}^{(m)} \end{bmatrix} \quad \text{and } \boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{\sigma}^{(o)} \\ \boldsymbol{\sigma}^{(m)} \\ \boldsymbol{\sigma}^{(c)} \end{bmatrix}$$

$$\begin{bmatrix} (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(m)})^T & (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(c)})^T \\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T & (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)})^T \\ (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(m)})^T & (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(c)})^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}^{(o)} \\ \boldsymbol{\sigma}^{(m)} \\ \boldsymbol{\sigma}^{(c)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^{(o)} & - \boldsymbol{M}^{(o,c)}\ddot{\boldsymbol{x}}^{(c)} & - \boldsymbol{M}^{(o,m)}\ddot{\boldsymbol{x}}^{(m)} \\ \boldsymbol{f}^{(c)} & - \boldsymbol{M}^{(c,c)}\ddot{\boldsymbol{x}}^{(c)} & - \boldsymbol{M}^{(c,m)}\ddot{\boldsymbol{x}}^{(m)} \\ \boldsymbol{f}^{(m)} & - \boldsymbol{M}^{(m,c)}\ddot{\boldsymbol{x}}^{(c)} & - \boldsymbol{M}^{(m,m)}\ddot{\boldsymbol{x}}^{(m)} \end{bmatrix}$$

If the square matrix $[(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T,\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T]$ is non-singular, then

$$\begin{bmatrix} \boldsymbol{\sigma}^{(o)} \\ \boldsymbol{\sigma}^{(m)} \end{bmatrix} = \tilde{D}_1 \left[\boldsymbol{f}^{(c)} - \boldsymbol{M}^{(c,c)}\ddot{\boldsymbol{x}}^{(c)} - \boldsymbol{M}^{(c,m)}\ddot{\boldsymbol{x}}^{(m)} - (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)})^T\boldsymbol{\sigma}^{(c)}\right],$$

with $\tilde{D}_1 = \left[(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T,\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T\right]^{-1}$.

Vector $\boldsymbol{\sigma}^{(c)}$ is known from the previous slide, so the reaction forces $\boldsymbol{f}^{(o)}$ and the driving forces $\boldsymbol{f}^{(m)}$ are then determined as well.

State equations

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}^d \\ \boldsymbol{q}^r \end{bmatrix}$$

$\boldsymbol{q}^d$: dynamic degrees of freedom (to be computed)
$\boldsymbol{q}^r$: rheonomic degrees of freedom (known)

$$\begin{bmatrix} \bar{\boldsymbol{M}}_{dd} & \bar{\boldsymbol{M}}_{dr} \\ \bar{\boldsymbol{M}}_{rd} & \bar{\boldsymbol{M}}_{rr} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}^d \\ \ddot{\boldsymbol{q}}^r \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)T} \\ \mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)T} \end{bmatrix} (\boldsymbol{f} - \boldsymbol{M}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}) - \begin{bmatrix} \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e)T} \\ \mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(e)T} \end{bmatrix} \boldsymbol{\sigma}$$

$$\bar{\boldsymbol{M}}_{dd}(\boldsymbol{q})\ddot{\boldsymbol{q}}_d = \bar{\boldsymbol{f}}_d(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - \bar{\boldsymbol{M}}_{dr}\ddot{\boldsymbol{q}}^r$$

$$\bar{\boldsymbol{M}}_{dd} = \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)T}\boldsymbol{M}\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}$$

$$\bar{\boldsymbol{M}}_{dr} = \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)T}\boldsymbol{M}\mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)}$$

$$\bar{\boldsymbol{f}}_d = \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)T}(\boldsymbol{f} - \boldsymbol{M}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}) - \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e)T}\boldsymbol{\sigma}$$

$\boldsymbol{f}$: nodal forces

$\boldsymbol{\sigma}$: stress resultants

$\boldsymbol{M}$: mass matrix

Non-linear state-space equations

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{q}^d \\ \dot{\boldsymbol{q}}^d \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{q}}^d \\ \bar{\boldsymbol{M}}_{dd}^{-1}(\bar{\boldsymbol{f}}_d - \bar{\boldsymbol{M}}_{dr}\ddot{\boldsymbol{q}}^r) \end{bmatrix}$$

with state vector $\boldsymbol{z} = \begin{bmatrix} \boldsymbol{q}^d \\ \dot{\boldsymbol{q}}^d \end{bmatrix}$

Linearised equations: prefix δ indicates small variations

$$\boldsymbol{x} = \boldsymbol{x}_0 + \delta\boldsymbol{x} \qquad \boldsymbol{q} = \boldsymbol{q}_0 + \delta\boldsymbol{q} \quad \text{so } \boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}^d \\ \boldsymbol{q}^r \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_0^d \\ \boldsymbol{q}_0^r \end{bmatrix} + \begin{bmatrix} \delta\boldsymbol{q}^d \\ \delta\boldsymbol{q}^r \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}_0 + \delta\dot{\boldsymbol{x}} \qquad \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_0 + \delta\dot{\boldsymbol{q}} \quad \text{so } \dot{\boldsymbol{q}} = \begin{bmatrix} \dot{\boldsymbol{q}}^d \\ \dot{\boldsymbol{q}}^r \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{q}}_0^d \\ \dot{\boldsymbol{q}}_0^r \end{bmatrix} + \begin{bmatrix} \delta\dot{\boldsymbol{q}}^d \\ \delta\dot{\boldsymbol{q}}^r \end{bmatrix}$$

$$\ddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}}_0 + \delta\ddot{\boldsymbol{x}} \qquad \ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}_0 + \delta\ddot{\boldsymbol{q}} \quad \text{so } \ddot{\boldsymbol{q}} = \begin{bmatrix} \ddot{\boldsymbol{q}}^d \\ \ddot{\boldsymbol{q}}^r \end{bmatrix} = \begin{bmatrix} \ddot{\boldsymbol{q}}_0^d \\ \ddot{\boldsymbol{q}}_0^r \end{bmatrix} + \begin{bmatrix} \delta\ddot{\boldsymbol{q}}^d \\ \delta\ddot{\boldsymbol{q}}^r \end{bmatrix}$$

Stresses $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \delta\boldsymbol{\sigma}_a$ and forces $\boldsymbol{f} = \boldsymbol{f}_0 + \delta\boldsymbol{f}$.

Linearised equations of kinematics

$$\begin{aligned}
\delta\boldsymbol{x} &= \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\delta\boldsymbol{q} \;, \\
\delta\dot{\boldsymbol{x}} &= \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\delta\dot{\boldsymbol{q}} + (\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}})\delta\boldsymbol{q} \;, \\
\delta\ddot{\boldsymbol{x}} &= \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\delta\ddot{\boldsymbol{q}} + 2(\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}})\delta\dot{\boldsymbol{q}} + (\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\ddot{\boldsymbol{q}} + \mathbf{D}^3\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}})\delta\boldsymbol{q}
\end{aligned}$$

with third-order geometric transfer function $\mathbf{D}^3\boldsymbol{\mathcal{F}}^{(x)}$.

Linearised equations of motion

$$\bar{\boldsymbol{M}}\delta\ddot{\boldsymbol{q}} + \left(\bar{\boldsymbol{C}} + \bar{\boldsymbol{D}}\right)\delta\dot{\boldsymbol{q}} + \left(\bar{\boldsymbol{K}} + \bar{\boldsymbol{N}} + \bar{\boldsymbol{G}}\right)\delta\boldsymbol{q} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}\delta\boldsymbol{f} - \mathbf{D}\boldsymbol{\mathcal{F}}^{(e)T}\delta\boldsymbol{\sigma}_a$$

with $\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}\delta\boldsymbol{f} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x,c)T}\delta\boldsymbol{f}^{(c)} + \mathbf{D}\boldsymbol{\mathcal{F}}^{(x,m)T}\delta\boldsymbol{f}^{(m)}$
and $\mathbf{D}\boldsymbol{\mathcal{F}}^{(e)T}\delta\boldsymbol{\sigma}_a = \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,m)T}\delta\boldsymbol{\sigma}_a^{(m)} + \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,c)T}\delta\boldsymbol{\sigma}_a^{(c)}$.

$$\bar{\boldsymbol{M}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}\boldsymbol{M}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}$$

$$\bar{\boldsymbol{C}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}\left((\mathbf{D}_{\dot{x}}\boldsymbol{f}_{in})\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)} + 2\boldsymbol{M}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\right)$$

$$\bar{\boldsymbol{D}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(e)T}\boldsymbol{S}_d\mathbf{D}\boldsymbol{\mathcal{F}}^{(e)}$$

$$\bar{\boldsymbol{K}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(e)T}\boldsymbol{S}\mathbf{D}\boldsymbol{\mathcal{F}}^{(e)}$$

$$\bar{\boldsymbol{N}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}\Big[\mathbf{D}_x(\boldsymbol{M}\ddot{\boldsymbol{x}} - \boldsymbol{f}_{in})\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)} + (\mathbf{D}_{\dot{x}}\boldsymbol{f}_{in})\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}$$
$$+\boldsymbol{M}\left(\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\ddot{\boldsymbol{q}} + (\mathbf{D}^3\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\right)\Big] + \mathbf{D}\boldsymbol{\mathcal{F}}^{(e)T}\boldsymbol{S}_d\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(e)}\dot{\boldsymbol{q}}$$

$$\bar{\boldsymbol{G}} = -\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)T}[\boldsymbol{f} - \boldsymbol{M}\ddot{\boldsymbol{x}}] - \mathbf{D}^2\boldsymbol{\mathcal{F}}^{(e)T}\boldsymbol{\sigma}$$

Linearised equations of reaction

First order terms in Taylor series expansion:

$$(\mathbf{D}_x\boldsymbol{\mathcal{D}})^T\delta\boldsymbol{\sigma} + ((\mathbf{D}_x^2\boldsymbol{\mathcal{D}})^T\boldsymbol{\sigma})\delta\boldsymbol{x} = \delta\boldsymbol{f} + (\mathbf{D}_x\boldsymbol{f}_{in})\delta\boldsymbol{x} + (\mathbf{D}_{\dot{x}}\boldsymbol{f}_{in})\delta\dot{\boldsymbol{x}} - \mathbf{D}_x(\boldsymbol{M}\ddot{\boldsymbol{x}})\delta\boldsymbol{x} - \boldsymbol{M}\delta\ddot{\boldsymbol{x}}$$

or $(\mathbf{D}_x\boldsymbol{\mathcal{D}})^T\delta\boldsymbol{\sigma} = \delta\boldsymbol{f} + \boldsymbol{M}^{(x)}\delta\ddot{\boldsymbol{q}} - \boldsymbol{C}^{(x)}\delta\dot{\boldsymbol{q}} - (\boldsymbol{N}^{(x)} + \boldsymbol{G}^{(x)})\delta\boldsymbol{q}$

$$\boldsymbol{M}^{(x)} = \boldsymbol{M}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}$$

$$\boldsymbol{C}^{(x)} = (\mathbf{D}_{\dot{x}}\boldsymbol{f}_{in})\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)} + 2\boldsymbol{M}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}$$

$$\boldsymbol{N}^{(x)} = \mathbf{D}_x(\boldsymbol{M}\ddot{\boldsymbol{x}} - \boldsymbol{f}_{in})\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)} + (\mathbf{D}_{\dot{x}}\boldsymbol{f}_{in})\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} + \boldsymbol{M}(\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\ddot{\boldsymbol{q}} + \mathbf{D}^3\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}})$$

$$\boldsymbol{G}^{(x)} = ((\mathbf{D}_x^2\boldsymbol{\mathcal{D}})^T\boldsymbol{\sigma})\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}$$

Partitioning:

$$\left[\begin{array}{cc} (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(m)})^T \\ \hline (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T \\ \hline (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(m)})^T \end{array}\right] \left[\begin{array}{c} \delta\boldsymbol{\sigma}^{(o)} \\ \delta\boldsymbol{\sigma}^{(m)} \end{array}\right] = \left[\begin{array}{c} \delta\boldsymbol{f}^{(o)} \\ \delta\boldsymbol{f}^{(c)} \\ \delta\boldsymbol{f}^{(m)} \end{array}\right] - \left[\begin{array}{c} (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(c)})^T \\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)})^T \\ (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(c)})^T \end{array}\right] \left[\delta\boldsymbol{\sigma}^{(c)}\right]$$

$$- \left[\begin{array}{ccc} \boldsymbol{M}^{(x,o)} & \boldsymbol{C}^{(x,o)} & (\boldsymbol{N}^{(x,o)} + \boldsymbol{G}^{(x,o)}) \\ \boldsymbol{M}^{(x,c)} & \boldsymbol{C}^{(x,c)} & (\boldsymbol{N}^{(x,c)} + \boldsymbol{G}^{(x,c)}) \\ \boldsymbol{M}^{(x,m)} & \boldsymbol{C}^{(x,m)} & (\boldsymbol{N}^{(x,m)} + \boldsymbol{G}^{(x,m)}) \end{array}\right] \left[\begin{array}{c} \delta\ddot{\boldsymbol{q}} \\ \delta\dot{\boldsymbol{q}} \\ \delta\boldsymbol{q} \end{array}\right]$$

- Stress resultants of redundant elastic elements $\delta\boldsymbol{\sigma}^{(c)}$ can be expressed in $\delta\boldsymbol{\sigma}_a^{(c)}$, $\delta\boldsymbol{q}$ and $\delta\dot{\boldsymbol{q}}$, see Eq. (41).
- If the square partitioned matrix $[(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T,\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T]$ is not singular, then the generalised stress resultant components of $\delta\boldsymbol{\sigma}^{(o)}$ and $\delta\boldsymbol{\sigma}^{(m)}$ can be computed with Eq. (43) from $\delta\boldsymbol{q}$, $\delta\dot{\boldsymbol{q}}$, $\delta\ddot{\boldsymbol{q}}$, $\delta\boldsymbol{f}^{(c)}$ and $\delta\boldsymbol{\sigma}^{(c)}$.
- Next the reaction forces $\delta\boldsymbol{f}^{(o)}$ and external driving forces $\delta\boldsymbol{f}^{(m)}$ follow from Eqs. (45) and (48).

Linearised state-space equations — state equations

$$\delta\dot{\boldsymbol{z}} = \boldsymbol{A}\delta\boldsymbol{z} + \boldsymbol{B}\delta\boldsymbol{u} \qquad \text{with state vector } \delta\boldsymbol{z} = \begin{bmatrix} \delta\boldsymbol{q}^d \\ \delta\dot{\boldsymbol{q}}^d \end{bmatrix} \quad \text{and}$$

$$\text{input vector } \delta\boldsymbol{u} = \left[\delta\boldsymbol{f}^{(c)T},\ \delta\boldsymbol{f}^{(m)T},\ \delta\boldsymbol{\sigma}_a^{(m)T},\ \delta\boldsymbol{\sigma}_a^{(c)T},\ \delta\ddot{\boldsymbol{q}}^{rT},\ \delta\dot{\boldsymbol{q}}^{rT},\ \delta\boldsymbol{q}^{rT}\right]^T .$$

$$\frac{d}{dt}\begin{bmatrix} \delta\boldsymbol{q}^d \\ \delta\dot{\boldsymbol{q}}^d \end{bmatrix} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{q}^d \\ \delta\dot{\boldsymbol{q}}^d \end{bmatrix} + \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{B}_2 \end{bmatrix} \delta\boldsymbol{u}$$

$$\boldsymbol{A}_{21} = -\bar{\boldsymbol{M}}_{dd}^{-1}\left(\bar{\boldsymbol{K}}_{dd} + \bar{\boldsymbol{N}}_{dd} + \bar{\boldsymbol{G}}_{dd}\right)$$

$$\boldsymbol{A}_{22} = -\bar{\boldsymbol{M}}_{dd}^{-1}\left(\bar{\boldsymbol{C}}_{dd} + \bar{\boldsymbol{D}}_{dd}\right)$$

$$\boldsymbol{B}_2 = \bar{\boldsymbol{M}}_{dd}^{-1}\left[\ \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x,c)T}\ \middle|\ \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x,m)T}\ \middle|\ -\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e,m)T}\ \middle|\ -\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e,c)T}\ \middle|\ -\bar{\boldsymbol{M}}_{dr}\ \middle|\ -\bar{\boldsymbol{C}}_{dr}\ \middle|\ -\left(\bar{\boldsymbol{N}}_{dr} + \bar{\boldsymbol{G}}_{dr}\right)\ \right]^T$$

Linearised state-space equations — output equations

$$\delta \boldsymbol{y} = \boldsymbol{C}\delta \boldsymbol{z} + \boldsymbol{D}\delta \boldsymbol{u}\ , \quad \begin{bmatrix} \delta \boldsymbol{y}^{(kin)} \\ \delta \boldsymbol{y}^{(dyn)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}^{(kin)} \\ \boldsymbol{C}^{(dyn)} \end{bmatrix} \delta \boldsymbol{z} + \begin{bmatrix} \boldsymbol{D}^{(kin)} \\ \boldsymbol{D}^{(dyn)} \end{bmatrix} \delta \boldsymbol{u}$$

$$\delta \boldsymbol{y} = \begin{bmatrix} \delta \boldsymbol{y}^{(kin)} \\ \delta \boldsymbol{y}^{(dyn)} \end{bmatrix}, \quad \delta \boldsymbol{y}^{(kin)} = \begin{bmatrix} \delta \boldsymbol{x} \\ \delta \dot{\boldsymbol{x}} \\ \delta \ddot{\boldsymbol{x}} \end{bmatrix}, \quad \boldsymbol{y}^{(dyn)} = \begin{bmatrix} \delta \boldsymbol{\sigma}^{(o)} \\ \delta \boldsymbol{\sigma}^{(m)} \\ \delta \boldsymbol{f}^{(o)} \\ \delta \boldsymbol{f}^{(m)} \end{bmatrix}$$

Linearised state-space equations — kinematic output matrices

$$\delta \boldsymbol{y}^{(kin)} = \begin{bmatrix} \delta \boldsymbol{x} \\ \delta \dot{\boldsymbol{x}} \\ \delta \ddot{\boldsymbol{x}} \end{bmatrix}$$

$$\boldsymbol{C}^{(kin)} = \left[\begin{array}{c:c} (\delta \boldsymbol{q}^d) & (\delta \dot{\boldsymbol{q}}^d) \\ \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)} & \boldsymbol{O} \\ \mathbf{D}_{q^d}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} & \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)} \\ \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\boldsymbol{A}_{21} + \mathbf{D}_{q^d}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\ddot{\boldsymbol{q}} + \mathbf{D}_{q^d}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}} & \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\boldsymbol{A}_{22} + 2\mathbf{D}_{q^d}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} \end{array}\right]$$

$$\boldsymbol{D}^{(kin)} = \begin{bmatrix} (\delta \boldsymbol{u}) \\ \boldsymbol{O} \\ \boldsymbol{O} \\ \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\boldsymbol{B}_2 \end{bmatrix} + \left[\begin{array}{c:c:c} (\delta \ddot{\boldsymbol{q}}^r) & (\delta \dot{\boldsymbol{q}}^r) & (\delta \boldsymbol{q}^r) \\ \boldsymbol{O} & \boldsymbol{O} & \mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)} \\ \boldsymbol{O} & \mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)} & \mathbf{D}_{q^r}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} \\ \mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)} & 2\mathbf{D}_{q^r}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} & \mathbf{D}_{q^r}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\ddot{\boldsymbol{q}} + \mathbf{D}_{q^r}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}} \end{array}\right]$$

Linearised state-space equations — dynamic output matrices

$$\boldsymbol{y}^{(dyn)} = \begin{bmatrix} \delta\boldsymbol{\sigma}^{(o)} \\ \delta\boldsymbol{\sigma}^{(m)} \\ \delta\boldsymbol{f}^{(o)} \\ \delta\boldsymbol{f}^{(m)} \end{bmatrix}$$

$$\boldsymbol{C}^{(dyn)} = \left[\tilde{\boldsymbol{M}}_d^{(x)}\boldsymbol{A}_2\right] + \Big[\overset{(\delta\boldsymbol{q}^d)}{\tilde{\boldsymbol{K}}_d^{(x)} + \tilde{\boldsymbol{N}}_d^{(x)} + \tilde{\boldsymbol{G}}_d^{(x)}} \;\Big|\; \overset{(\delta\dot{\boldsymbol{q}}^d)}{\tilde{\boldsymbol{C}}_d^{(x)} + \tilde{\boldsymbol{D}}_d^{(x)}}\Big]$$

$$\boldsymbol{D}^{(dyn)} = \left[\tilde{\boldsymbol{M}}_d^{(x)}\boldsymbol{B}_2\right] + \Big[\overset{(\delta\boldsymbol{f}^{(c)})}{\tilde{\boldsymbol{D}}_{\boldsymbol{f}^{(c)}}} \;\Big|\; \overset{(\delta\boldsymbol{f}^{(m)})}{\boldsymbol{O}} \;\Big|\; \overset{(\delta\boldsymbol{\sigma}_a^{(m)})}{\boldsymbol{O}} \;\Big|\; \overset{(\delta\boldsymbol{\sigma}_a^{(c)})}{\tilde{\boldsymbol{D}}_{\boldsymbol{\sigma}^{(c)}}} \;\Big|$$
$$\Big|\; \overset{(\delta\ddot{\boldsymbol{q}}^r)}{\tilde{\boldsymbol{M}}_r^{(x)}} \;\Big|\; \overset{(\delta\dot{\boldsymbol{q}}^r)}{\tilde{\boldsymbol{C}}_r^{(x)} + \tilde{\boldsymbol{D}}_r^{(x)}} \;\Big|\; \overset{(\delta\boldsymbol{q}^r)}{\tilde{\boldsymbol{K}}_r^{(x)} + \tilde{\boldsymbol{N}}_r^{(x)} + \tilde{\boldsymbol{G}}_r^{(x)}}\Big]$$

where the matrices are given in § 8.4.

Stationary and equilibrium solutions

$$\bar{\boldsymbol{M}}_{dd}(\boldsymbol{q})\ddot{\boldsymbol{q}}_d = \bar{\boldsymbol{f}}_d(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - \bar{\boldsymbol{M}}_{dr}\ddot{\boldsymbol{q}}^r$$

$$\ddot{\boldsymbol{q}}^d = \boldsymbol{0}, \quad \dot{\boldsymbol{q}}^d = \boldsymbol{0}, \quad \ddot{\boldsymbol{q}}^r = \boldsymbol{0}$$

$$\begin{bmatrix} \dot{\boldsymbol{q}}^d \\ \bar{\boldsymbol{f}}_d(\boldsymbol{q}^d, \dot{\boldsymbol{q}}^d) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

Stability of stationary solution is determined by eigenvalues of state matrix $\boldsymbol{A}$.

Frequency equation:

$$\det\left(-\omega_i^2 \bar{\boldsymbol{M}}_{dd} + \bar{\boldsymbol{K}}_{dd} + \bar{\boldsymbol{N}}_{dd} + \bar{\boldsymbol{G}}_{dd}\right) = 0$$

Solutions of eigenvalue problem are natural frequencies ω_i and mode shapes.

Stability of equilibrium equation:

$$\det\left(\bar{\boldsymbol{K}}_{dd} + \lambda_i \bar{\boldsymbol{G}}_{dd}\right) = 0$$

$\lambda_i = \boldsymbol{f}_i / \boldsymbol{f}_0$, $\quad \boldsymbol{f}_0$ is reference load and λ_i the critical mode multipliers

From state space equations to transfer function(s)

Linearised state-space equations with state vector $\delta \boldsymbol{z} = \begin{bmatrix} \delta \boldsymbol{q}^d \\ \delta \dot{\boldsymbol{q}}^d \end{bmatrix}$:

$$\delta \dot{\boldsymbol{z}} = \boldsymbol{A} \delta \boldsymbol{z} + \boldsymbol{B} \delta \boldsymbol{u}$$

$$\delta \boldsymbol{y} = \boldsymbol{C} \delta \boldsymbol{z} + \boldsymbol{D} \delta \boldsymbol{u}$$

with general output vector $\delta \boldsymbol{y} = \begin{bmatrix} \delta \boldsymbol{y}^{(kin)} \\ \delta \boldsymbol{y}^{(dyn)} \end{bmatrix}$ and input vector

input vector $\delta \boldsymbol{u} = \left[\delta \boldsymbol{f}^{(c)T},\ \delta \boldsymbol{f}^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(c)T},\ \delta \ddot{\boldsymbol{q}}^{rT},\ \delta \dot{\boldsymbol{q}}^{rT},\ \delta \boldsymbol{q}^{rT} \right]^T$

The state space representation can be translated into a transfer function matrix with the standard expression

$$\tilde{\boldsymbol{G}}(s) = \boldsymbol{C}(sI - \boldsymbol{A})^{-1} \boldsymbol{B} + \boldsymbol{D}$$

which is straightforward for the inputs $\delta \boldsymbol{f}^{(c)}$, $\delta \boldsymbol{f}^{(m)}$, $\delta \boldsymbol{\sigma}_a^{(m)}$ and $\delta \boldsymbol{\sigma}_a^{(c)}$.

Note however the occurrence of $\delta \boldsymbol{q}^r$ **and** the time derivatives $\delta \dot{\boldsymbol{q}}^r$ and $\delta \ddot{\boldsymbol{q}}^r$!!!

A transfer function relates the *Laplace transforms* of the system's input $\delta\boldsymbol{u} = \left[\delta\boldsymbol{f}^{(c)T},\ \delta\boldsymbol{f}^{(m)T},\ \delta\boldsymbol{\sigma}_a^{(m)T},\ \delta\boldsymbol{\sigma}_a^{(c)T},\ \delta\ddot{\boldsymbol{q}}^{rT},\ \delta\dot{\boldsymbol{q}}^{rT},\ \delta\boldsymbol{q}^{rT}\right]^T$ and output $\delta\boldsymbol{y}$:

$$\mathcal{L}\{\delta\boldsymbol{y}(t)\} = \delta\boldsymbol{y}(s) = \tilde{\boldsymbol{G}}(s)\delta\boldsymbol{u}(s) = \tilde{\boldsymbol{G}}(s)\mathcal{L}\{\delta\boldsymbol{u}(t)\}$$

The Laplace transforms of $\delta\boldsymbol{q}^r$, $\delta\dot{\boldsymbol{q}}^r$ and $\delta\ddot{\boldsymbol{q}}^r$ are not independent as

$$\mathcal{L}\{\delta\dot{\boldsymbol{q}}^r(t)\} = s\mathcal{L}\{\delta\boldsymbol{q}^r(t)\} \quad \text{and} \quad \mathcal{L}\{\delta\ddot{\boldsymbol{q}}^r(t)\} = s^2\mathcal{L}\{\delta\boldsymbol{q}^r(t)\}$$

and have to be taken into account to specify e.g. the transfer function (part) from $\delta\boldsymbol{q}^r$ to $\delta\boldsymbol{y}(t)$.

Case 1: Define the input vector as

$$\delta\boldsymbol{u}_1 = \left[\delta\boldsymbol{f}^{(c)T},\ \delta\boldsymbol{f}^{(m)T},\ \delta\boldsymbol{\sigma}_a^{(m)T},\ \delta\boldsymbol{\sigma}_a^{(c)T},\ \delta\boldsymbol{q}^{rT}\right]^T$$

and determine the transfer function in

$$\delta\boldsymbol{y}(s) = \boldsymbol{G}_1(s)\delta\boldsymbol{u}_1(s)$$

Case 1 (position input): Define the input vector as

$$\delta \boldsymbol{u}_1 = \left[\delta \boldsymbol{f}^{(c)T},\ \delta \boldsymbol{f}^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(c)T},\ \delta \boldsymbol{q}^{rT}\right]^T$$

then $\delta \boldsymbol{u}(s) = \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & s^2 I \\ 0 & 0 & 0 & 0 & s\, I \\ 0 & 0 & 0 & 0 & I \end{array}\right] \delta \boldsymbol{u}_1(s)$

and the transfer function from $\delta \boldsymbol{u}_1$ to $\delta \boldsymbol{y}$ is

$$\boldsymbol{G}_1(s) = \left\{\boldsymbol{C}(sI - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right\} \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & s^2 I \\ 0 & 0 & 0 & 0 & s\, I \\ 0 & 0 & 0 & 0 & I \end{array}\right]$$

Case 2 (velocity input): Define the input vector as

$$\delta \boldsymbol{u}_2 = \left[\delta \boldsymbol{f}^{(c)T},\ \delta \boldsymbol{f}^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(c)T},\ \delta \dot{\boldsymbol{q}}^{rT}\right]^T$$

then $$\delta \boldsymbol{u}(s) = \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & s\,I \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 1/s\,I \end{array}\right] \delta \boldsymbol{u}_2(s)$$

and the transfer function from $\delta \boldsymbol{u}_2$ to $\delta \boldsymbol{y}$ is

$$\boldsymbol{G}_2(s) = \left\{\boldsymbol{C}(sI - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right\} \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & s\,I \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 1/s\,I \end{array}\right]$$

Case 3 (acceleration input): Define the input vector as

$$\delta \boldsymbol{u}_3 = \left[\delta \boldsymbol{f}^{(c)T},\ \delta \boldsymbol{f}^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(c)T},\ \delta \ddot{\boldsymbol{q}}^{rT} \right]^T$$

then $\delta \boldsymbol{u}(s) = \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 1/s\ I \\ 0 & 0 & 0 & 0 & 1/s^2\ I \end{array} \right] \delta \boldsymbol{u}_3(s)$

and the transfer function from $\delta \boldsymbol{u}_3$ to $\delta \boldsymbol{y}$ is

$$\boldsymbol{G}_3(s) = \left\{ \boldsymbol{C}(sI - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \right\} \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 1/s\ I \\ 0 & 0 & 0 & 0 & 1/s^2\ I \end{array} \right]$$

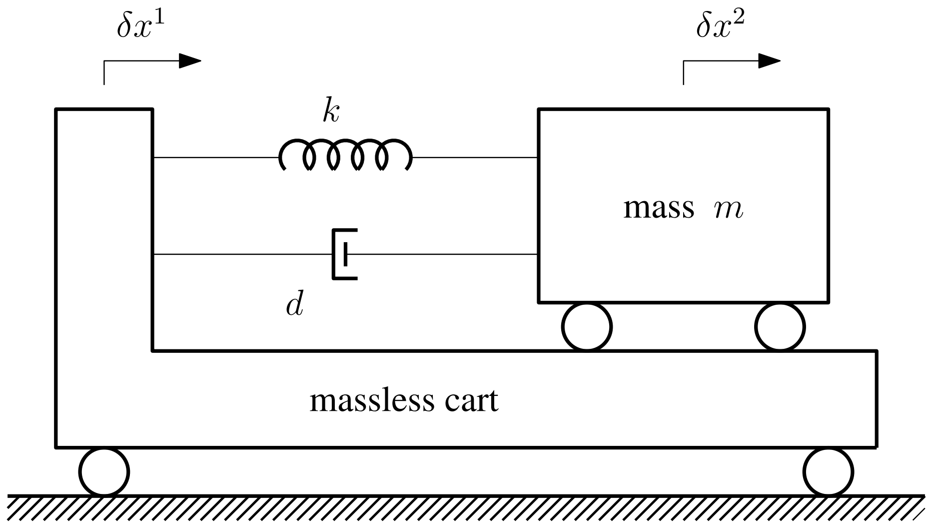
Note 1: The three cases can be combined, e.g. to define an input $\delta\boldsymbol{u}$ with the position of one rheonomic degree of freedom and the acceleration of another rheonomic degree of freedom.

Note 2: In the case only accelerations $\delta\ddot{\boldsymbol{q}}^r$ and no velocities $\delta\dot{\boldsymbol{q}}^r$ and positions $\delta\boldsymbol{q}^r$ appear in the input, the transfer function matrix can also be obtained by adding the velocities and positions to the state vector $\delta\boldsymbol{x}$.

Spring-mass-damper system mounted on a cart

$$\delta q = \delta x^2 - \delta x^1$$

$$\delta \boldsymbol{u} = \left[\delta \ddot{x}^1,\ \delta \dot{x}^1,\ \delta x^1\right]^T$$



$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -m^{-1}k & -m^{-1}d \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \boldsymbol{D} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Transfer functions

$$\widetilde{\boldsymbol{G}}(s) = \frac{\delta x^2(s)}{\delta u(s)} = \boldsymbol{C}(sI - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

$$= \begin{bmatrix} \dfrac{-1}{s^2 + \frac{d}{m}s + \frac{k}{m}} & 0 & 1 \end{bmatrix}$$

Single input $\delta x^1(s)$ to the output $\delta x^2(s)$

$$\delta \boldsymbol{u}(s) = \begin{bmatrix} s^2 & s & 1 \end{bmatrix}^T \delta x^1(s)$$

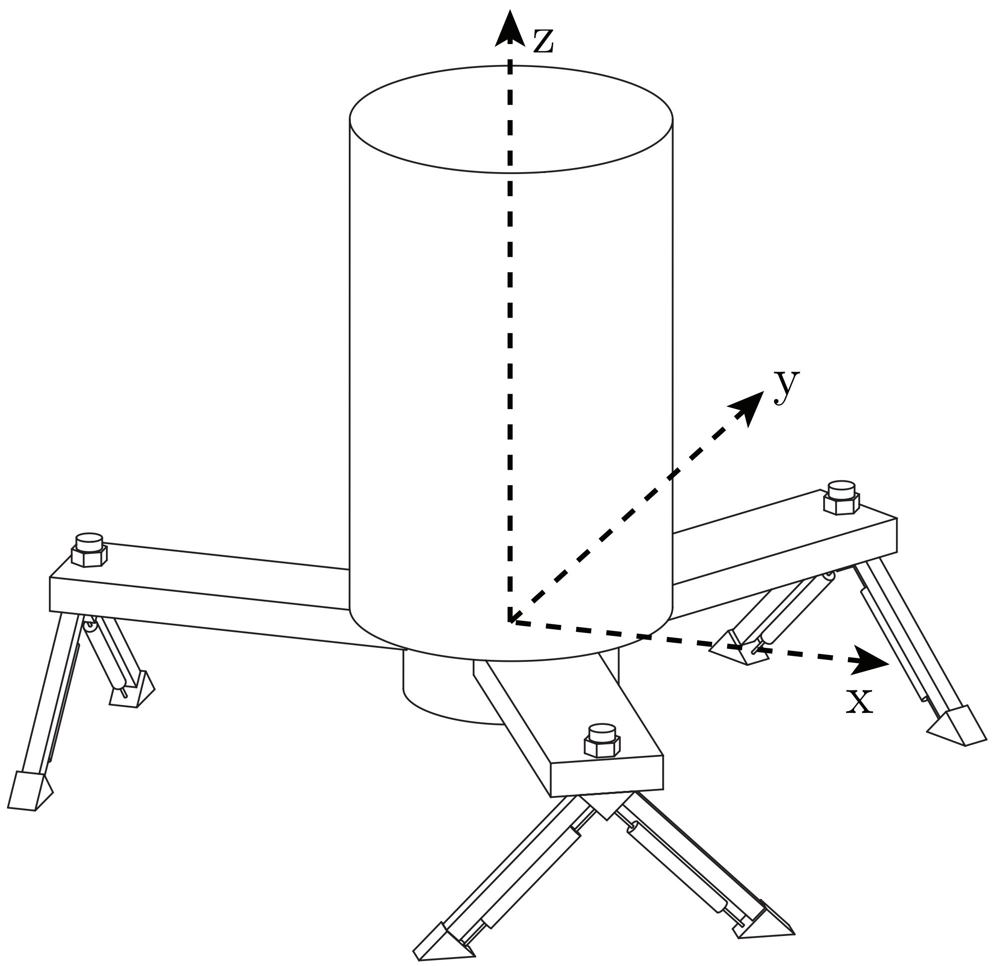
$$G_1(s) = \frac{\delta x^2(s)}{\delta x^1(s)} = \widetilde{\boldsymbol{G}}(s) \begin{bmatrix} s^2 \\ s \\ 1 \end{bmatrix} = \frac{\frac{d}{m}s + \frac{k}{m}}{s^2 + \frac{d}{m}s + \frac{k}{m}}$$

Single input $\delta \dot{x}^1(s) = \delta v^1$ to output $\delta x^2(s)$

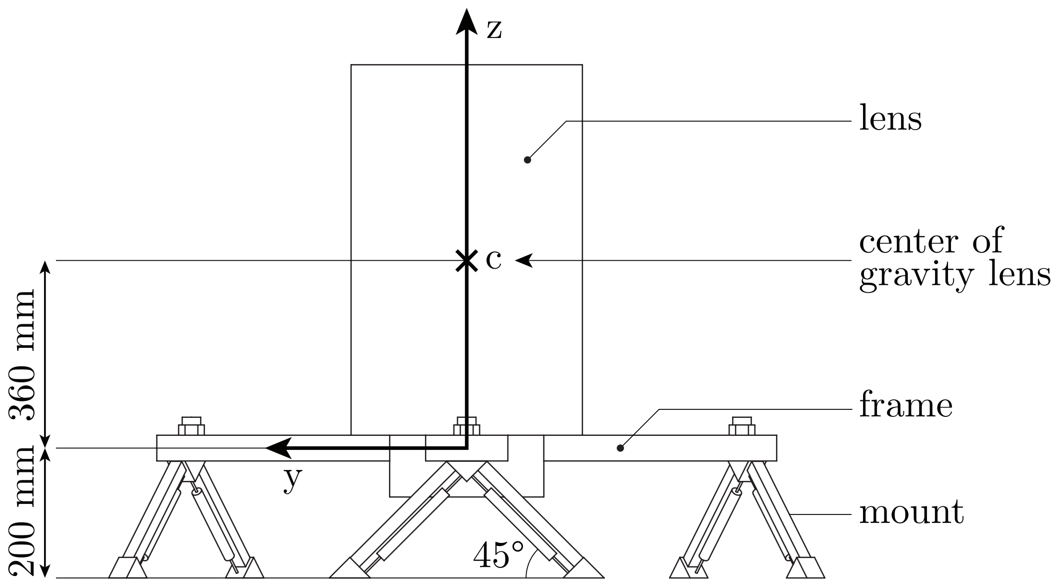
$$\delta \boldsymbol{u}(s) = [s \quad 1 \quad 1/s]^T \delta v^1(s)$$

$$G_2(s) = \frac{\delta x^2(s)}{\delta v^1(s)} = \widetilde{\boldsymbol{G}}(s) \begin{bmatrix} s \\ 1 \\ 1/s \end{bmatrix} = \frac{\frac{d}{m}s + \frac{k}{m}}{s(s^2 + \frac{d}{m}s + \frac{k}{m})}$$

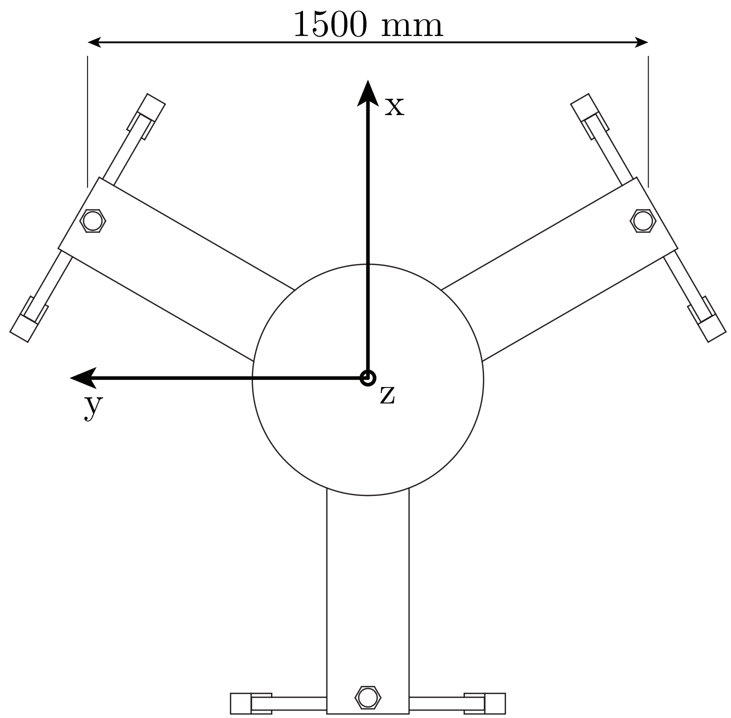
Active vibration isolation of a metrology frame



3D view of lens suspension frame of a wafer stepper/scanner

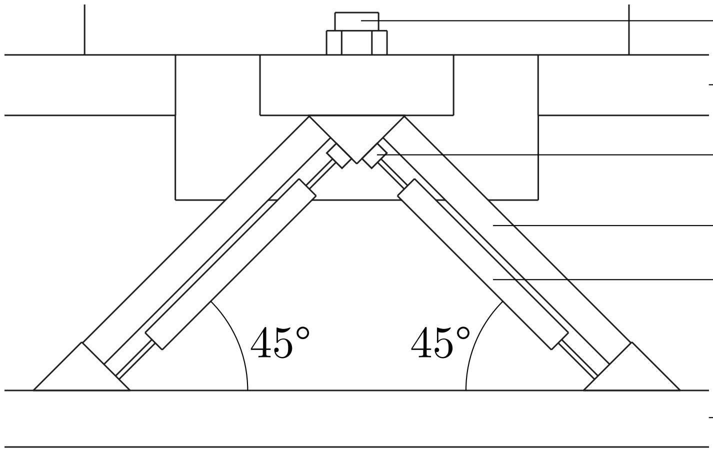


Front view

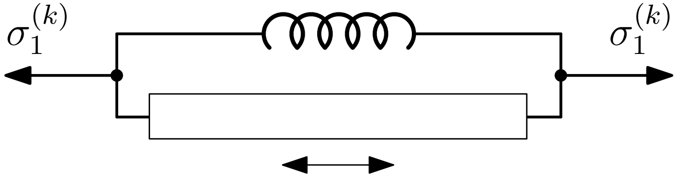


Top view

Detailed view of a mount



Piezo actuator (force $\sigma_a^{(k)}$) with parallel spring (stiffness $s^{(k)}$)



$$\sigma_1^{(k)} = \sigma_a^{(k)} + s^{(k)} e_1^{(k)}$$

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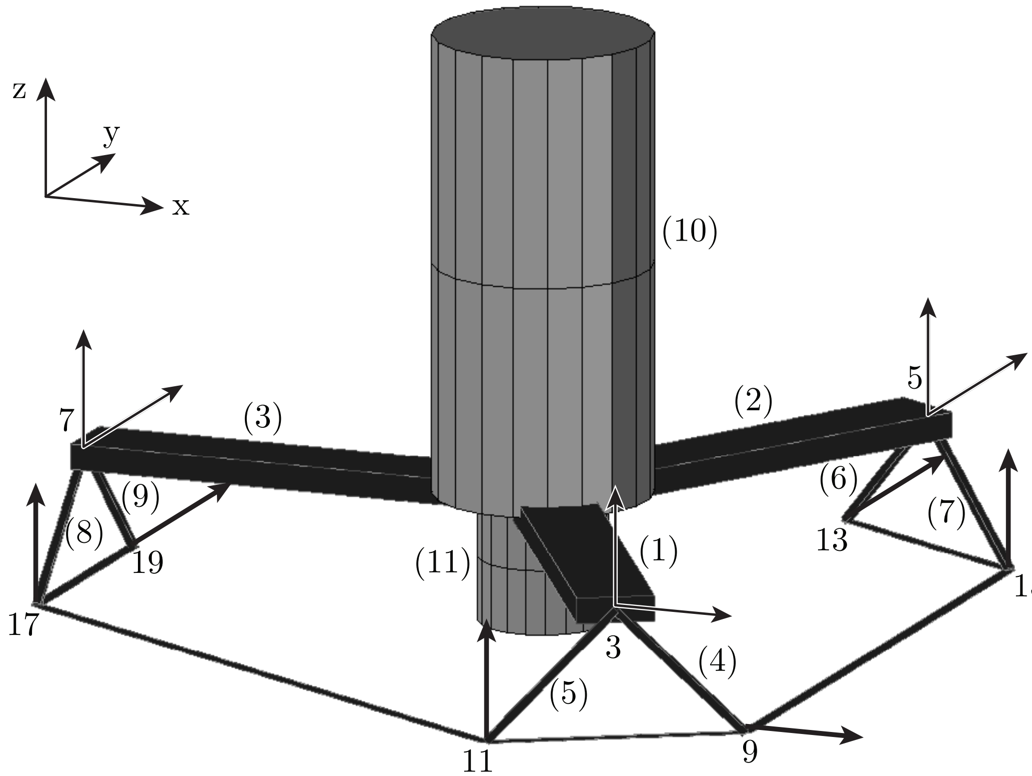
Finite element model of metrology frame and floor

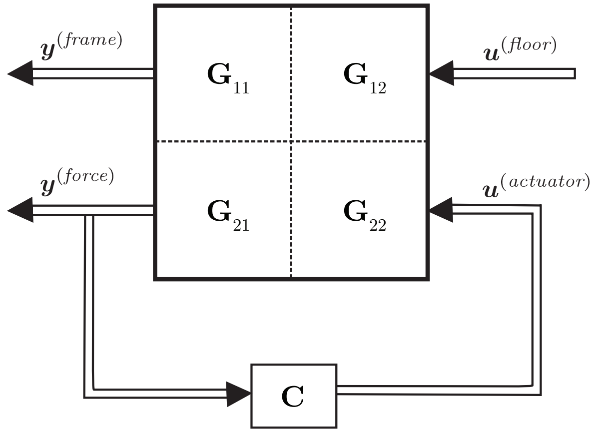


$$\boldsymbol{q}^{(d)} = \left[e_1^{(4)}, e_1^{(5)}, e_1^{(6)}, e_1^{(7)}, e_1^{(8)}, e_1^{(9)}\right]^T$$

$$\boldsymbol{u}^{(floor)} = \left[\ddot{x}^9, \ddot{z}^{11}, \ddot{y}^{13}, \ddot{z}^{15}, \ddot{z}^{17}, \ddot{y}^{19}\right]^T \qquad \boldsymbol{y}^{(frame)} = \left[\ddot{x}^3, \ddot{z}^3, \ddot{y}^5, \ddot{z}^5, \ddot{z}^7, \ddot{y}^7\right]^T$$

$$\boldsymbol{u}^{(actuator)} = \left[\sigma_a^{(4)}, \sigma_a^{(5)}, \sigma_a^{(6)}, \sigma_a^{(7)}, \sigma_a^{(8)}, \sigma_a^{(9)}\right]^T \qquad \boldsymbol{y}^{(force)} = \left[\sigma_1^{(4)}, \sigma_1^{(5)}, \sigma_1^{(6)}, \sigma_1^{(7)}, \sigma_1^{(8)}, \sigma_1^{(9)}\right]^T$$

Generalised plant $\boldsymbol{G}$ with 12 inputs and 12 outputs and controller $\boldsymbol{C}$ with 6 inputs and 6 outputs



Feedback control equations

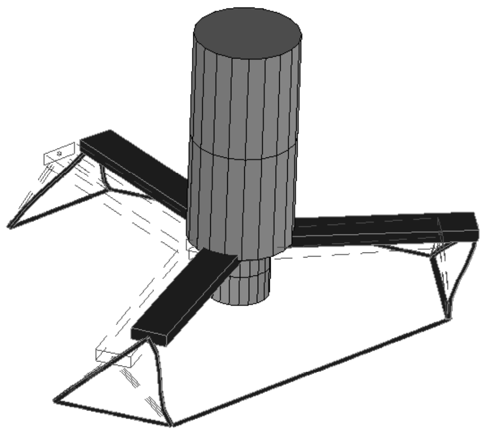
$$\boldsymbol{u}(s)^{(actuator)} = \boldsymbol{C}(s)\boldsymbol{y}(s)^{(force)}$$

$$\boldsymbol{C}(s) = -\left(\boldsymbol{K}^{(P)} + \frac{\boldsymbol{K}^{(I)}}{s}\right)$$

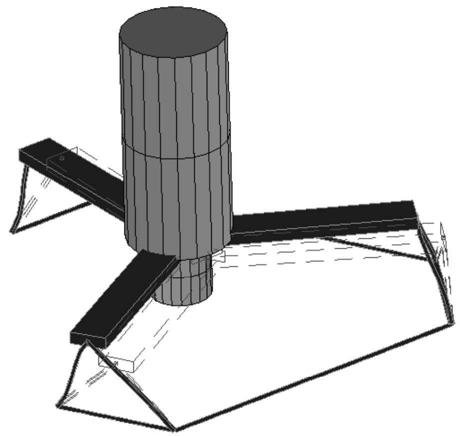
$$\boldsymbol{K}^{(P)} = \left(\omega_d^2 \boldsymbol{I}\bar{\boldsymbol{M}}_{dd}\right)^{-1}\bar{\boldsymbol{K}}_{dd} - \boldsymbol{I}$$

$$\boldsymbol{K}^{(I)} = 2\zeta\omega_d\left(\boldsymbol{I} + \boldsymbol{K}^{(P)}\right)$$

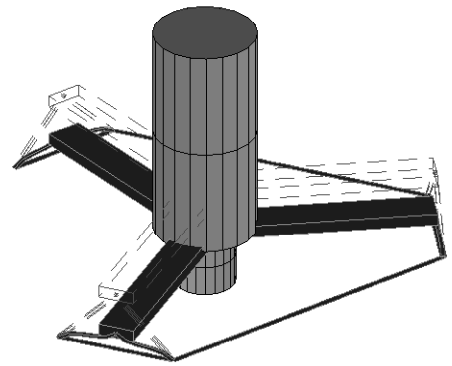
Mode shapes and natural frequencies of the passive system



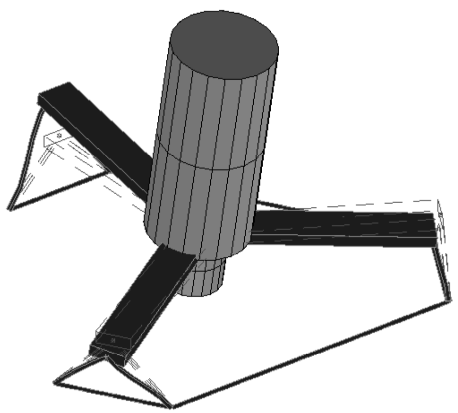
Mode 1: 13.9 Hz



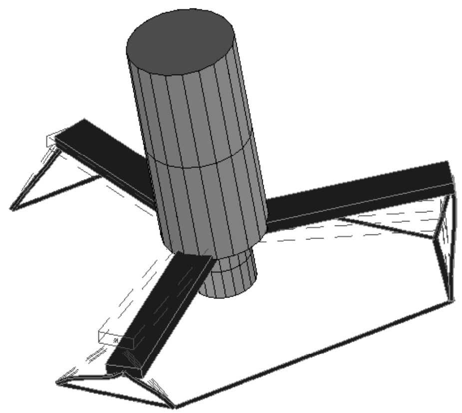
Mode 2: 13.9 Hz



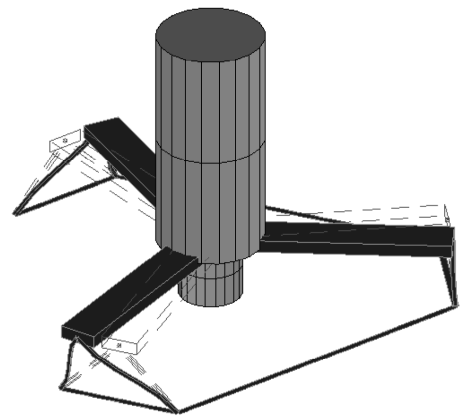
Mode 3: 20.0 Hz



Mode 4: 31.3 Hz



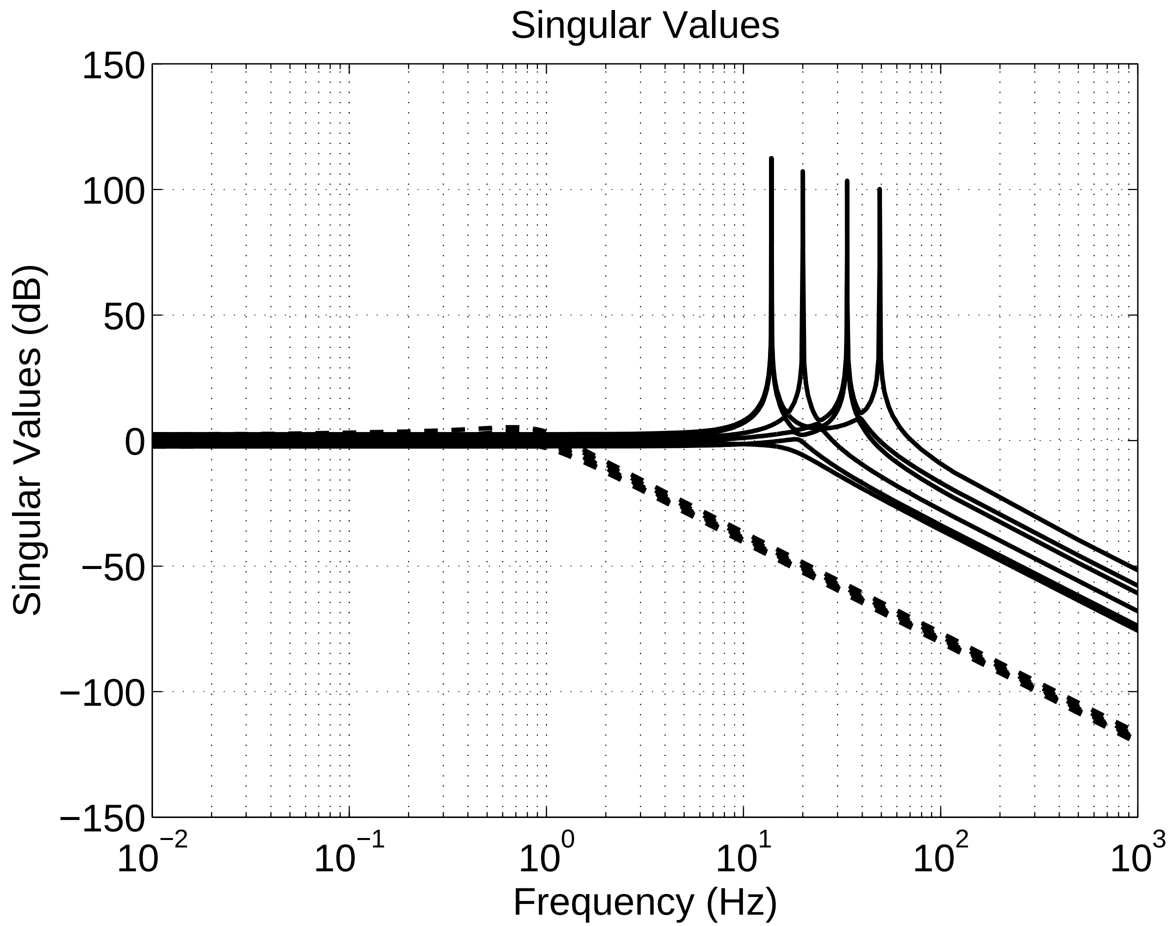
Mode 5: 31.3 Hz



Mode 6: 49.0 Hz

Closed loop transfer function T

$$T = G_{11} + G_{12} \cdot C \cdot (I - G_{22} \cdot C)^{-1} \cdot G_{21}$$



Singular values; dashed line is closed loop, solid line is open loop

Conclusions

- Linearised state-space formulation for flexible multibody systems.
- Arbitrary combination of positions, velocities, accelerations and forces can be taken as input variables and as output variables.
- Finite element based multibody concept enables a low dimensional description of prototype models suitable for design purposes.
- Insight into the relations between component properties and dominant system behaviour.