A Linearised Input-Output Representation for Control Synthesis in Flexible Multibody System Dynamics

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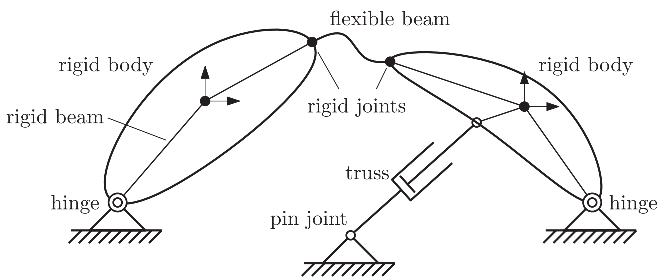
Layout

- Finite element representation of flexible multibody systems
- Equations of motion and reaction
- Linearised equations of motion and reaction
- Linearised state-space equations
- Stationary and equilibrium solutions
- From state-space equations to transfer function(s)
- Illustrative examples
- Conclusions

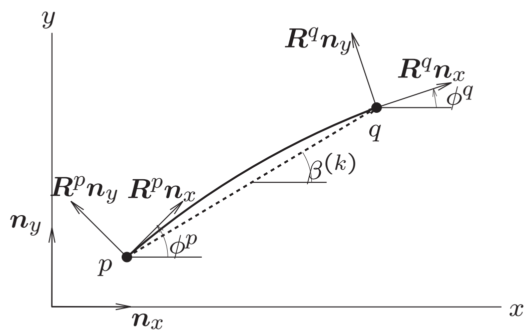
Finite element representation of multibody systems



Physical description of a flexible multibody system

Element k with set of nodal coordinates $\boldsymbol{x}^{(k)}$ (Cartesian and rotational) in a fixed inertial coordinate system and deformation modes specified by a vector of deformation parameters $\boldsymbol{e}^{(k)}$.

Planar flexible beam element



$$\boldsymbol{R}^p \equiv \begin{bmatrix} \cos\phi^p & -\sin\phi^p \\ \sin\phi^p & \cos\phi^p \end{bmatrix}$$

$$\boldsymbol{R}^q \equiv \begin{bmatrix} \cos\phi^q & -\sin\phi^q \\ \sin\phi^q & \cos\phi^q \end{bmatrix}$$

$$\boldsymbol{l}^{(k)} \equiv \boldsymbol{x}^q - \boldsymbol{x}^p = [x^q - x^p,\ y^q - y^p]^T$$

Elongation: $\varepsilon_1^{(k)} = \mathcal{D}_1^{(k)}(\boldsymbol{x}^{(k)}) = \left((x^q - x^p)^2 + (y^q - y^p)^2\right)^{1/2} - l_0^{(k)}$

Bending: $\varepsilon_2^{(k)} = \mathcal{D}_2^{(k)}(\boldsymbol{x}^{(k)}) = -(\boldsymbol{R}^p\boldsymbol{n}_y, \boldsymbol{l}^{(k)})$

$\varepsilon_3^{(k)} = \mathcal{D}_3^{(k)}(\boldsymbol{x}^{(k)}) = (\boldsymbol{R}^q\boldsymbol{n}_y, \boldsymbol{l}^{(k)})$

Kinematic analysis

Deformation equations

$$\boldsymbol{e} = \boldsymbol{\mathcal{D}}(\boldsymbol{x}) \qquad \boldsymbol{x}\text{: nodal coordinates}$$

$$\dot{\boldsymbol{e}} = \frac{\partial \boldsymbol{\mathcal{D}}}{\partial \boldsymbol{x}} = \mathbf{D}_x\boldsymbol{\mathcal{D}}\dot{\boldsymbol{x}} \qquad \boldsymbol{e}\text{: deformation mode coordinates}$$

Partitioning:

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^{(o)} \\ \boldsymbol{x}^{(c)} \\ \boldsymbol{x}^{(m)} \end{bmatrix} \quad \begin{array}{l} \text{fixed coordinates} \\ \text{dependent nodal coordinates} \\ \text{absolute generalized / independent coordinates} \end{array}$$

$$\boldsymbol{e} = \begin{bmatrix} \boldsymbol{e}^{(o)} \\ \boldsymbol{e}^{(m)} \\ \boldsymbol{e}^{(c)} \end{bmatrix} \quad \begin{array}{l} \text{rigid / zero deformations} \\ \text{relative generalized /independent coordinates} \\ \text{dependent deformations} \end{array}$$

Generalised coordinates $\boldsymbol{x}^{(m)}, \boldsymbol{e}^{(m)}$ collected in vector $\boldsymbol{q}$ with $ndof$ kinematic degrees of freedom.

Geometric transfer functions

$$\boldsymbol{x} = \boldsymbol{\mathcal{F}}^{(x)}(\boldsymbol{q}) \qquad \boldsymbol{q}\text{: generalised coordinates } \boldsymbol{x}^{(m)} \text{ and } \boldsymbol{e}^{(m)}$$

$$\boldsymbol{e} = \boldsymbol{\mathcal{F}}^{(e)}(\boldsymbol{q})$$

Velocities

$$\dot{\boldsymbol{x}} = \mathbf{D}_q\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} \qquad \mathbf{D}_q\boldsymbol{\mathcal{F}}\text{: first-order geometric transfer function}$$

$$\dot{\boldsymbol{e}} = \mathbf{D}_q\boldsymbol{\mathcal{F}}^{(e)}\dot{\boldsymbol{q}} \qquad \mathbf{D}_q^2\boldsymbol{\mathcal{F}}\text{: second-order geometric transfer function}$$

Accelerations

$$\ddot{\boldsymbol{x}} = \mathbf{D}_q^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}} + \mathbf{D}_q\boldsymbol{\mathcal{F}}^{(x)}\ddot{\boldsymbol{q}}$$

$$\ddot{\boldsymbol{e}} = \mathbf{D}_q^2\boldsymbol{\mathcal{F}}^{(e)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}} + \mathbf{D}_q\boldsymbol{\mathcal{F}}^{(e)}\ddot{\boldsymbol{q}}$$

Equations of motion expressed the kinematic degrees of freedom $\boldsymbol{q}$:

$$\bar{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}(\boldsymbol{f} - M\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x,c)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}) - \mathbf{D}\boldsymbol{\mathcal{F}}^{(e)T}\boldsymbol{\sigma}$$

$$\bar{M} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}M\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)} \qquad \text{system mass matrix}$$

$$\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)T}\boldsymbol{f} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(x,c)T}\boldsymbol{f}^{(c)} + \mathbf{D}\boldsymbol{\mathcal{F}}^{(x,m)T}\boldsymbol{f}^{(m)} \qquad \text{nodal forces}$$

$$\mathbf{D}\boldsymbol{\mathcal{F}}^{(e)T}\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,m)T}\boldsymbol{\sigma}^{(m)} + \mathbf{D}\boldsymbol{\mathcal{F}}^{(e,c)T}\boldsymbol{\sigma}^{(c)} \qquad \text{stress resultants}$$

$$\begin{bmatrix} \boldsymbol{\sigma}^{(m)} \\ \boldsymbol{\sigma}^{(c)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}_a^{(m)} \\ \boldsymbol{\sigma}_a^{(c)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{S}^{(m,m)} & \boldsymbol{S}^{(m,c)} \\ \boldsymbol{S}^{(c,m)} & \boldsymbol{S}^{(c,c)} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}^{(m)} \\ \boldsymbol{e}^{(c)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{S}_d^{(m,m)} & \boldsymbol{S}_d^{(m,c)} \\ \boldsymbol{S}_d^{(c,m)} & \boldsymbol{S}_d^{(c,c)} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{e}}^{(m)} \\ \dot{\boldsymbol{e}}^{(c)} \end{bmatrix}$$

Elastic coefficients $\boldsymbol{S}^{(m,m)}$, $\boldsymbol{S}^{(m,c)}$ and $\boldsymbol{S}^{(c,c)}$ (symmetric matrices)

Viscous damping coefficients $\boldsymbol{S}_d^{(m,m)}$, $\boldsymbol{S}_d^{(m,c)}$ and $\boldsymbol{D}_d^{(c,c)}$ (symmetric matrices)

Driving forces and torques $\boldsymbol{\sigma}_a^{(m)}$ and $\boldsymbol{\sigma}_a^{(c)}$.

Equations of reaction for unknown stress resultants and reaction forces

$$(\mathbf{D}_x\boldsymbol{\mathcal{D}})^T\boldsymbol{\sigma} = \boldsymbol{f} - M\ddot{\boldsymbol{x}} \quad \text{with partitioning } \boldsymbol{f} = \begin{bmatrix} \boldsymbol{f}^{(o)} \\ \boldsymbol{f}^{(c)} \\ \boldsymbol{f}^{(m)} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{\sigma}^{(o)} \\ \boldsymbol{\sigma}^{(m)} \\ \boldsymbol{\sigma}^{(c)} \end{bmatrix}$$

$$\begin{bmatrix} (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(m)})^T & (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(c)})^T \\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T & (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)})^T \\ (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(m)})^T & (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(c)})^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}^{(o)} \\ \boldsymbol{\sigma}^{(m)} \\ \boldsymbol{\sigma}^{(c)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^{(o)} - M^{(o,c)}\ddot{\boldsymbol{x}}^{(c)} - M^{(o,m)}\ddot{\boldsymbol{x}}^{(m)} \\ \boldsymbol{f}^{(c)} - M^{(c,c)}\ddot{\boldsymbol{x}}^{(c)} - M^{(c,m)}\ddot{\boldsymbol{x}}^{(m)} \\ \boldsymbol{f}^{(m)} - M^{(m,c)}\ddot{\boldsymbol{x}}^{(c)} - M^{(m,m)}\ddot{\boldsymbol{x}}^{(m)} \end{bmatrix}$$

If the square matrix $[(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T,\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T]$ is non-singular, then

$$\begin{bmatrix} \boldsymbol{\sigma}^{(o)} \\ \boldsymbol{\sigma}^{(m)} \end{bmatrix} = \tilde{D}_1\left[\boldsymbol{f}^{(c)} - M^{(c,c)}\ddot{\boldsymbol{x}}^{(c)} - M^{(c,m)}\ddot{\boldsymbol{x}}^{(m)} - (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)})^T\boldsymbol{\sigma}^{(c)}\right],$$

with $\tilde{D}_1 = \left[(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T,\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T\right]^{-1}$.

Vector $\boldsymbol{\sigma}^{(c)}$ is known from the previous slide, so the reaction forces $\boldsymbol{f}^{(o)}$ and the driving forces $\boldsymbol{f}^{(m)}$ are then determined as well.

State equations

$$q = \begin{bmatrix} q^d \\ q^r \end{bmatrix}$$

q^d : dynamic degrees of freedom (to be computed)
q^r : rheonomic degrees of freedom (known)

$$\begin{bmatrix} \bar{M}_{dd} & \bar{M}_{dr} \\ \bar{M}_{rd} & \bar{M}_{rr} \end{bmatrix} \begin{bmatrix} \ddot{q}^d \\ \ddot{q}^r \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{q^d}\mathcal{F}^{(x)T} \\ \mathbf{D}_{q^r}\mathcal{F}^{(x)T} \end{bmatrix} (f - M\mathbf{D}^2\mathcal{F}^{(x)}\dot{q}\dot{q}) - \begin{bmatrix} \mathbf{D}_{q^d}\mathcal{F}^{(e)T} \\ \mathbf{D}_{q^r}\mathcal{F}^{(e)T} \end{bmatrix} \sigma$$

$$\bar{M}_{dd}(q)\ddot{q}_d = \bar{f}_d(q, \dot{q}, t) - \bar{M}_{dr}\ddot{q}^r$$

f: nodal forces

$$\bar{M}_{dd} = \mathbf{D}_{q^d}\mathcal{F}^{(x)T} M \mathbf{D}_{q^d}\mathcal{F}^{(x)}$$

σ: stress resultants

$$\bar{M}_{dr} = \mathbf{D}_{q^d}\mathcal{F}^{(x)T} M \mathbf{D}_{q^r}\mathcal{F}^{(x)}$$

M: mass matrix

$$\bar{f}_d = \mathbf{D}_{q^d}\mathcal{F}^{(x)T}(f - M\mathbf{D}^2\mathcal{F}^{(x)}\dot{q}\dot{q}) - \mathbf{D}_{q^d}\mathcal{F}^{(e)T}\sigma$$

Non-linear state-space equations

$$\frac{d}{dt}\begin{bmatrix} q^d \\ \dot{q}^d \end{bmatrix} = \begin{bmatrix} \dot{q}^d \\ \bar{M}_{dd}^{-1}(\bar{f}_d - \bar{M}_{dr}\ddot{q}^r) \end{bmatrix}$$

with state vector $z = \begin{bmatrix} q^d \\ \dot{q}^d \end{bmatrix}$

Linearised equations: prefix δ indicates small variations

$$x = x_0 + \delta x \qquad q = q_0 + \delta q \quad \text{so } q = \begin{bmatrix} q^d \\ q^r \end{bmatrix} = \begin{bmatrix} q_0^d \\ q_0^r \end{bmatrix} + \begin{bmatrix} \delta q^d \\ \delta q^r \end{bmatrix}$$

$$\dot{x} = \dot{x}_0 + \delta\dot{x} \qquad \dot{q} = \dot{q}_0 + \delta\dot{q} \quad \text{so } \dot{q} = \begin{bmatrix} \dot{q}^d \\ \dot{q}^r \end{bmatrix} = \begin{bmatrix} \dot{q}_0^d \\ \dot{q}_0^r \end{bmatrix} + \begin{bmatrix} \delta\dot{q}^d \\ \delta\dot{q}^r \end{bmatrix}$$

$$\ddot{x} = \ddot{x}_0 + \delta\ddot{x} \qquad \ddot{q} = \ddot{q}_0 + \delta\ddot{q} \quad \text{so } \ddot{q} = \begin{bmatrix} \ddot{q}^d \\ \ddot{q}^r \end{bmatrix} = \begin{bmatrix} \ddot{q}_0^d \\ \ddot{q}_0^r \end{bmatrix} + \begin{bmatrix} \delta\ddot{q}^d \\ \delta\ddot{q}^r \end{bmatrix}$$

Stresses $\sigma = \sigma_0 + \delta\sigma_a$ and forces $f = f_0 + \delta f$.

Linearised equations of kinematics

$$\delta x = \mathbf{D}\mathcal{F}^{(x)}\delta q \,,$$
$$\delta\dot{x} = \mathbf{D}\mathcal{F}^{(x)}\delta\dot{q} + (\mathbf{D}^2\mathcal{F}^{(x)}\dot{q})\delta q \,,$$
$$\delta\ddot{x} = \mathbf{D}\mathcal{F}^{(x)}\delta\ddot{q} + 2(\mathbf{D}^2\mathcal{F}^{(x)}\dot{q})\delta\dot{q} + (\mathbf{D}^2\mathcal{F}^{(x)}\ddot{q} + \mathbf{D}^3\mathcal{F}^{(x)}\dot{q}\dot{q})\delta q$$

with third-order geometric transfer function $\mathbf{D}^3\mathcal{F}^{(x)}$.

Linearised equations of motion

$$\bar{M}\delta\ddot{q} + \left(\bar{C} + \bar{D}\right)\delta\dot{q} + \left(\bar{K} + \bar{N} + \bar{G}\right)\delta q = \mathbf{D}\mathcal{F}^{(x)T}\delta f - \mathbf{D}\mathcal{F}^{(e)T}\delta\sigma_a$$

with $\mathbf{D}\mathcal{F}^{(x)T}\delta f = \mathbf{D}\mathcal{F}^{(x,c)T}\delta f^{(c)} + \mathbf{D}\mathcal{F}^{(x,m)T}\delta f^{(m)}$
and $\mathbf{D}\mathcal{F}^{(e)T}\delta\sigma_a = \mathbf{D}\mathcal{F}^{(e,m)T}\delta\sigma_a^{(m)} + \mathbf{D}\mathcal{F}^{(e,c)T}\delta\sigma_a^{(c)}$.

$$\bar{M} = \mathbf{D}\mathcal{F}^{(x)T} M \mathbf{D}\mathcal{F}^{(x)}$$
$$\bar{C} = \mathbf{D}\mathcal{F}^{(x)T}\left((\mathbf{D}_{\dot{x}} f_{in})\mathbf{D}\mathcal{F}^{(x)} + 2M\mathbf{D}^2\mathcal{F}^{(x)}\dot{q}\right)$$
$$\bar{D} = \mathbf{D}\mathcal{F}^{(e)T} S_d \mathbf{D}\mathcal{F}^{(e)}$$
$$\bar{K} = \mathbf{D}\mathcal{F}^{(e)T} S \mathbf{D}\mathcal{F}^{(e)}$$
$$\bar{N} = \mathbf{D}\mathcal{F}^{(x)T}\left[\mathbf{D}_x(M\ddot{x} - f_{in})\mathbf{D}\mathcal{F}^{(x)} + (\mathbf{D}_{\dot{x}} f_{in})\mathbf{D}^2\mathcal{F}^{(x)}\dot{q} + M\left(\mathbf{D}^2\mathcal{F}^{(x)}\ddot{q} + (\mathbf{D}^3\mathcal{F}^{(x)}\dot{q})\dot{q}\right)\right] + \mathbf{D}\mathcal{F}^{(e)T} S_d \mathbf{D}^2\mathcal{F}^{(e)}\dot{q}$$
$$\bar{G} = -\mathbf{D}^2\mathcal{F}^{(x)T}[f - M\ddot{x}] - \mathbf{D}^2\mathcal{F}^{(e)T}\sigma$$

Linearised equations of reaction

First order terms in Taylor series expansion:

$$(\mathbf{D}_x\mathcal{D})^T\delta\sigma + ((\mathbf{D}_x^2\mathcal{D})^T\sigma)\delta x = \delta f + (\mathbf{D}_x f_{in})\delta x + (\mathbf{D}_{\dot{x}} f_{in})\delta\dot{x} - \mathbf{D}_x(M\ddot{x})\delta x - M\delta\ddot{x}$$

or $(\mathbf{D}_x\mathcal{D})^T\delta\sigma = \delta f + M^{(x)}\delta\ddot{q} - C^{(x)}\delta\dot{q} - (N^{(x)} + G^{(x)})\delta q$

$$M^{(x)} = M\mathbf{D}\mathcal{F}^{(x)}$$
$$C^{(x)} = (\mathbf{D}_{\dot{x}} f_{in})\mathbf{D}\mathcal{F}^{(x)} + 2M\mathbf{D}^2\mathcal{F}^{(x)}\dot{q}$$
$$N^{(x)} = \mathbf{D}_x(M\ddot{x} - f_{in})\mathbf{D}\mathcal{F}^{(x)} + (\mathbf{D}_{\dot{x}} f_{in})\mathbf{D}^2\mathcal{F}^{(x)}\dot{q} + M(\mathbf{D}^2\mathcal{F}^{(x)}\ddot{q} + \mathbf{D}^3\mathcal{F}^{(x)}\dot{q}\dot{q})$$
$$G^{(x)} = ((\mathbf{D}_x^2\mathcal{D})^T\sigma)\mathbf{D}\mathcal{F}^{(x)}$$

Partitioning:

$$\left[\begin{array}{cc} (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(m)})^T \\ \hdashline (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T \\ \hdashline (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(o)})^T & (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(m)})^T \end{array}\right] \begin{bmatrix} \delta\boldsymbol{\sigma}^{(o)} \\ \delta\boldsymbol{\sigma}^{(m)} \end{bmatrix} = \begin{bmatrix} \delta\boldsymbol{f}^{(o)} \\ \delta\boldsymbol{f}^{(c)} \\ \delta\boldsymbol{f}^{(m)} \end{bmatrix} - \begin{bmatrix} (\mathbf{D}^{(o)}\boldsymbol{\mathcal{D}}^{(c)})^T \\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(c)})^T \\ (\mathbf{D}^{(m)}\boldsymbol{\mathcal{D}}^{(c)})^T \end{bmatrix} \left[\delta\boldsymbol{\sigma}^{(c)}\right]$$

$$- \begin{bmatrix} \boldsymbol{M}^{(x,o)} & \boldsymbol{C}^{(x,o)} & (\boldsymbol{N}^{(x,o)} + \boldsymbol{G}^{(x,o)}) \\ \boldsymbol{M}^{(x,c)} & \boldsymbol{C}^{(x,c)} & (\boldsymbol{N}^{(x,c)} + \boldsymbol{G}^{(x,c)}) \\ \boldsymbol{M}^{(x,m)} & \boldsymbol{C}^{(x,m)} & (\boldsymbol{N}^{(x,m)} + \boldsymbol{G}^{(x,m)}) \end{bmatrix} \begin{bmatrix} \delta\ddot{\boldsymbol{q}} \\ \delta\dot{\boldsymbol{q}} \\ \delta\boldsymbol{q} \end{bmatrix}$$

- Stress resultants of redundant elastic elements $\delta\boldsymbol{\sigma}^{(c)}$ can be expressed in $\delta\boldsymbol{\sigma}_a^{(c)}$, $\delta\boldsymbol{q}$ and $\delta\dot{\boldsymbol{q}}$, see Eq. (41).
- If the square partitioned matrix $[(\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(o)})^T,\ (\mathbf{D}^{(c)}\boldsymbol{\mathcal{D}}^{(m)})^T]$ is not singular, then the generalised stress resultant components of $\delta\boldsymbol{\sigma}^{(o)}$ and $\delta\boldsymbol{\sigma}^{(m)}$ can be computed with Eq. (43) from $\delta\boldsymbol{q}$, $\delta\dot{\boldsymbol{q}}$, $\delta\ddot{\boldsymbol{q}}$, $\delta\boldsymbol{f}^{(c)}$ and $\delta\boldsymbol{\sigma}^{(c)}$.
- Next the reaction forces $\delta\boldsymbol{f}^{(o)}$ and external driving forces $\delta\boldsymbol{f}^{(m)}$ follow from Eqs. (45) and (48).

Linearised state-space equations — state equations

$$\delta\dot{\boldsymbol{z}} = \boldsymbol{A}\delta\boldsymbol{z} + \boldsymbol{B}\delta\boldsymbol{u} \qquad \text{with state vector } \delta\boldsymbol{z} = \begin{bmatrix} \delta\boldsymbol{q}^d \\ \delta\dot{\boldsymbol{q}}^d \end{bmatrix} \quad \text{and}$$

input vector $\delta\boldsymbol{u} = \left[\delta\boldsymbol{f}^{(c)T},\ \delta\boldsymbol{f}^{(m)T},\ \delta\boldsymbol{\sigma}_a^{(m)T},\ \delta\boldsymbol{\sigma}_a^{(c)T},\ \delta\ddot{\boldsymbol{q}}^{rT},\ \delta\dot{\boldsymbol{q}}^{rT},\ \delta\boldsymbol{q}^{rT}\right]^T$.

$$\frac{d}{dt}\begin{bmatrix} \delta\boldsymbol{q}^d \\ \delta\dot{\boldsymbol{q}}^d \end{bmatrix} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{q}^d \\ \delta\dot{\boldsymbol{q}}^d \end{bmatrix} + \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{B}_2 \end{bmatrix} \delta\boldsymbol{u}$$

$$\boldsymbol{A}_{21} = -\bar{\boldsymbol{M}}_{dd}^{-1}\left(\bar{\boldsymbol{K}}_{dd} + \bar{\boldsymbol{N}}_{dd} + \bar{\boldsymbol{G}}_{dd}\right)$$
$$\boldsymbol{A}_{22} = -\bar{\boldsymbol{M}}_{dd}^{-1}\left(\bar{\boldsymbol{C}}_{dd} + \bar{\boldsymbol{D}}_{dd}\right)$$

$$\boldsymbol{B}_2 = \bar{\boldsymbol{M}}_{dd}^{-1}\left[\ \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x,c)T} \ \middle|\ \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x,m)T} \ \middle|\ -\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e,m)T} \ \middle|\ -\mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(e,c)T} \ \middle|\ -\bar{\boldsymbol{M}}_{dr} \ \middle|\ -\bar{\boldsymbol{C}}_{dr} \ \middle|\ -\left(\bar{\boldsymbol{N}}_{dr} + \bar{\boldsymbol{G}}_{dr}\right)\ \right]^T$$

Linearised state-space equations — output equations

$$\delta\boldsymbol{y} = \boldsymbol{C}\delta\boldsymbol{z} + \boldsymbol{D}\delta\boldsymbol{u}\ , \quad \begin{bmatrix} \delta\boldsymbol{y}^{(kin)} \\ \delta\boldsymbol{y}^{(dyn)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}^{(kin)} \\ \boldsymbol{C}^{(dyn)} \end{bmatrix} \delta\boldsymbol{z} + \begin{bmatrix} \boldsymbol{D}^{(kin)} \\ \boldsymbol{D}^{(dyn)} \end{bmatrix} \delta\boldsymbol{u}$$

$$\delta\boldsymbol{y} = \begin{bmatrix} \delta\boldsymbol{y}^{(kin)} \\ \delta\boldsymbol{y}^{(dyn)} \end{bmatrix}\ , \quad \delta\boldsymbol{y}^{(kin)} = \begin{bmatrix} \delta\boldsymbol{x} \\ \delta\dot{\boldsymbol{x}} \\ \delta\ddot{\boldsymbol{x}} \end{bmatrix}\ , \quad \boldsymbol{y}^{(dyn)} = \begin{bmatrix} \delta\boldsymbol{\sigma}^{(o)} \\ \delta\boldsymbol{\sigma}^{(m)} \\ \delta\boldsymbol{f}^{(o)} \\ \delta\boldsymbol{f}^{(m)} \end{bmatrix}$$

Linearised state-space equations — kinematic output matrices

$$\delta\boldsymbol{y}^{(kin)} = \begin{bmatrix} \delta\boldsymbol{x} \\ \delta\dot{\boldsymbol{x}} \\ \delta\ddot{\boldsymbol{x}} \end{bmatrix}$$

$$\boldsymbol{C}^{(kin)} = \overset{\qquad\qquad(\delta\boldsymbol{q}^d)\qquad\qquad\qquad\qquad\qquad(\delta\dot{\boldsymbol{q}}^d)}{\left[\begin{array}{c:c} \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)} & \boldsymbol{O} \\ \mathbf{D}_{q^d}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} & \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)} \\ \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\boldsymbol{A}_{21} + \mathbf{D}_{q^d}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\ddot{\boldsymbol{q}} + \mathbf{D}_{q^d}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}} & \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\boldsymbol{A}_{22} + 2\mathbf{D}_{q^d}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} \end{array}\right]}$$

$$\boldsymbol{D}^{(kin)} = \overset{(\delta\boldsymbol{u})}{\begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{O} \\ \mathbf{D}_{q^d}\boldsymbol{\mathcal{F}}^{(x)}\boldsymbol{B}_2 \end{bmatrix}} + \overset{\quad(\delta\ddot{\boldsymbol{q}}^r)\qquad\qquad(\delta\dot{\boldsymbol{q}}^r)\qquad\qquad\qquad(\delta\boldsymbol{q}^r)}{\left[\begin{array}{c:c:c} \boldsymbol{O} & \boldsymbol{O} & \mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)} \\ \boldsymbol{O} & \mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)} & \mathbf{D}_{q^r}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} \\ \mathbf{D}_{q^r}\boldsymbol{\mathcal{F}}^{(x)} & 2\mathbf{D}_{q^r}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}} & \mathbf{D}_{q^r}\mathbf{D}\boldsymbol{\mathcal{F}}^{(x)}\ddot{\boldsymbol{q}} + \mathbf{D}_{q^r}\mathbf{D}^2\boldsymbol{\mathcal{F}}^{(x)}\dot{\boldsymbol{q}}\dot{\boldsymbol{q}} \end{array}\right]}$$

Linearised state-space equations — dynamic output matrices

$$y^{(dyn)} = \begin{bmatrix} \delta\sigma^{(o)} \\ \delta\sigma^{(m)} \\ \delta f^{(o)} \\ \delta f^{(m)} \end{bmatrix}$$

$$C^{(dyn)} = \left[\tilde{M}_d^{(x)} A_2\right] + \left[\overset{(\delta q^d)}{\tilde{K}_d^{(x)} + \tilde{N}_d^{(x)} + \tilde{G}_d^{(x)}} \;\middle|\; \overset{(\delta \dot{q}^d)}{\tilde{C}_d^{(x)} + \tilde{D}_d^{(x)}}\right]$$

$$D^{(dyn)} = \left[\tilde{M}_d^{(x)} B_2\right] + \left[\overset{(\delta f^{(c)})}{\tilde{D}_{f^{(c)}}} \;\middle|\; \overset{(\delta f^{(m)})}{O} \;\middle|\; \overset{(\delta \sigma_a^{(m)})}{O} \;\middle|\; \overset{(\delta \sigma_a^{(c)})}{\tilde{D}_{\sigma^{(c)}}} \;\middle|\right.$$
$$\left.\middle|\; \overset{(\delta \ddot{q}^r)}{\tilde{M}_r^{(x)}} \;\middle|\; \overset{(\delta \dot{q}^r)}{\tilde{C}_r^{(x)} + \tilde{D}_r^{(x)}} \;\middle|\; \overset{(\delta q^r)}{\tilde{K}_r^{(x)} + \tilde{N}_r^{(x)} + \tilde{G}_r^{(x)}}\right]$$

where the matrices are given in § 8.4.

Stationary and equilibrium solutions

$$\bar{M}_{dd}(q)\ddot{q}_d = \bar{f}_d(q, \dot{q}, t) - \bar{M}_{dr}\ddot{q}^r$$
$$\ddot{q}^d = 0, \quad \dot{q}^d = 0, \quad \ddot{q}^r = 0$$
$$\begin{bmatrix} \dot{q}^d \\ \bar{f}_d(q^d, \dot{q}^d) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Stability of stationary solution is determined by eigenvalues of state matrix A.

Frequency equation:

$$\det\left(-\omega_i^2 \bar{M}_{dd} + \bar{K}_{dd} + \bar{N}_{dd} + \bar{G}_{dd}\right) = 0$$

Solutions of eigenvalue problem are natural frequencies ω_i and mode shapes.

Stability of equilibrium equation:

$$\det\left(\bar{K}_{dd} + \lambda_i \bar{G}_{dd}\right) = 0$$

$\lambda_i = f_i / f_0$, $\quad f_0$ is reference load and λ_i the critical mode multipliers

From state space equations to transfer function(s)

Linearised state-space equations with state vector $\delta z = \begin{bmatrix} \delta q^d \\ \delta \dot{q}^d \end{bmatrix}$:

$$\delta \dot{z} = A\delta z + B\delta u$$

$$\delta y = C\delta z + D\delta u$$

with general output vector $\delta y = \begin{bmatrix} \delta y^{(kin)} \\ \delta y^{(dyn)} \end{bmatrix}$ and input vector

input vector $\delta u = \left[\delta f^{(c)T}, \; \delta f^{(m)T}, \; \delta \sigma_a^{(m)T}, \; \delta \sigma_a^{(c)T}, \; \delta \ddot{q}^{rT}, \; \delta \dot{q}^{rT}, \; \delta q^{rT}\right]^T$

The state space representation can be translated into a transfer function matrix with the standard expression

$$\tilde{G}(s) = C(sI - A)^{-1}B + D$$

which is straightforward for the inputs $\delta f^{(c)}$, $\delta f^{(m)}$, $\delta \sigma_a^{(m)}$ and $\delta \sigma_a^{(c)}$.

Note however the occurrence of δq^r **and** the time derivatives $\delta \dot{q}^r$ and $\delta \ddot{q}^r$!!!

A transfer function relates the *Laplace transforms* of the system's input $\delta u = \left[\delta f^{(c)T}, \; \delta f^{(m)T}, \; \delta \sigma_a^{(m)T}, \; \delta \sigma_a^{(c)T}, \; \delta \ddot{q}^{rT}, \; \delta \dot{q}^{rT}, \; \delta q^{rT}\right]^T$ and output δy:

$$\mathcal{L}\{\delta y(t)\} = \delta y(s) = \tilde{G}(s)\delta u(s) = \tilde{G}(s)\mathcal{L}\{\delta u(t)\}$$

The Laplace transforms of δq^r, $\delta \dot{q}^r$ and $\delta \ddot{q}^r$ are not independent as

$$\mathcal{L}\{\delta \dot{q}^r(t)\} = s\mathcal{L}\{\delta q^r(t)\} \quad \text{and} \quad \mathcal{L}\{\delta \ddot{q}^r(t)\} = s^2\mathcal{L}\{\delta q^r(t)\}$$

and have to be taken into account to specify e.g. the transfer function (part) from δq^r to $\delta y(t)$.

Case 1: Define the input vector as

$$\delta u_1 = \left[\delta f^{(c)T}, \; \delta f^{(m)T}, \; \delta \sigma_a^{(m)T}, \; \delta \sigma_a^{(c)T}, \; \delta q^{rT}\right]^T$$

and determine the transfer function in

$$\delta y(s) = G_1(s)\delta u_1(s)$$

Case 1 (position input): Define the input vector as

$$\delta \boldsymbol{u}_1 = \left[\delta \boldsymbol{f}^{(c)T},\ \delta \boldsymbol{f}^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(c)T},\ \delta \boldsymbol{q}^{rT}\right]^T$$

then $\delta \boldsymbol{u}(s) = \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & s^2\, I \\ 0 & 0 & 0 & 0 & s\, I \\ 0 & 0 & 0 & 0 & I \end{array}\right] \delta \boldsymbol{u}_1(s)$

and the transfer function from $\delta \boldsymbol{u}_1$ to $\delta \boldsymbol{y}$ is

$$\boldsymbol{G}_1(s) = \left\{\boldsymbol{C}(sI - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right\} \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & s^2\, I \\ 0 & 0 & 0 & 0 & s\, I \\ 0 & 0 & 0 & 0 & I \end{array}\right]$$

Case 2 (velocity input): Define the input vector as

$$\delta \boldsymbol{u}_2 = \left[\delta \boldsymbol{f}^{(c)T},\ \delta \boldsymbol{f}^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(c)T},\ \delta \dot{\boldsymbol{q}}^{rT}\right]^T$$

then $\delta \boldsymbol{u}(s) = \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & s\, I \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 1/s\, I \end{array}\right] \delta \boldsymbol{u}_2(s)$

and the transfer function from $\delta \boldsymbol{u}_2$ to $\delta \boldsymbol{y}$ is

$$\boldsymbol{G}_2(s) = \left\{\boldsymbol{C}(sI - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right\} \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & s\, I \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 1/s\, I \end{array}\right]$$

Case 3 (acceleration input): Define the input vector as

$$\delta \boldsymbol{u}_3 = \left[\delta \boldsymbol{f}^{(c)T},\ \delta \boldsymbol{f}^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(m)T},\ \delta \boldsymbol{\sigma}_a^{(c)T},\ \delta \ddot{\boldsymbol{q}}^{rT}\right]^T$$

then $\delta \boldsymbol{u}(s) = \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 1/s\, I \\ 0 & 0 & 0 & 0 & 1/s^2\, I \end{array}\right] \delta \boldsymbol{u}_3(s)$

and the transfer function from $\delta \boldsymbol{u}_3$ to $\delta \boldsymbol{y}$ is

$$\boldsymbol{G}_3(s) = \left\{\boldsymbol{C}(sI - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right\} \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 1/s\, I \\ 0 & 0 & 0 & 0 & 1/s^2\, I \end{array}\right]$$

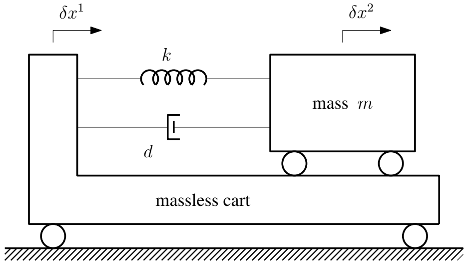
Note 1: The three cases can be combined, e.g. to define an input $\delta \boldsymbol{u}$ with the position of one rheonomic degree of freedom and the acceleration of another rheonomic degree of freedom.

Note 2: In the case only accelerations $\delta \ddot{\boldsymbol{q}}^r$ and no velocities $\delta \dot{\boldsymbol{q}}^r$ and positions $\delta \boldsymbol{q}^r$ appear in the input, the transfer function matrix can also be obtained by adding the velocities and positions to the state vector $\delta \boldsymbol{x}$.

Spring-mass-damper system mounted on a cart

$$\delta q = \delta x^2 - \delta x^1$$

$$\delta \boldsymbol{u} = \left[\delta \ddot{x}^1,\ \delta \dot{x}^1,\ \delta x^1\right]^T$$



$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -m^{-1}k & -m^{-1}d \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \boldsymbol{D} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Transfer functions

$$\widetilde{\boldsymbol{G}}(s) = \frac{\delta x^2(s)}{\delta u(s)} = \boldsymbol{C}(sI - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

$$= \left[\frac{-1}{s^2 + \frac{d}{m}s + \frac{k}{m}} \quad 0 \quad 1\right]$$

Single input $\delta x^1(s)$ to the output $\delta x^2(s)$

$$\delta \boldsymbol{u}(s) = \left[s^2 \quad s \quad 1\right]^T \delta x^1(s)$$

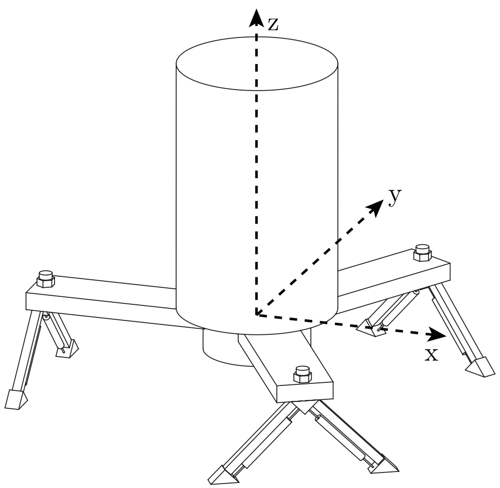
$$G_1(s) = \frac{\delta x^2(s)}{\delta x^1(s)} = \widetilde{\boldsymbol{G}}(s)\begin{bmatrix} s^2 \\ s \\ 1 \end{bmatrix} = \frac{\frac{d}{m}s + \frac{k}{m}}{s^2 + \frac{d}{m}s + \frac{k}{m}}$$

Single input $\delta \dot{x}^1(s) = \delta v^1$ to output $\delta x^2(s)$

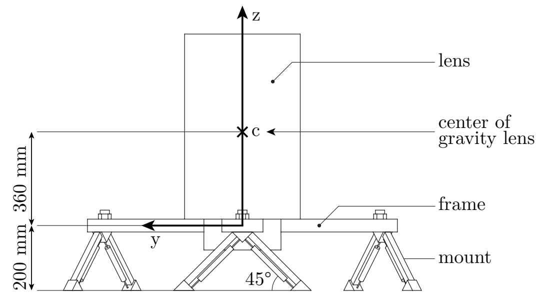
$$\delta \boldsymbol{u}(s) = [s \quad 1 \quad 1/s]^T \delta v^1(s)$$

$$G_2(s) = \frac{\delta x^2(s)}{\delta v^1(s)} = \widetilde{\boldsymbol{G}}(s)\begin{bmatrix} s \\ 1 \\ 1/s \end{bmatrix} = \frac{\frac{d}{m}s + \frac{k}{m}}{s(s^2 + \frac{d}{m}s + \frac{k}{m})}$$

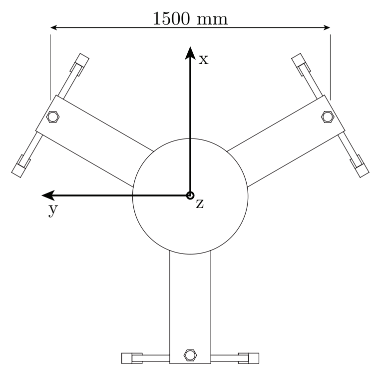
Active vibration isolation of a metrology frame



3D view of lens suspension frame of a wafer stepper/scanner

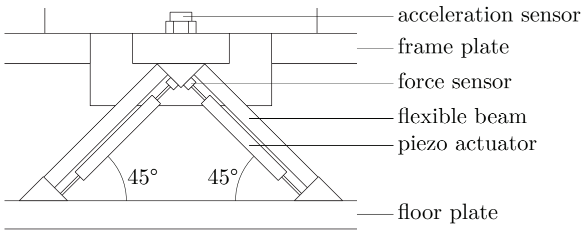


Front view

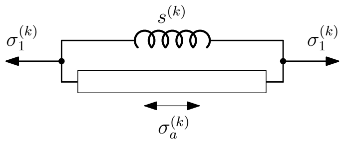


Top view

Detailed view of a mount

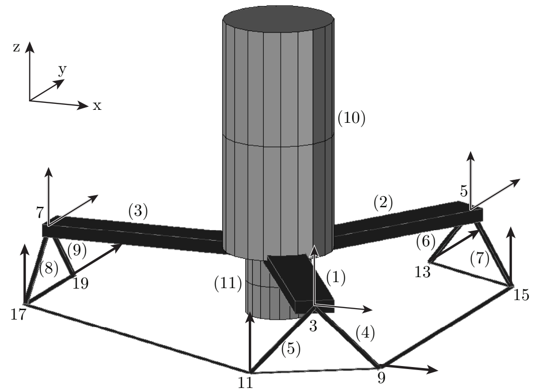


Piezo actuator (force $\sigma_a^{(k)}$) with parallel spring (stiffness $s^{(k)}$)



$$\sigma_1^{(k)} = \sigma_a^{(k)} + s^{(k)} e_1^{(k)}$$

Finite element model of metrology frame and floor

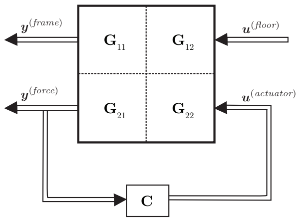


$$q^{(d)} = \left[e_1^{(4)}, e_1^{(5)}, e_1^{(6)}, e_1^{(7)}, e_1^{(8)}, e_1^{(9)}\right]^T$$

$$\boldsymbol{u}^{(floor)} = \left[\ddot{x}^9, \ddot{z}^{11}, \ddot{y}^{13}, \ddot{z}^{15}, \ddot{z}^{17}, \ddot{y}^{19}\right]^T \qquad \boldsymbol{y}^{(frame)} = \left[\ddot{x}^3, \ddot{z}^3, \ddot{y}^5, \ddot{z}^5, \ddot{z}^7, \ddot{y}^7\right]^T$$

$$\boldsymbol{u}^{(actuator)} = \left[\sigma_a^{(4)}, \sigma_a^{(5)}, \sigma_a^{(6)}, \sigma_a^{(7)}, \sigma_a^{(8)}, \sigma_a^{(9)}\right]^T \qquad \boldsymbol{y}^{(force)} = \left[\sigma_1^{(4)}, \sigma_1^{(5)}, \sigma_1^{(6)}, \sigma_1^{(7)}, \sigma_1^{(8)}, \sigma_1^{(9)}\right]^T$$

Generalised plant $\boldsymbol{G}$ with 12 inputs and 12 outputs and controller $\boldsymbol{C}$ with 6 inputs and 6 outputs



Feedback control equations

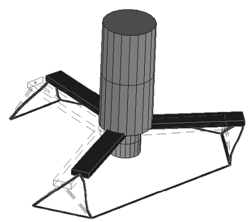
$$\boldsymbol{u}(s)^{(actuator)} = \boldsymbol{C}(s)\boldsymbol{y}(s)^{(force)}$$

$$\boldsymbol{C}(s) = -\left(\boldsymbol{K}^{(P)} + \frac{\boldsymbol{K}^{(I)}}{s}\right)$$

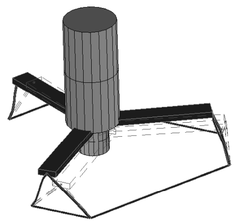
$$\boldsymbol{K}^{(P)} = \left(\omega_d^2 \boldsymbol{I} \bar{\boldsymbol{M}}_{dd}\right)^{-1} \bar{\boldsymbol{K}}_{dd} - \boldsymbol{I}$$

$$\boldsymbol{K}^{(I)} = 2\zeta\omega_d \left(\boldsymbol{I} + \boldsymbol{K}^{(P)}\right)$$

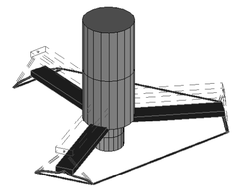
Mode shapes and natural frequencies of the passive system



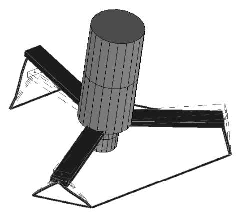
Mode 1: 13.9 Hz



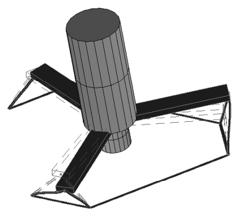
Mode 2: 13.9 Hz



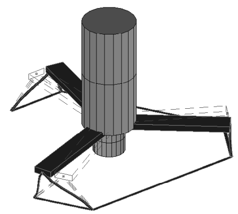
Mode 3: 20.0 Hz



Mode 4: 31.3 Hz



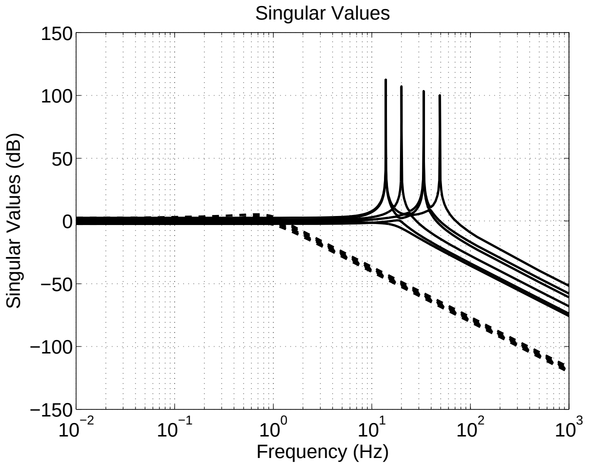
Mode 5: 31.3 Hz



Mode 6: 49.0 Hz

Closed loop transfer function T

$$T = G_{11} + G_{12} \cdot C \cdot (I - G_{22} \cdot C)^{-1} \cdot G_{21}$$



Singular values; dashed line is closed loop, solid line is open loop

Conclusions

- Linearised state-space formulation for flexible multibody systems.
- Arbitrary combination of positions, velocities, accelerations and forces can be taken as input variables and as output variables.
- Finite element based multibody concept enables a low dimensional description of prototype models suitable for design purposes.
- Insight into the relations between component properties and dominant system behaviour.