5. **Classical lamination theory-2**

5.1. Introduction

The classical lamination theory as summarised in chapter 4 allows us to find a relation between the laminated plate deformation on one side, and the loading on the plate on the other side. The deformations considered were the mid-plane in-plane strains ($\varepsilon_x^0, \varepsilon_y^0, \varepsilon_{xy}^0$) and the plate curvatures ($\kappa_x, \kappa_y, \kappa_{xy}$). The loading studied were in-plane force ($N_x, N_y, N_{xy}$) and moments ($M_x, M_y, M_{xy}$).

From this relation, in-plane stresses and strains can be deduced in any layer of the laminate.

From this displacement-loading relation, engineering constants of the laminate can also be derived. As for the single layers in chapter 3, this will be done by performing simple experiments.

As shown in chapter 3, anisotropy of composites is also present in hygrothermal properties. It means stresses and strains are sensitive to changes in temperature as well as in moisture contents. The influence of temperature is already noticeable during fabrication, as most composites are consolidated above room temperature.

Taking hygrothermal “loading” into account in the deformation-loading relation will be developed in the second part of this chapter. An example on the calculation of thermal residual stresses in a laminated cross-ply beam ([0/90]s) will also be given.

5.2. Engineering constants of a laminate

Laminate engineering constants can be obtained by performing simple experiments on the laminate. For example, the longitudinal elasticity modulus of the laminate $E_x$ can be obtained by applying a force per unit width $N_x$. From the laminate force-deformation relation:

$$\begin{pmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{pmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & a_{16} \\
a_{21} & a_{22} & a_{26} \\
a_{61} & a_{61} & a_{61}
\end{bmatrix}
\begin{pmatrix}
N_x \\
N_y \\
N_{xy}
\end{pmatrix}
or
\begin{pmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{pmatrix} = [a][N]$$

(5.1)

With $[a]$ the laminate extensional compliance matrix. $E_x$ is then:

$$E_x = \frac{\sigma_x}{\varepsilon_x^0} = \frac{N_x}{a_{11}N_x} = \frac{1}{h \cdot a_{11}}$$

(5.2)

Using the same loading case, an expression for the in-plane Poisson's ratio can be written:

$$\nu_{xy} = -\frac{\varepsilon_y^0}{\varepsilon_x^0} = \frac{a_{21}N_x}{a_{11}N_x} = \frac{-a_{21}}{a_{11}}$$

(5.3)

Similarly, the transverse elasticity modulus of the laminate $E_y$ is obtained by applying a force $N_y$ and is then:

$$E_y = \frac{1}{h \cdot a_{22}}$$

(5.4)
And the shear modulus of the laminate is:

\[ G_{xy} = \frac{1}{h \cdot a_{66}} \]  

(5.5)

5.3. Hygrothermal effects in the theory of laminated plates

The stress-strain relationship of an orthotropic layer with hygrothermal effects was developed in 3.6. It was based on the assumption that strain having mechanical, thermal or moisture origins can be treated separately and then added using superposition. The strain in a layer \( k \) subjected to a mechanical stress, a temperature change \( \Delta T \) and a moisture concentration \( c \) is therefore written as (3.33):

\[ \{e\}_k = \left[ S \right]_k \cdot \{\sigma\}_k + \{\alpha\}_k \Delta T + \{\beta\}_k c \]  

(5.6)

With the thermal expansion vector \( \{\alpha\} \) and the moisture swelling vector \( \{\beta\} \) as defined in (3.32). The resulting stresses are therefore:

\[ \{\sigma\}_k = \left[ C \right]_k \cdot \{e\}_k - \{\alpha\}_k \Delta T - \{\beta\}_k c \]  

(5.7)

In the global laminate coordinate system, relation (5.7) becomes:

\[ \{\sigma^*\}_k = \left[ C^* \right]_k \cdot \{e^*\}_k - \{\alpha^*\}_k \Delta T - \{\beta^*\}_k c \]  

(5.8)

According to (4.12), this expression can be written as a function of the mid-plane strain \( e^0 \) and the plate curvature \( \kappa \):

\[ \{\sigma^*\}_k = \left[ C^* \right]_k \cdot \{e^0\}_k + \kappa \cdot \{\alpha^*\}_k \Delta T - \{\beta^*\}_k c \]  

(5.9)

Following a similar procedure as in 4.3.3, external forces and moments acting on a laminated plate can be related to the stress in the layer, and then to the laminate deformation. For example, the axial forces \( N_x \) per unit width can be obtained by summing the axial stresses \( \sigma_x \) acting on each layer:

\[ N_x = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} (\sigma_{xx})_k \cdot dz \]  

(5.10)

where \((\sigma_{xx})_k\) is the stress in the \( k^{th} \) layer in the \( 1^{st} \) direction (global coordinate system). A similar expression can be written for the normal force in the \( y \) direction as well as for the in-plane shear force \( N_{xy} \). Substituting (5.9) in the force resultants gives in matrix form:
\[
\{N\} = \sum_{k=1}^{N} \left( \int_{z_{k-1}}^{z_k} \left( \{C^*\}_k \cdot [\varepsilon^0] + z[C^*]_k \cdot \{\kappa\} - \Delta T \cdot \{C^*\}_k \cdot \{\alpha^*\}_k - c.[C^*]_k \cdot \{\beta^*\}_k \right) dz \right)
\]

\[
\{N\} = \left( \sum_{k=1}^{N} [C^*]_k (z_k - z_{k-1}) \right) \{\varepsilon^0\} + \left( \frac{1}{2} \sum_{k=1}^{N} [C^*]_k (z_k^2 - z_{k-1}^2) \right) \{\kappa\} - \Delta T \cdot \left( \sum_{k=1}^{N} [C^*]_k (z_k - z_{k-1}) \right) \{\alpha^*\}_k - c \cdot \left( \sum_{k=1}^{N} [C^*]_k (z_k - z_{k-1}) \right) \{\beta^*\}_k
\]

(5.11)

This is mostly rewritten in the following way:

\[
\{N\} = [A] \cdot \{\varepsilon^0\} + [B] \cdot \{\kappa\} - \{N^{Th}\} - \{N^H\}
\]

(5.12)

Where the \([A]-\) and \([B]-\)matrix are defined in (4.16) and (4.17). The components of vectors \(\{N^{Th}\}\) and \(\{N^H\}\) are fictive thermal and hygroscopic forces respectively defined as:

\[
N_i^{Th} = \Delta T \sum_{k=1}^{N} (C_{gj}^*)_k (\alpha_{j^*})_k (z_k - z_{k-1})
\]

(5.13)

And:

\[
N_i^H = c \sum_{k=1}^{N} (C_{gj}^*)_k (\beta_{j^*})_k (z_k - z_{k-1})
\]

(5.14)

The term “fictive” emphasises that the components in \(\{N^{Th}\}\) and \(\{N^H\}\) are not actual forces. It only means that for the deformations to be zero, \(\{\varepsilon^0\}=\{0\}\) and \(\{\kappa\}=\{0\}\), a force vector \(\{N\}\) equal and opposite to \(\{N^{Th}\}\) must be applied to the system.

A similar development can be performed for the moment resultants. This gives as end result the following relations:

\[
\{M\} = [B] \cdot \{\varepsilon^0\} + [D] \cdot \{\kappa\} - \{M^{Th}\} - \{M^H\}
\]

(5.15)

Where the D-matrix is defined in (4.19). The components of vectors \(\{M^{Th}\}\) and \(\{M^H\}\) are fictive thermal and hygroscopic moments respectively defined as:

\[
M_i^{Th} = \frac{1}{2} \Delta T \sum_{k=1}^{N} (C_{gj}^*)_k (\alpha_{j^*})_k (z_k^2 - z_{k-1}^2)
\]

(5.16)

And:

\[
M_i^H = \frac{1}{2} c \sum_{k=1}^{N} (C_{gj}^*)_k (\beta_{j^*})_k (z_k^2 - z_{k-1}^2)
\]

(5.17)

Now the laminate stiffness matrix and the fictive force and moment vector are known \((\{A\}, \{B\}, \{D\}, \{N^{Th}\}, \{N^H\}, \{M^{Th}\}\) & \(\{M^H\})\), it is possible to express the laminate deformations:
\[ \begin{pmatrix} \varepsilon^0 \\ \kappa \end{pmatrix} = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} N \\ M \end{pmatrix} + \begin{pmatrix} N^{Th} \\ M^{Th} \end{pmatrix} + \begin{pmatrix} N^H \\ M^H \end{pmatrix} \] (5.18)

5.4. Example: Thermal residual stress in a cross-ply laminate: 1D solution.

Roughly said, residual stresses are stresses, which are present in a material even before it is loaded. These stresses will build up during fabrication, when the matrix is consolidated to form the composite. The following example is meant to illustrate the formation of thermal residual stresses in a simple laminate on a macroscopic scale. The matrix used consolidates at a temperature higher than the temperature at use. A cross-ply [90/0/90] beam is shown in fig. 5.1, where three situations are depicted:

- The situation at a temperature where the matrix starts to consolidate. In the case of an amorphous thermoplastic material at the glass transition temperature \( T_g \). It is assumed that the layers have the same initial length.
- The situation at a lower temperature, in the case the layers can freely contract, i.e. with a strain \( \alpha_i \Delta T \).
- The situation at room temperature, in the most common case that the layers were not able to freely contract due to the temperature change.

\[
\begin{array}{c}
\text{--- at } T_g \\
\text{--- } \alpha_i \Delta T \\
\text{--- at } RT \\
\end{array}
\]

fig. 5.1: 1D macroscopic residual strain field model of a cross-ply laminate.

The residual stress profile on the macroscopic scale can be obtained from a simple strain compatibility equation:

\[
\frac{u_{00}^{\Delta T}}{L} + \frac{u_{90}^{m}}{L} = \frac{u_{0}^{\Delta T}}{L} + \frac{u_{0}^{m}}{L}
\]

where \( u_{i}^{\Delta T} \) (with \( i \) being 0° or 90° stands for the ply orientation) are the thermal displacements due to the ply free expansion only, and \( u_{i}^{m} \) are the mechanical displacements necessary to compensate the free expansion of the different plies. This equation can then be rewritten as:

\[
\alpha_{00} \Delta T + \frac{\sigma_{00}^r}{E_{00}} = \alpha_{0} \Delta T + \frac{\sigma_{0}^r}{E_{0}}
\]
where $\alpha_i$ and $E_i$ are respectively the coefficients of thermal expansion and the Young’s moduli of the unidirectional $0^\circ$ and $90^\circ$ layers. The stress equilibrium equation is:

$$\sigma_{00}t_{00} + \sigma_{0}t_{0} = 0$$

with $t_i$ the layer thickness. This equation then leads to the following relation for the thermal stress in $90^\circ$ layer:

$$\sigma'_{90} = -\frac{E_{90}t_{0}E_{0}(\alpha_{90} - \alpha_{0})\Delta T}{t_{0}E_{0} + t_{90}E_{90}}$$

### 5.5. Summary of the calculation of strain and stresses in a laminated plate.

<table>
<thead>
<tr>
<th>Input</th>
<th>Ply level</th>
<th>Laminate level</th>
</tr>
</thead>
<tbody>
<tr>
<td>k-ply engineering constants: $E_1$, $E_2$, $G_{12}$, $\nu_{12}$, $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$</td>
<td>k-ply stiffness in ply local CS: $[C]_k$, $[\alpha]_k$, $[\beta]_k$</td>
<td>Laminate stiffness: $[A]$, $[B]$, $[D]$</td>
</tr>
<tr>
<td>k-ply orientation $\theta_k$</td>
<td>k-ply stiffness in ply global CS: $[C^<em>]_k$, $[\alpha^</em>]_k$, $[\beta^*]_k$</td>
<td>Laminate compliance: $[a]$, $[b]$, $[d]$</td>
</tr>
<tr>
<td>Ply lay-up</td>
<td></td>
<td>Laminate deformation: $[\varepsilon]^l$, $[\kappa]$</td>
</tr>
<tr>
<td>Loading (mechanical, thermal, moisture)</td>
<td>k-ply strains in ply global CS: $[\varepsilon^*]_k$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>k-ply stresses in ply global CS: $[\sigma^*]_k$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>k-ply stresses in ply local CS: $[\sigma]_k$</td>
<td></td>
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</tbody>
</table>