On the Energy Benefit of Network Coding for Wireless Multiple Unicast

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***Abstract*—We consider energy savings offered by network coding for multiple unicast in wireless networks. For d-dimensional wireless networks we show that the maximum possible benefit is at least $2d/\lfloor\sqrt{d}\rfloor$.**

I. Introduction

Network coding has the potential of reducing energy consumption in wireless networks by exploiting the broadcast nature of the wireless medium. This has been demonstrated for multiple unicast traffic [1]–[5], multicast traffic [6], as well as many-to-many communication [7]. Lower bounds on the maximum possible energy savings of network coding are presented in [1]–[5]. Some design principles for constructing efficient network codes are presented in [2], a linear programming approach to finding efficient codes in [6] and practical algorithms in [3] and [7].

In this paper we are interested in the energy savings that network coding can offer for wireless multiple unicast problems. More precisely, we bound the maximum ratio of the energy consumption of routing to the energy consumption of network coding, where the maximum is over all possible multiple unicast configurations. We call this ratio the energy benefit of network coding. The best known lower bound on the energy benefit of network coding is 3 for two dimensional networks [5]. Our main result is a new lower of $2d/\lfloor\sqrt{d}\rfloor$ for d-dimensional networks.

For 2-dimensional networks our lower bound equals 4, in 3 dimensions it equals 6. It is interesting to compare this with the upper bound of 3 presented in [8], which is obtained under the restriction that only the type of network codes introduced in [3] are allowed. These codes follow a *decode-and-recombine* strategy, *i.e.*, nodes transmit linear combinations of only those symbols that they have successfully decoded by themselves. Note, that in general, it is also possible to retransmit linear combinations of coded symbols without decoding the corresponding source symbols. Our lower bound shows that it can be beneficial to consider also hese coding strategies.

This paper is organized as follows. In Section II the model is defined more precisely. The main results of the work are presented in Section III. The network code that achieves a high benefit is constructed in Section IV. Section V, finally, provides a discussion of the work.

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II. Model and Notation

Let $V \subset \mathbb{R}^d$ be the nodes of a d-dimensional wireless network. We consider a wireless network model with broadcast, where all nodes within range r of a transmitting node can receive, and nodes outside this range cannot. The energy required to transmit one unit of information to all other nodes within range r equals cr^α, where α is the path loss exponent and c some constant. We will fix the transmission range r and compare network coding and routing solutions on the resulting topology, *i.e.*, a node $\mathbf{v}$ is broadcasting to all nodes in the set

$$\{\mathbf{u} \in V \mid \|\mathbf{u} - \mathbf{v}\| \leq r\},$$

where $\|\mathbf{u} - \mathbf{v}\|$ denotes the Euclidean norm of $\mathbf{u} - \mathbf{v}$.

The traffic pattern that we consider is multiple unicast. All symbols are from the field $\mathbb{F}_2$, *i.e.*, they are bits and addition corresponds to the xor operation. The source of each unicast session has a sequence of source symbols that need to be delivered to the corresponding receiver. Let M be the set of unicast sessions. We will call $C = \{V, r, M\}$ a wireless multiple unicast configuration.

We measure energy consumption by the total energy required to deliver one symbol for each unicast session. Our goal is to establish lower bounds on

$$\text{energy benefit} = \max_C \frac{\begin{array}{l}\text{minimum energy consumption of any}\\ \text{routing solution on } C\end{array}}{\begin{array}{l}\text{minimum energy consumption of any}\\ \text{network coding solution on } C\end{array}},$$

where the maximum is over all wireless multiple unicast configurations. Since r is fixed, the energy per transmission is a constant and the benefit is equivalent to the ratio of the number of transmissions required in routing and network coding solutions.

Since we are interested in energy consumption only, we can assume that all transmissions are scheduled sequentially and/or that there is no interference. Time is slotted. To simplify notation in Section IV we allow nodes to transmit more than

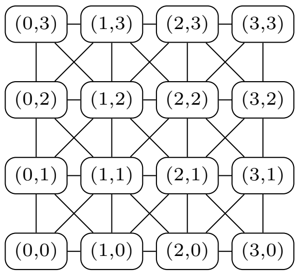

Fig. 1. $C(2,3)$: Nodes $V = \mathbb{Z}_3^2$ and connectivity induced by $r = \sqrt{2}$.

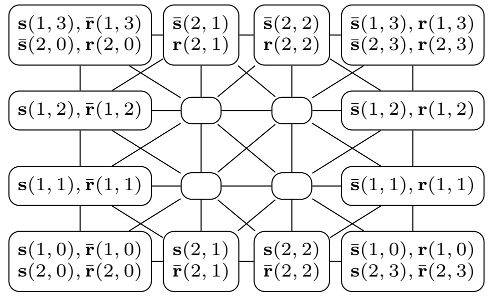

Fig. 2. $C(2,3)$: Locations of sources and receivers.

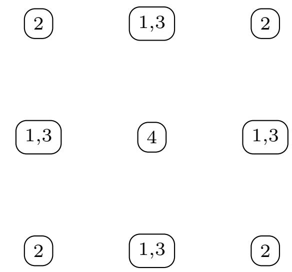

Fig. 3. The linear combination transmitted by the center node at time t. Symbols received at time $t-1$ and $t-3$ are included from, *e.g.*, the middle left node.

once in each time slot. Alternatively, we could have rescaled time such that only one transmission from each node occurs in a time slot.

For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, let $\mathbf{v}_i^j = (v_i, \ldots, v_j)$, $(\mathbf{u}, \mathbf{v})$ the concatenation of $\mathbf{u}$ and $\mathbf{v}$ and $\mathbf{v}^{\setminus i} = \left(\mathbf{v}_1^{i-1}, \mathbf{v}_{i+1}^d\right)$. For vector $\mathbf{v} \in \mathbb{R}^d$ and scalar $u \in \mathbb{R}$, let $\mathbf{v}^i\left[u\right] = \left(\mathbf{v}_1^{i-1}, u, \mathbf{v}_{i+1}^d\right)$. Finally, let $\mathbb{Z}_K^d = \{\mathbf{v} \in \mathbb{Z}^d : 0 \le v_i \le K\}$ and, for $V = \mathbb{Z}_K^d$, let $\mathring{V} = \{\mathbf{v} \in V : 0 < v_i < K \text{ for all } i = 1, \ldots, d\}$.

III. Results

We construct a set of multiple unicast configurations $\{C(d,K) | d \ge 1, K > 1\}$, that will be used in the remainder of the paper. Let $C(d,K) = \{V, r, M\}$, with $r = \sqrt{d}$, $V = \mathbb{Z}_K^d$ and the set of unicast sessions M defined as follows. There are $2d(K+1)^{d-1}$ sessions in total. We have sessions $x(i, \mathbf{v})$ and $\bar{x}(i, \mathbf{v})$ for each $1 \le i \le d$ and $\mathbf{v} \in \mathbb{Z}_K^{d-1}$. Session $x(i, \mathbf{v})$ has source $\mathbf{s}(i, \mathbf{v}) = (\mathbf{v}_1^{i-1}, 0, \mathbf{v}_i^{d-1})$ and receiver $\mathbf{r}(i, \mathbf{v}) = (\mathbf{v}_1^{i-1}, K, \mathbf{v}_i^{d-1})$. Session $\bar{x}(i, \mathbf{v})$ has source $\bar{\mathbf{s}}(i, \mathbf{v}) = \mathbf{r}(i, \mathbf{v})$ and receiver $\bar{\mathbf{r}}(i, \mathbf{v}) = \mathbf{s}(i, \mathbf{v})$. The information symbols to be transmitted by $x(i, \mathbf{v})$ and $\bar{x}(i, \mathbf{v})$ are $\{x_t(i, \mathbf{v})\}_{t>0}$ and $\{\bar{x}_t(i, \mathbf{v})\}_{t>0}$ respectively. Note, that in general, we will omit dependence on d and K from the notation. As an example, Figures 1 and 2 depict $C(2,3)$.

Lemma 1. *The optimal routing solution on* $C(d,K)$ *requires* $\lceil K / \lfloor \sqrt{d} \rfloor \rceil 2d(K+1)^{d-1}$ *transmissions.*

Proof: The optimal routing solution on $C(d,K)$ takes the shortest paths for all sessions. For each session, the shortest path takes $\lceil K / \lfloor \sqrt{d} \rfloor \rceil$ hops, hence $\lceil K / \lfloor \sqrt{d} \rfloor \rceil 2d(K+1)^{d-1}$ transmissions are required in total. ■

In Section IV we will prove the following result.

Lemma 2. *On* $C(d,K)$ *there is a network coding solution using* $(K-1)^d + 2d\left((K+1)^d - (K-1)^d\right)$ *transmissions.*

Our main result is the following.

Theorem 1. *The energy benefit of network coding in* d *dimensional wireless networks is at least* $2d / \lfloor \sqrt{d} \rfloor$.

Proof: From Lemmas 1 and 2 it follows that

$$\begin{aligned} \text{benefit} &\ge \lim_{K \to \infty} \frac{\lceil K / \lfloor \sqrt{d} \rfloor \rceil 2d(K+1)^{d-1}}{(K-1)^d + 2d\left((K+1)^d - (K-1)^d\right)} \\ &= 2d / \lfloor \sqrt{d} \rfloor. \end{aligned}$$

■

In two dimensions this gives a new lower bound of 4. For three dimensions it is 6.

Note that we have defined the energy benefit of network coding by fixing both the node positions and the transmission range. Alternatively, we could have optimized the transmission range independently for routing and network coding solutions. In this case, one can observe that an optimal routing solution uses transmission range 1. This increases the number of hops per session to K, but the cost per transmission reduces from $cd^{\alpha/2}$ to c. The energy benefit of the proposed network coding solution (still with $r = \sqrt{d}$) would hence be

$$\lim_{K \to \infty} \frac{c2dK(K+1)^{d-1}}{cd^{\alpha/2}\left[(K-1)^d + 2d\left((K+1)^d - (K-1)^d\right)\right]},$$

which equals $2d^{1-\alpha/2}$. Therefore, under this model, since $\alpha \ge 2$, the benefit of our coding solution reduces to at most 2. Note, that for $r = 1$, there exists a network coding solution achieving a benefit 2, independent of α, by coding only among pairs of oppositely directed sessions, see *e.g.*, [1]. The benefit of network coding on the configuration constructed in [5] is 3 under both models, since the transmission range that is used for the network coding solution is the minimum required for connectivity. Also, the lower bound of 2.4 obtained in [2] holds under both models. The codes that are constructed in [2] follow the decode-and-recombine strategy.

IV. Network Code Construction

In this section we prove Lemma 2 by constructing a network code using the indicated number of transmissions. Before giving the general construction, we provide an example of our construction in two dimensions in Section IV-A. In Section IV-B we specify the coding operations performed by nodes at the border of the network. In Section IV-C we specify the coding operation of internal nodes. In Section IV-D we specify how receivers can decode the required source symbols. Finally, in Section IV-E we connect the parts and prove Lemma 2.

A. Example

To demonstrate the main idea of our construction in two dimensions, we first ignore the effects of borders. Suppose, that at time 1 the only non-zero symbol that is being transmitted, is the symbol x by node (i, j). Our code has the property that at time $t \geq 1$, the only non-zero symbols that are transmitted in the network are x by the nodes $(i \pm (t-1), j)$ and $(i, j \pm (t-1))$. One can verify, that this property is satisfied by having each node in the network code according to Figure 3. The figure depicts the linear combination that is transmitted by the center node in each of the time slots $1 \leq t' \leq t$. The middle left node (1,3), for instance, indicates that the data received from the left neighbour at times $t'-1$ and $t'-3$ is included in this linear combination.

Now, we include the effects of the border. By properly coding at the borders of the network, we can ensure that data transmitted by the sources propagate in the network only along the shortest paths (straight lines) connecting sources and receivers. The above will be made more precise in the following sections.

B. Operation at the Border

We assume that for $t \leq 0$, for all $i = 1, \ldots, d$ and $\mathbf{v} \in \mathbb{Z}_K^{d-1}$, source symbols $x_t(i, \mathbf{v})$, $\bar{x}_t(i, \mathbf{v})$ and all transmitted data symbols are zero. The code that we construct is such that at the end of time slot $t-1$, receivers are able to decode the source symbols that have been generated by the sources at time $t - K$.

Nodes at the border of the network transmit $2d$ symbols each time slot. At time t, a node $\mathbf{v} \in V \setminus \mathring{V}$ transmits symbols $\mathbf{v}_t(i)$ and $\bar{\mathbf{v}}_t(i)$, $i = 1, \ldots, d$. The $\mathbf{v}_t(i)$ are created as follows

$$\boxed{\mathbf{v}_t(i) = \begin{cases} x_t(i, \mathbf{v}^{\setminus i}), & \text{if } v_i = 0, \\ \mathbf{v}^i\left[v_i - 1\right]_{t-1}(i), & \text{if } 0 < v_i < K, \\ x_{t-K}(i, \mathbf{v}^{\setminus i}) & \text{if } v_i = K, \end{cases}} \tag{1}$$

where $\mathbf{v}^i\left[v_i - 1\right] = (\mathbf{v}_1^{i-1}, v_i - 1, \mathbf{v}_{i+1}^d)$ as defined in Section II.

Note, that if $v_i = 0$, $\mathbf{v}$ is the source of $x(i, \mathbf{v}^{\setminus i})$ and, therefore, has $x_t(i, \mathbf{v}^{\setminus i})$ available as a source symbol. Also, if $\mathbf{v} \in V \setminus \mathring{V}$ and $0 < v_i < K$, then also $\mathbf{v}^i\left[v_i - 1\right] \in V \setminus \mathring{V}$ and, therefore $\mathbf{v}^i\left[v_i - 1\right]_{t-1}(i)$ is one of the $2d$ symbols it is transmitting in time slot $t-1$. Finally, if $v_i = K$, $\mathbf{v}$ is the receiver of $x(v, \mathbf{v}^{\setminus i})$. In that case $x_{t-K}(i, \mathbf{v}^{\setminus i})$ is the symbol decoded by $\mathbf{v}$ at the end of time slot $t-1$. The $\bar{\mathbf{v}}_t(i)$ are created as follows

$$\boxed{\bar{\mathbf{v}}_t(i) = \begin{cases} \bar{x}_{t-K}(i, \mathbf{v}^{\setminus i}), & \text{if } v_i = 0, \\ \overline{\mathbf{v}^i\left[v_i + 1\right]_{t-1}}(i), & \text{if } 0 < v_i < K, \\ \bar{x}_t(i, \mathbf{v}^{\setminus i}) & \text{if } v_i = K. \end{cases}} \tag{2}$$

For notational convenience, for $\mathbf{v} \in V \setminus \mathring{V}$, let $\mathbf{v}_t = \sum_{i=1}^d (\mathbf{v}_t(i) + \bar{\mathbf{v}}_t(i))$.

Note that by operating according to (1) and (2), nodes at the border of the network transmit uncoded packets. Moreover, this is done in such a way that information only propagates along shortest paths between sources and receivers. This is made precise in the next lemma.

Lemma 3. *Assume that for all $t' < t$, $\mathbf{u} \in V \setminus \mathring{V}$ and $i = 1, \ldots, d$*

$$\mathbf{u}_{t'}(i) = x_{t'-u_i}(i, \mathbf{u}^{\setminus i}) \;\; \textit{and} \;\; \bar{\mathbf{u}}_{t'}(i) = \bar{x}_{t'-K+u_i}(i, \mathbf{u}^{\setminus i}), \tag{3}$$

then for all $i = 1, \ldots, d$ and any $\mathbf{v} \in V \setminus \mathring{V}$, by coding according to (1) and (2), we have

$$\mathbf{v}_t(i) = x_{t-v_i}(i, \mathbf{v}^{\setminus i}) \;\; \textit{and} \;\; \bar{\mathbf{v}}_t(i) = \bar{x}_{t-K+v_i}(i, \mathbf{v}^{\setminus i}).$$

Proof: For i such that v_i satisfies $0 < v_i < K$ we have $\mathbf{v}_t(i) = \mathbf{v}^i\left[v_i - 1\right]_{t-1}(i) = x_{t-v_i}(i, \mathbf{v}^{\setminus i})$ and $\bar{\mathbf{v}}_t(i) = \overline{\mathbf{v}^i\left[v_i + 1\right]_{t-1}}(i) = \bar{x}_{t-K+v_i}(i, \mathbf{v}^{\setminus i})$. For the other cases the result follows directly from (1) and (2). ■

C. Operation of Internal Nodes

Internal nodes in the network transmit only once in each time slot. In order to describe the coding operation performed by internal nodes we introduce some notation. Let

$$N_{\mathbf{v}} = \{\mathbf{u} \in V : |u_i - v_i| \leq 1 \; \forall i\}$$

and $\operatorname{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|_1 = \sum_{i=1}^d |u_i - v_i|$, *i.e.*, $\operatorname{dist}(\mathbf{u}, \mathbf{v})$ denotes the Manhattan distance from $\mathbf{u}$ to $\mathbf{v}$.

Also, we introduce sets $\Theta_\delta \subset \{1, \ldots, 2d\}$, $0 \leq \delta \leq d$. Let $\Theta_d = \{d\}$. The remaining sets are defined recursively, by means of the corresponding indicator vectors. Let $\mathfrak{I}_\delta \in \mathbb{F}_2^{2d}$ be the indicator vector of Θ_δ. For $0 \leq \delta \leq d-1$ let

$$\mathfrak{I}_\delta = \text{shift left}(\mathfrak{I}_{\delta+1}) + \text{shift right}(\mathfrak{I}_{\delta+1}),$$

where addition corresponds to the elementwise XOR operation and the shift operation is performed by shifting in zeros and discarding symbols that are shifted out. As an example for $d = 2$ we have $\Theta_2 = \{2\}$, $\Theta_1 = \{1, 3\}$ and $\Theta_0 = \{4\}$, see Section IV-A. In the remainder of the paper we will repeatedly make use of the fact that

$$\sum_{\tau \in \Theta_{\delta+1}} y_{t-\tau-1} + \sum_{\tau \in \Theta_\delta} y_{t-\tau} + \sum_{\tau \in \Theta_{\delta+1}} y_{t-\tau+1} = 0, \tag{4}$$

for $0 < \delta < d$ and

$$\sum_{\tau \in \Theta_1} y_{t-\tau-1} + \sum_{\tau \in \Theta_0} y_{t-\tau} + \sum_{\tau \in \Theta_1} y_{t-\tau+1} = y_t, \tag{5}$$

where $y_t = x_t(i, \mathbf{v})$ or $y_t = \bar{x}_t(i, \mathbf{v})$ for some $i = 1, \ldots, d$ and $\mathbf{v} \in \mathbb{Z}_K^{d-1}$.

At time t, a node $\mathbf{v} \in \mathring{V}$ transmits one symbol $\mathbf{v}_t$, where

$$\boxed{\mathbf{v}_t = \sum_{\mathbf{u} \in N_{\mathbf{v}}} \sum_{\tau \in \Theta_{\operatorname{dist}(\mathbf{u}, \mathbf{v})}} \mathbf{u}_{t-\tau}.} \tag{6}$$

We show that all symbols transmitted by $\mathbf{v}$ are linear combinations of exactly one source symbol from each of the sessions for which $\mathbf{v}$ is on its shortest path.

Lemma 4. *Assume that for all $t' < t$ and $\mathbf{u} \in V$, $\mathbf{u}_{t'}$ satisfies*

$$\mathbf{u}_{t'} = \sum_{i=1}^{d} \left(x_{t'-u_i}(i, \mathbf{u}^{\setminus i}) + \bar{x}_{t'-K+u_i}(i, \mathbf{u}^{\setminus i}) \right), \qquad (7)$$

then, for any $\mathbf{v} \in \mathring{V}$, by coding according to (6), $\mathbf{v}_t$ satisfies

$$\mathbf{v}_t = \sum_{i=1}^{d} \left(x_{t-v_i}(i, \mathbf{v}^{\setminus i}) + \bar{x}_{t-K+v_i}(i, \mathbf{v}^{\setminus i}) \right). \qquad (8)$$

Proof: By the assumption in the lemma and (6) we have

$$\mathbf{v}_t = \sum_{i=1}^{d} \sum_{\substack{\mathbf{u} \in N_{\mathbf{v}} \\ \tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})}}} \left(x_{t-\tau-u_i}(i, \mathbf{u}^{\setminus i}) + \bar{x}_{t-\tau-K+u_i}(i, \mathbf{u}^{\setminus i}) \right).$$

We rewrite this as

$$\mathbf{v}_t = \sum_{i=1}^{d} \sum_{\mathbf{u} \in N_{\mathbf{v}} : u_i = v_i} \left(x_i^*(\mathbf{u}) + \bar{x}_i^*(\mathbf{u}) \right), \qquad (9)$$

where

$$x_i^*(\mathbf{u}) = \sum_{\tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})+1}} x_{t-u_i+1-\tau}(i, \mathbf{u}^{\setminus i}) + \sum_{\tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})}} x_{t-u_i-\tau}(i, \mathbf{u}^{\setminus i}) + \sum_{\tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})+1}} x_{t-u_i-1-\tau}(i, \mathbf{u}^{\setminus i})$$

and

$$\bar{x}_i^*(\mathbf{u}) = \sum_{\tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})+1}} \bar{x}_{t-K+u_i-1-\tau}(i, \mathbf{u}^{\setminus i}) + \sum_{\tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})}} \bar{x}_{t-K+u_i-\tau}(i, \mathbf{u}^{\setminus i}) + \sum_{\tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})+1}} \bar{x}_{t-K+u_i+1-\tau}(i, \mathbf{u}^{\setminus i}).$$

Now, by (4), we have, for $\mathbf{u} \neq \mathbf{v}$ in (9), $x_i^*(\mathbf{u}) = \bar{x}_i^*(\mathbf{u}) = 0$. Moreover, by (5) we have $x_i^*(\mathbf{v}) = x_{t-v_i}(i, \mathbf{v}^{\setminus i})$ and $\bar{x}_i^*(\mathbf{v}) = \bar{x}_{t-K+v_i}(i, \mathbf{v}^{\setminus i})$. This shows that $\mathbf{v}_t$ satisfies (8). ■

D. Decoding

In this section we present the decoding operations that are performed at the receivers. First we consider decoding of the $x_{t-K}(i, \mathbf{v}^{\setminus i})$, $\mathbf{v}$ such that $v_i = K$, at the end of time slot $t-1$. We will see that if there exists $j \neq i$ such that $v_j \in \{0, K\}$, the required symbol is simply transmitted by one of the neighbors. Otherwise, a more complicated decoding procedure is required. This procedure is based on the assumption that symbols transmitted by neighbors satisfy the relations given in Lemmas 3 and 4.

In Section IV-E we will finalize the proof of Lemma 2 by showing that the conditions for Lemmas 3–6 are satisfied for all time slots.

Lemma 5. *Let $t > K$, $\mathbf{v} \in V \setminus \mathring{V}$ and i such that $v_i = K$. Assume that for all $t' < t$ and $\mathbf{u} \in \mathring{V}$, $\mathbf{u}_{t'}$ satisfies (7), and, that for all $t' < t$, $1 \leq j \leq d$ and $\mathbf{u} \in V \setminus \mathring{V}$, $\mathbf{u}_{t'}(j)$ and $\bar{\mathbf{u}}_{t'}(j)$ satisfy (3). At the end of time slot $t-1$, $\mathbf{v}$ can decode $x_{t-K}(i, \mathbf{v}^{\setminus i})$. If $\exists j \neq i$ s.t. $v_j \in \{0, K\}$ then take $x_{t-K}(i, \mathbf{v}^{\setminus i}) = \mathbf{v}^i \left[K-1\right]_{t-1}(i)$. Otherwise, take*

$$\begin{aligned} x_{t-K}(i, \mathbf{v}^{\setminus i}) = & \sum_{\substack{\mathbf{u} \in N_v \setminus \{\mathbf{v}\} \\ \tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})}}} \mathbf{u}_{t-\tau} + \sum_{\substack{j \neq i \\ \tau \in \Theta_0}} \left(\mathbf{v}_{t-\tau}(j) + \bar{\mathbf{v}}_{t-\tau}(j) \right) \\ & + \sum_{j \neq i} \left(\mathbf{v}^j \left[v_j - 1\right]_{t-1}(j) + \overline{\mathbf{v}^j \left[v_j + 1\right]}_{t-1}(j) \right) \\ & + \sum_{\substack{\mathbf{u} \in N_{\mathbf{v}} : u_i = K \\ 0 < \mathrm{dist}(\mathbf{u},\mathbf{v}) < d \\ \tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})+1}}} \left(\mathbf{u}_{t-\tau-1}(i) + \bar{\mathbf{u}}_{t-\tau+1}(i) \right) \\ & + \sum_{\tau \in \Theta_1 \setminus \{1\}} \mathbf{v}_{t-\tau+1}(i) + \sum_{\tau \in \Theta_1} \bar{\mathbf{v}}_{t-\tau-1}(i). \end{aligned} \qquad (10)$$

Proof: If $\exists j \neq i$ s.t. $v_j \in \{0, K\}$, then $\mathbf{v}^i \left[K-1\right] \in V \setminus \mathring{V}$ and is, therefore, transmitting $x_{t-K}(i, \mathbf{v}^{\setminus i})$ in time slot $t-1$. For the other case, we first observe that in (10) all terms correspond to symbols that have been received by $\mathbf{v}$ in time slots before t. Now, denote the RHS of (10) as $\hat{x}_{t-K}(i, \mathbf{v}^{\setminus i})$. By the assumptions in the lemma this can be rewritten as

$$\hat{x}_{t-K}(i, \mathbf{v}^{\setminus i}) = \sum_{j \neq i} \left(\mathbf{v}_A(j) + \mathbf{v}_{\bar{A}}(j) \right) + \mathbf{v}_B + \mathbf{v}_{\bar{B}}, \qquad (11)$$

where

$$\mathbf{v}_A(j) = \sum_{\substack{\mathbf{u} \in N_{\mathbf{v}} \\ \tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})}}} x_{t-\tau-u_j}(j, \mathbf{u}^{\setminus j}) + x_{t-v_j}(j, \mathbf{v}^{\setminus j}), \qquad (12)$$

$$\mathbf{v}_{\bar{A}}(j) = \sum_{\substack{\mathbf{u} \in N_{\mathbf{v}} \\ \tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})}}} \bar{x}_{t-\tau-K+u_j}(j, \mathbf{u}^{\setminus j}) + \bar{x}_{t-K+v_j}(j, \mathbf{v}^{\setminus j}), \qquad (13)$$

$$\mathbf{v}_B = \sum_{\substack{\mathbf{u} \in N_v \setminus \{\mathbf{v}\} \\ \tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})}}} x_{t-\tau-u_i}(i, \mathbf{u}^{\setminus i}) + \sum_{\substack{\mathbf{u} \in N_{\mathbf{v}} : u_i = K \\ 0 < \mathrm{dist}(\mathbf{u},\mathbf{v}) < d \\ \tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})+1}}} x_{t-\tau-1-u_i}(i, \mathbf{u}^{\setminus i}) + \sum_{\tau \in \Theta_1 \setminus \{1\}} x_{t-\tau+1-v_i}(i, \mathbf{v}^{\setminus i}), \qquad (14)$$

$$\mathbf{v}_{\bar{B}} = \sum_{\substack{\mathbf{u} \in N_v \setminus \{\mathbf{v}\} \\ \tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})}}} \bar{x}_{t-\tau-K+u_i}(i, \mathbf{u}^{\setminus i}) + \sum_{\substack{\mathbf{u} \in N_{\mathbf{v}} : u_i = K \\ 0 < \mathrm{dist}(\mathbf{u},\mathbf{v}) < d \\ \tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})+1}}} \bar{x}_{t-\tau+1-K+u_i}(i, \mathbf{u}^{\setminus i}) + \sum_{\tau \in \Theta_1} \bar{x}_{t-\tau-1-K+v_i}(i, \mathbf{v}^{\setminus i}). \qquad (15)$$

We will show that in (11), $\mathbf{v}_B = x_{t-K}(i, \mathbf{v}^{\setminus i})$ and that $\mathbf{v}_A(j) = \mathbf{v}_{\bar{A}}(j) = \mathbf{v}_{\bar{B}} = 0$ for all $j \neq i$.

For $\mathbf{v}_A(j)$, $j \neq i$, following the proof of Lemma 4, we have

$$\sum_{\substack{\mathbf{u} \in N_{\mathbf{v}} \\ \tau \in \Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})}}} x_{t-\tau-u_j}(j, \mathbf{u}^{\setminus j}) = x_{t-v_j}(j, \mathbf{v}^{\setminus j}).$$

$$
\begin{aligned}
\mathbf{v}_B =& \sum_{\tau\in\Theta_1} x_{t-\tau-(v_i-1)}(i,\mathbf{v}^{\setminus i}) + \sum_{\substack{\mathbf{u}\in N_{\mathbf{v}}:u_i=K\\ 0<\mathrm{dist}(\mathbf{u},\mathbf{v})<d}} \left(\sum_{\tau\in\Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})}} x_{t-\tau-u_i}(i,\mathbf{u}^{\setminus i}) + \sum_{\tau\in\Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})+1}} x_{t-\tau-(u_i-1)}(i,\mathbf{u}^{\setminus i}) \right) + \\
& \sum_{\substack{\mathbf{u}\in N_{\mathbf{v}}:u_i=K\\ 0<\mathrm{dist}(\mathbf{u},\mathbf{v})<d\\ \tau\in\Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})+1}}} x_{t-\tau-1-u_i}(i,\mathbf{u}^{\setminus i}) + \sum_{\tau\in\Theta_1\setminus\{1\}} x_{t-\tau+1-v_i}(i,\mathbf{v}^{\setminus i}) \\
=& \sum_{\substack{\mathbf{u}\in N_{\mathbf{v}}:u_i=K\\ 0<\mathrm{dist}(u,v)<d}} \left(\sum_{\tau\in\Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})+1}} x_{t-\tau-1-u_i}(i,\mathbf{u}^{\setminus i}) + \sum_{\tau\in\Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})}} x_{t-\tau-u_i}(i,\mathbf{u}^{\setminus i}) + \sum_{\tau\in\Theta_{\mathrm{dist}(\mathbf{u},\mathbf{v})+1}} x_{t-\tau+1-u_i}(i,\mathbf{u}^{\setminus i}) \right) + x_{t-v_i}(i,\mathbf{v}^{\setminus i}) \\
=& \, x_{t-K}(i,\mathbf{v}^{\setminus i}). \qquad (16)
\end{aligned}
$$

Therefore, $\mathbf{v}_A(j) = 0$. Similarly one can show that $\mathbf{v}_{\bar{A}}(j) = 0$. For $\mathbf{v}_B$ it follows from (16) that $\mathbf{v}_B = x_{t-K}(i,\mathbf{v}^{\setminus i})$. The last equality in (16) follows from (4) and the fact that $v_i = K$. Similarly, we have $\mathbf{v}_{\bar{B}} = 0$. Therefore, $\hat{x}_{t-K}(i,\mathbf{v}^{\setminus i}) = x_{t-K}(i,\mathbf{v}^{\setminus i})$. ■

The decoding procedure for the $\bar{x}_{t-K}(i,\mathbf{v}^{\setminus i})$, $\mathbf{v}$ such that $v_i = 0$, can be obtained by considering the symmetry of the network topology and the coding operations.

Lemma 6. *Let $t > K$ and $\mathbf{v} \in V \setminus \mathring{V}$ and i such that $v_i = 0$. Assume that for all $t' < t$ and $\mathbf{u} \in \mathring{V}$, $\mathbf{u}_{t'}$ satisfies (7), and, that for all $t' < t$, $1 \le j \le d$ and $\mathbf{u} \in V \setminus \mathring{V}$, $\mathbf{u}_{t'}(j)$ and $\bar{\mathbf{u}}_{t'}(j)$ satisfy (3). At the end of time slot $t-1$, node $\mathbf{v}$ can decode $\bar{x}_{t-K}(i,\mathbf{v}^{\setminus i})$.*

Proof: Follows from Lemma 5 by considering the symmetry of the configuration and the coding operations (1), (2) and (6). ■

E. Proof of Lemma 2

For $t \le 0$ all symbols are assumed zero and therefore satisfy (3) and (7). Also, at $t = 1$, no non-zero decoded symbols are required in (1) and (2). The conditions to Lemmas 3–6 for $t = 1$ are, therefore, satisfied. By using induction over time, it follows that in all time slots, the source symbols required in (1) and (2) have been successfully decoded and that all transmitted symbols satisfy (3) and (7), hence, the code is valid.

To count the number of transmissions, note that there are $(K+1)^d$ nodes in total, of which the $(K-1)^d$ internal ones transmit once. The remaining nodes transmit $2d$ times in each time slot. Moreover, one source symbol for each unicast session is decoded in each time slot. ■

V. Discussion

We have obtained a lower bound on the energy benefit of network coding for multiple unicast in d-dimensional wireless networks. For 2 and 3 dimensional networks the new bound improves upon previous results. For higher dimensions our results might lead to a better insight in the energy benefit of network coding for wireless networks. These insights could in turn lead to new results for lower dimensions.

Note, that the energy benefit of network coding restricted to decode-and-recombine strategies is upper bounded by 3 [8]. In the network code that has been constructed in this paper, nodes retransmit linear combination of symbols that have not been decoded at that node. The code, therefore, does not qualify as decode-and-recombine [3]. This shows, that energy can be saved by considering coding strategies other than decode-and-recombine. As a final remark, note that applying the general bounding techniques from [4] to our configuration leads to a trivial lower bound of 1 on the energy benefit. By explicitly constructing a network code we obtain a better bound.

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