Random Access with Physical-layer Network Coding

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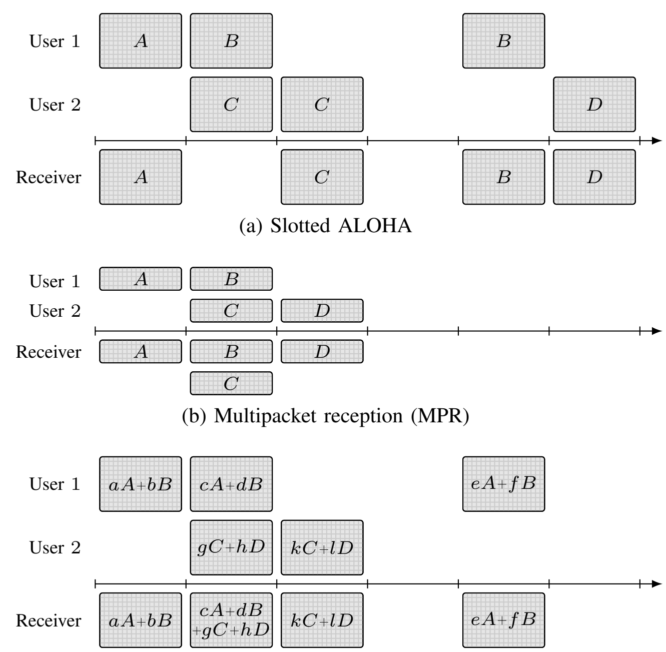
***Abstract*—Leveraging recent progress in compute-and-forward we propose an approach to random access that is based on physical-layer network coding: When packets collide, it is possible to recover a linear combination of the packets at the receiver. Over many rounds of transmission, the receiver can thus obtain many linear combinations and eventually recover all original packets. This is by contrast to slotted ALOHA where packet collisions lead to complete erasures. The throughput of the proposed strategy is derived for a system with two users and shown to be significantly superior to the best known strategies, including multipacket reception.**

I. Introduction

Consider a multipe-access channel with two users transmitting according to a random access mechanism. Users are in one of two states, active or inactive, and do not have knowledge of the states of other users. The receiver has complete knowledge of the states of all users. The network is operated in rounds. In each round each user chooses his state at random independently of the state of other rounds and independent of the other users. The *contribution of the current work* is an approach to random access based on *physical-layer network coding*. The basic idea of the approach is that the receiver is decoding in each round a linear combination of messages. Once enough linear combinations are obtained the original messages can be retrieved.

The concept of physical-layer network coding is studied in, for instance [1]–[7]. See [8] for a survey of recent results. In [1]–[3] the aim is to obtain linear combinations reliably. In contrast, in [4]–[7] one is satisfied with a noisy version of these linear combinations. In [9] the approach of [4]–[7] is used in the random access setting. Our strategy is based on *reliable* physical layer network coding, also known as compute-and-forward. We demonstrate that in the random-access setting with two users it can be useful to decode linear combinations of messages, even if the receiver is ultimately interested in the messages themselves.

The most elementary approach to random access is slotted ALOHA [10], *cf.* [11]. In ALOHA, if more than one user is active, a packet collision occurs and the receiver does not obtain any information about the transmitted packets.



(a) Slotted ALOHA

(b) Multipacket reception (MPR)

(c) Physical-layer network coding (PLNC)

Fig. 1: Illustration of various approaches to random access. The height of the packets reflects the amount of information contained in the packet.

This is illustrated in Figure 1a, in which different rounds are represented along the horizontal axis. Packets transmitted by the users are depicted above the axis, the packets below the axis represent the information obtained by the receiver. In order to compare various strategies, we let the height of a packet reflect the amount of information contained in the packet, *i.e.*, it reflects the rate of the underlying forward-error correcting code.

It is well known, see for instance [12], that if the communication rates of the users are chosen carefully the receiver can decode all, or a subset of, the packets of the active users. This is often referred to as *multipacket reception* and its use for random access was considered in [13]. It is illustrated in Figure 1b. As reflected in the figure, multipacket reception requires the rate to be adjusted. More recent results on multipacket reception for random access are given in [14] and [15]. One of the aspects studied in [14] is the tradeoff between the rate of the code and the maximum number of packets that can be decoded.

The *strategy* that we propose in the current paper is based on another way of dealing with collisions. Instead of trying to decode any of the packets transmitted by the users, the receiver attempts to decode a linear combination of these packets, using the compute-and-forward approach. In the proposed strategy, the users transmit in each time slot a packet that is itself a linear combination of the messages intended for the receiver. After obtaining a sufficient number of linear combinations the receiver can retrieve the original messages. The physical-layer network coding strategy is illustrated in Figure 1c. For the example in the figure, the information obtained by the receiver can be represented as

$$\begin{bmatrix} a & b & 0 & 0 \\ c & d & g & h \\ 0 & 0 & k & l \\ e & f & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \tag{1}$$

where $A, \cdots, D$ denote the messages and $b_1, \ldots, b_4$ the information obtained by the receiver at the channel output.

The outline of the remainder of this paper is as follows. In Section III we define the model. Reliable physical-layer network coding is briefly introduced in Section II. The main contributions of the current paper, a description of our approach and an analysis of the resulting throughput are given in Section IV. Section V provides an analysis of the throughput of some other approaches as well as an upper bound on the achievable throughput. A numerical evaluation of results is given in VI. Finally, in Section VII we an outlook on future work.

II. Reliable Physical-layer Network Coding

One key ingredient of the strategy proposed in this paper is the technique of computation codes. We provide a short introduction to this technique and a result from [1] that will be needed later. We refer the reader to [1], [2] or [8] for technical details.

To set the stage for the technique, we consider two transmitters, each having an independent message. Moreover, we think of the two messages as being represented as strings over an appropriately chosen finite field $\mathbb{F}_q$. That is, we denote the message of transmitter i as

$$M_i = (M_i(1), M_i(2), \ldots, M_i(L_c)), \tag{2}$$

where $M_i(j) \in \mathbb{F}_q$.

Each transmitter can encode its message into a string of B real numbers satisfying an average power constraint P. We define the rate of the resulting two codes, which is the same for both transmitters, by

$$R = R_1 = R_2 = \frac{L_c \log_2 q}{B} \text{ bits per channel use.} \tag{3}$$

The real-valued strings of length B, denoted as X_1 and X_2, are then transmitted element-wise across the standard AWGN multiple-access channel, whose channel output is given by

$$Y = X_1 + X_2 + Z, \tag{4}$$

where Z is additive white Gaussian noise with unit variance.

The decoder, upon observing the real-valued string Y of length B, is asked to provide an estimate sequence $(\hat{M}(1), \hat{M}(2), \ldots, \hat{M}(L_c))$ in such a way as to minimize the probability of the event

$$\begin{aligned} &(\hat{M}(1), \hat{M}(2), \ldots, \hat{M}(L_c)) \\ &\quad \neq \quad (M_1(1) + M_2(1), \ldots, M_1(L_c) + M_2(L_c)). \end{aligned} \tag{5}$$

In this sense, the receiver recovers a function (namely, the sum) of the original messages, which is why this approach is referred to as computation coding. We refer to the rate $R_1 = R_2$ as the *computation rate* and say that it is achievable if the probability of the above event can be made arbitrarily small by increasing B. The next result provides the best known achievable computation rate.

Theorem 1 ([2], Thm. 2)**.** *For the standard AWGN multiple-access channel, the following computation rate is achievable:*

$$R \quad = \quad \frac{1}{2} \log_2 \left(\frac{1}{2} + P \right). \tag{6}$$

The scheme we will employ in the present paper achieves the above computation rate and was developed in [1], [2]. It involves using one and the same code at both encoders, namely,

$$X_1 = F(M_1) \quad \text{and} \quad X_2 = F(M_2), \tag{7}$$

and we refer to $F(\cdot)$ as the *computation code.*

III. Model and Problem Formulation

We consider a system with two users. Time is slotted. The system is operated in blocks of B time slots. The length of a block, B, is a design parameter. Let $s(t) = 1 + (t-1) \bmod B$ and $n(t) = \lceil t/B \rceil$, *i.e.*, time slot t is the $s(t)$-th time slot in block $n(t)$. Also, let $t(n, s) = s + (n-1)B$, *i.e.*, the s-th time slot of block n is $t(n, s)$.

The random access feature of the model is captured by state variables S_i which can be zero or one, depending on whether a user is active (1) or inactive (0). Let $S_i(n)$ denote the state of user i in block n. The state of a user is independently and identically distributed over all blocks and independent of the state of other users. Users are active with probability a, *i.e.*, $\Pr(S_i(n) = 1) = a$ for all $i = 1, 2$ and all $n = 1, \ldots, N$.

Let $X_i[t] \in \mathbb{R}$ and $Y[t] \in \mathbb{R}$ denote the signal transmitted by user i and the signal obtained by the receiver, respectively, in time slot t. We consider an AWGN channel without fading, *i.e.*,

$$Y[t] = S_1[t]X_1[t] + S_2[t]X_2[t + Z[t], \tag{8}$$

where $S_i[t] \triangleq S_i(n(t))$ and $\{Z[t]\}$ is white Gaussian noise with unit variance.

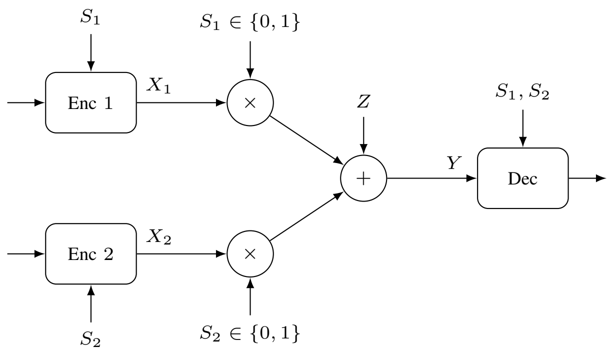


Fig. 2: Model

We consider transmission over N blocks of length B, and will denote the real-valued sequences of length NB transmitted by the transmitters and the real-valued sequence of length NB received at the destination by

$$X_i = (X_i[1], X_i[2], \cdots, X_i[NB]), \tag{9}$$
$$Y = (Y[1], Y[2], \cdots, Y[NB]), \tag{10}$$

respectively.

In the description of our proposed strategy, we will also find it convenient to index individual elements in each block. To this end, using the shorthand $X_i(n,s) = X_i[t(n,s)]$, $Y(n,s) = Y[t(n,s)]$ and $Z(n,s) = Z[t(n,s)]$, we introduce the notation

$$X_i(n) = (X_i(n,1), X_i(n,2), \cdots, X_i(n,B)), \tag{11}$$
$$Y(n) = (Y(n,1), Y(n,2), \cdots, Y(n,B)). \tag{12}$$

The channel model, (8), can alternatively be written as

$$Y(n,s) = \sum_{i=1}^{2} S_i(n) X_i(n,s) + Z(n,s). \tag{13}$$

The model is illustrated in Figure 2.

Definition 1 (Strategy)**.** *A strategy defines encoders E_i, $i = 1,2$, that map the user message M_i and channel states S_i to a signal X_i, i.e.,*

$$E_i : \left\{1, \ldots, 2^{NBR_i}\right\} \times \{0,1\}^N \to \mathbb{R}^{NB}, \tag{14}$$

where we require these mappings to satisfy the following average power constraint:

$$\frac{1}{NB} \sum_{t=1}^{NB} \mathbb{E}\left[X_i^2[t]\right] \le P_i, \tag{15}$$

for $i = 1,2$, where the expectation is over all messages. We denote the encoder mapping by $X_i = E_i(M_i, S_i)$. Finally, a strategy also defines a decoder D that uses knowledge of user states to map the received signal to an estimate of the user messages, i.e.,

$$D : \mathbb{R}^{NB} \times \{0,1\}^N \times \{0,1\}^N \to \left\{1, \ldots, 2^{NBR_1}\right\} \times \left\{1, \ldots, 2^{NBR_2}\right\}, \tag{16}$$

We denote the decoder mapping by $\hat{M} \triangleq (\hat{M}_1, \hat{M}_2) = D(Y, S_1, S_2)$.

Note that the long-term average power constraint allows to perform power control: Transmit at power $a^{-1}P$ in a block in which a user is active and with zero power otherwise, leading to long-term average power P.

For a given strategy, we define the resulting average error probability by

$$P_e = \Pr\left[(\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)\right]. \tag{17}$$

In the present paper, we restrict attention to the *symmetric* scenario where both rates and both constraints are equal, i.e.,

$$R = R_1 = R_2 \quad \text{and} \quad P = P_1 = P_2,$$

and we will refer to the *throughput* T of a strategy simply as the sum of both user rates, namely,

$$T = 2R. \tag{18}$$

The goal of the present paper is to characterize the achievable throughput.

Definition 2 (Achievable throughput)**.** *Throughput T is achievable if there exists for every $\epsilon_1 > 0$ and $\epsilon_2 > 0$ a strategy with $R = T/2 - \epsilon_1$ for which $P_e \le \epsilon_2$.*

IV. Proposed Strategy

The strategy presented in this section forms the main contribution of the current paper.

A. Message structure

Remember from Section III that user i has a message M_i to transmit, where

$$M_i \in \left\{1, \ldots, 2^{NBR}\right\}. \tag{19}$$

The first step of the proposed strategy consists of expressing the message M_i as a string of $NBR/\log_2 q$ symbols from $\mathbb{F}_q$, where q will be suitably chosen. These symbols are grouped in L_b message substrings $M_i(\ell)$, each of length L_c, such that we can express

$$M_i = (M_i(1), \ldots M_i(L_b)), \tag{20}$$

where $M_i(\ell) \in \mathbb{F}_q^{L_c}$, with

$$L_c = \frac{NBR}{L_b \log_2 q}. \tag{21}$$

The values of L_b and L_c need to be chosen carefully; an analysis is provided in subsection IV-D.

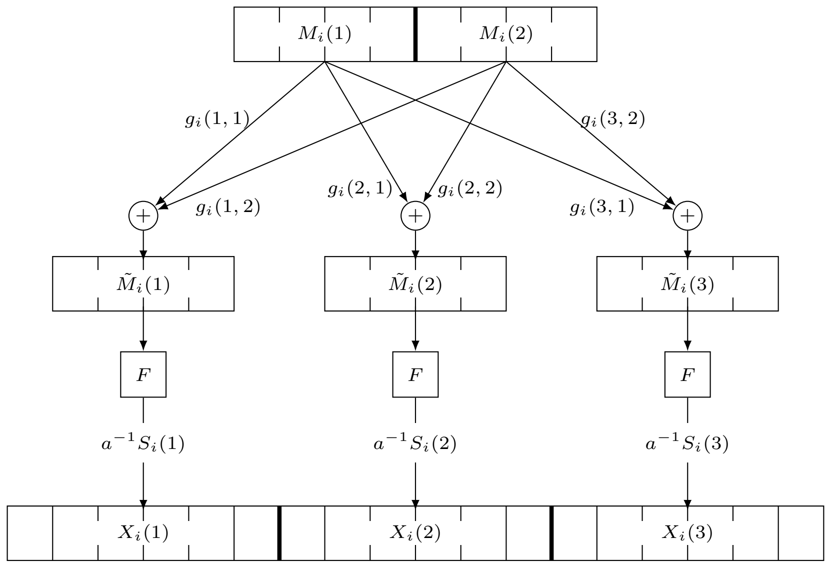


Fig. 3: The encoder for user i, in the case that $N = 3$, $B = 6$, $L_c = 4$, $L_b = 2$.

B. Encoder

The encoder E_i at user i consists of:

- Matrix $\mathbf{G}_i = [g_i(n, \ell)]$ of size $N \times L_b$ with elements from $\mathbb{F}_q$,
- Computation code $F : \mathbb{F}_q^{L_c} \to \mathbb{R}^B$, *i.e.*, a code of blocklength B that takes L_c message symbols.

Encoder i constructs the signal X_i, by performing the following steps for each block $n = 1, \ldots, N$:

1) The encoder first computes new equivalent message substrings $\tilde{M}_i(n)$ by mixing the original message substrings $M_i(\ell)$, as follows:
$$\tilde{M}_i(n) = \sum_{\ell=1}^{L_b} g_i(n, \ell) M_i(\ell), \tag{22}$$
where all operations on $M_i(\ell)$ are componentwise.
2) Use $\tilde{M}_i(n)$ as the input of computation code F (as in Eqn. (7)) and take the user state into consideration, *i.e.*,
$$X_i(n) = a^{-1} S_i(n) F(\tilde{M}_i(n)). \tag{23}$$

The encoder strategy is illustrated in Figure 3.

C. Decoder

The receiver decodes as follows:

1) In block n the receiver observes the signal
$$Y(n) = a^{-1} \sum_{i=1}^{2} \mathbb{1}_{S_i(n)=1} F(\tilde{M}_i(n)) + Z(n). \tag{24}$$
2) Assuming that L_c is chosen properly such that the computation rate is achievable, the computation code thus enables the decoder to recover
$$b(n) = \sum_{i=1}^{2} \mathbb{1}_{S_i(n)=1} \tilde{M}_i(n)$$
$$= \sum_{i=1}^{2} \sum_{\ell=1}^{L_b} \mathbb{1}_{S_i(n)=1} g_i(n, \ell) M_i(\ell), \tag{25}$$
for all $n = 1, \ldots, N$.
3) It remains to retrieve the messages by solving the system of linear equations given by (25). It is shown in the next subsection that a carefully chosen L_b results in a full rank system of equations.

D. Achievable throughput

First of all, we consider the use of the computation code, which in our strategy is done separately in each block of B channel uses, and involves strings of L_c symbols from $\mathbb{F}_q$. By combining Theorem 1 with Eqn. (3), we know that if we choose
$$\frac{L_c \log_2 q}{B} = \frac{1}{2} \log_2 \left[\frac{1}{2} + a^{-1} P \right], \tag{26}$$
we can make the error probability arbitrarily small if B is chosen sufficiently large.

Moreover, we have to choose L_b such that the system of linear equations (over $\mathbb{F}_q$) given by (25) is full rank. We demonstrate that by choosing
$$\frac{L_b}{N} = a - \frac{1}{2} a^2, \tag{27}$$
we can make this probability arbitrarily close to one if N is chosen sufficiently large.

By (18) and (21), we have that
$$T = 2R = \frac{2 L_b L_c \log_2 q}{NB}. \tag{28}$$

The above choices of L_b and L_c thus lead to the following theorem, which forms the main contribution of the current paper.

Theorem 2. *Throughput*

$$T = \left(a - \frac{1}{2}a^2\right) \log_2 \left[\frac{1}{2} + a^{-1}P\right] \tag{29}$$

is achievable using a physical-layer network coding strategy.

The proof of the theorem is deferred to the end of this section and consists mostly of demonstrating that the system of linear equations given by (25) is full rank. We find it convenient to express the system of linear equations (25) as

$$\mathbf{AM} = \mathbf{b}, \tag{30}$$

where

$$\mathbf{M} = \begin{bmatrix}\mathbf{M}_1 \\ \mathbf{M}_2\end{bmatrix}, \quad \mathbf{M}_i = \begin{bmatrix} M_i(1) \\ \vdots \\ M_i(L_b)\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b(1) \\ \vdots \\ b(N)\end{bmatrix} \tag{31}$$

and

$$\mathbf{A} = \mathbf{SG}, \tag{32}$$

with $\mathbf{S}$ the horizontal concatenation of $\mathbf{S}_1$ and $\mathbf{S}_2$, *i.e.*, $\mathbf{S} = [\mathbf{S}_1\ \mathbf{S}_2]$, where $\mathbf{S}_i = \operatorname{diag}(S_i(1), \ldots, S_i(N))$, and $\mathbf{G}$ is the block diagonal matrix $\mathbf{G} = \operatorname{diag}(\mathbf{G}_1, \mathbf{G}_2)$. Alternatively, we can express $\mathbf{A}$ as

$$\mathbf{A} = \begin{bmatrix}\mathbf{S}_1\mathbf{G}_1 & \mathbf{S}_2\mathbf{G}_2\end{bmatrix}. \tag{33}$$

Both representations of $\mathbf{A}$ will be useful in the remainder.

For notational convenience, we define for a subset $U \subset \{1, 2\}$ with elements $U = \{U_1, \ldots, U_{|U|}\}$,

$$\mathbf{S}_U = \begin{bmatrix}\mathbf{S}_{U_1} \cdots \mathbf{S}_{U_{|U|}}\end{bmatrix}, \tag{34}$$

In addition, we define for $I \subset \{1, \ldots, N\}$, $\mathbf{A}_I$ as the submatrix of $\mathbf{A}$ obtained by selecting rows I.

First, we present a necessary condition on $\mathbf{S}$, for $\mathbf{A}$ to be full rank.

Lemma 1. *If there exists $U \subset \{1, 2\}$ for which $\operatorname{rank} \mathbf{S}_U < |U|L_b$ then $\mathbf{A}$ cannot be full rank, i.e., $\mathbf{A}$ can be full rank only if*

$$\operatorname{rank} \mathbf{S}_U \geq |U|L_b, \quad \textit{for all } U \subset \{1, 2\}. \tag{35}$$

The proof of the above lemma follows readily and is omitted here. Note that the rank of $\mathbf{S}_U$ is determined by the number of rounds that all users in U are inactive, *i.e.*, if there are exactly M rounds in which all users in U are inactive then $\operatorname{rank} \mathbf{S}_U = N - M$.

Condition (35) of Lemma 1 will be satisfied with high probability, *i.e.*, the probability that it is not satisfied is approaching zero as N increases. This is addressed by the following lemma. Observe that this lemma is stated in terms of $\tilde{L}_b$ which is slightly smaller than the value of L_b given in (27). This slack is required to make the probability of getting a full rank matrix sufficiently high. Observe also that by taking a large value of N the influence on the rate is vanishing.

Lemma 2. *Suppose, as in our model, that the diagonal entries in every matrix $\mathbf{S}_i$ are i.i.d. distributed with Bernoulli distribution with parameter a. Moreover,*

$$\tilde{L}_b = \left(a - \frac{1}{2}a^2\right) N - N^{\gamma}, \tag{36}$$

where $1/2 < \gamma < 1$. Then, we have

$$\Pr\left[\operatorname{rank} \mathbf{S}_U \geq |U|\tilde{L}_b,\ \forall U \subset \{1, 2\}\right] \geq 1 - \kappa N^{1-2\gamma} \tag{37}$$

for some constant κ that is independent of $|U|$.

Proof: Fix $U \subset \{1, 2\}$. The rank of matrix $\mathbf{S}_U$ is equal to the number of nonzero rows, the number of which is binomially distributed with sample size N and parameter $1 - (1 - a)^{|U|}$. Now

$$\begin{aligned}
&\Pr[\operatorname{rank} \mathbf{S}_U < |U|\tilde{L}_b] && (38)\\
&= \Pr\left[\operatorname{rank} \mathbf{S}_U < \frac{|U|\left(1 - (1-a)^2\right)}{2} N - |U|N^{\gamma}\right] && (39)\\
&\leq \Pr\left[\operatorname{rank} \mathbf{S}_U < \left(1 - (1-a)^{|U|}\right) N - |U|N^{\gamma}\right] && (40)\\
&\leq \tilde{\kappa} N^{1-2\gamma}, && (41)
\end{aligned}$$

for some constant $\tilde{\kappa}$, where the last step is an application of Chebyshev's inequality. The result follows from a union bound over the sets U. ■

Next, we demonstrate that condition (35) of Lemma 1 is not only a necessary, but also a sufficient condition for $\mathbf{A}$ to be full rank. Before giving the exact result and a proof, note that if $\mathbf{S}_1$ and $\mathbf{S}_2$ satisfy condition (35) we can select L_b rounds, *i.e.*, $I \subset \{1, \ldots, N\}$, $|I| = L_b$ such that $\mathbf{A}_I$ also satisfies condition (35). By properly ordering I, the matrix $\mathbf{A}_I$ will have form

$$\mathbf{A}_I = \left[\begin{array}{c|c} \multirow{2}{*}{C_1} & C_3 \\ & 0 \\ \hline C_2 & 0 \\ 0 & C_4 \end{array}\right], \tag{42}$$

where each of the four parts has dimension $L_b \times L_b$, and the sum of the number of rows in C_3 and C_4 is at least L_b. We demonstrate below that there exist $\mathbf{G}_1$ and $\mathbf{G}_2$ such that $\mathbf{A}_I$ as given in (42) will be full rank.

Lemma 3. *There exist matrices $\mathbf{G}_1$ and $\mathbf{G}_2$, such that, if condition (35) of Lemma 1 is satisfied, the matrix $\mathbf{A}$ is full rank.*

Proof: It is readily verified that for any particular $\mathbf{A}_I$ of the form (42) we can choose $\mathbf{G}_1$ and $\mathbf{G}_2$ such that $\mathbf{A}_I$ is full rank. This can be achieved, for instance, by choosing $\mathbf{G}_1$ such that there are additional zero rows in C_1, effectively decoupling both users. Now, there are a finite number of realizations of $\mathbf{S}_1$ and $\mathbf{S}_2$ that satisfy condition (35). For any such realization there is a finite number of sets $I \subset \{1, \ldots, N\}$ for which $\mathbf{A}_I$ is of the form (42). Now, using the algebraic techniques developed by Koetter and Médard [16] it follows directly that by choosing the alphabet $\mathbb{F}_q$ sufficiently large, there exist $\mathbf{G}_1$ and $\mathbf{G}_2$ that result in full rank $\mathbf{A}_I$ for all

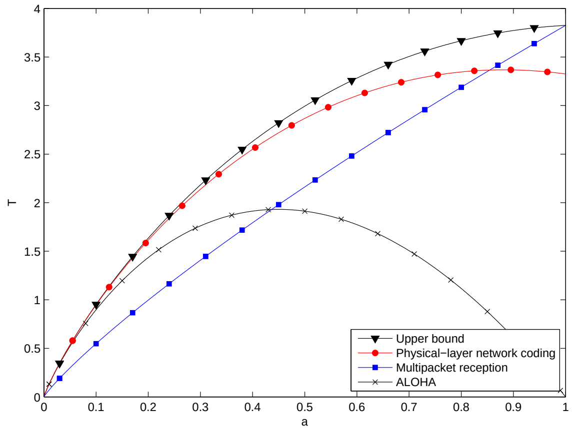


Fig. 4: $P = 10^2$

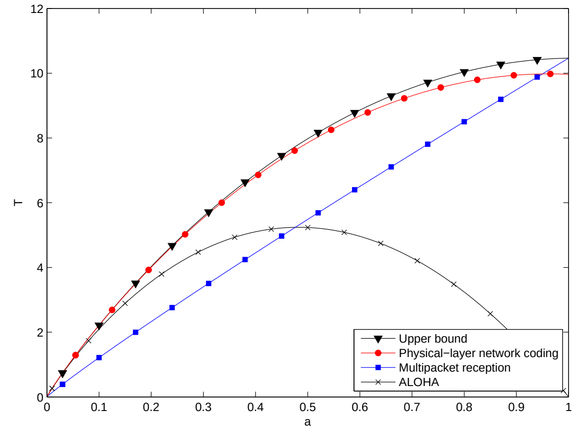


Fig. 5: $P = 10^6$

realizations $\mathbf{S}_1$ and $\mathbf{S}_2$ that satisfy condition (35) and all corresponding subsets I. ■

We conclude this section with a proof of Theorem 2.

Proof of Theorem 2: To prove that the claimed rate is achievable, we show that under the proposed parameter choices, the decoding error probability can be made arbitrarily small. The decoding error probability is given by

$$P_e = Pr(\mathcal{E}_c \cup \mathcal{E}_r), \quad (43)$$

where $\mathcal{E}_c$ is the event that one (or more) of the computation-coded blocks are wrongly decoded, and $\mathcal{E}_r$ is the event that the linear system of equations in Eqn. (25) is not full rank. By the union bound,

$$P_e \leq Pr(\mathcal{E}_c) + Pr(\mathcal{E}_r). \quad (44)$$

It follows from Theorem 1 that by choosing L_c as in Eqn. (26) we can make $Pr(\mathcal{E}_c)$ arbitrarily small by choosing B sufficiently large.

Moreover, whenever the matrix $\mathbf{A}$ has rank $2L_b$, the linear system of equations in Eqn. (25) can be solved. However, if we select the coding matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ properly, we know from Lemma 3 that $\text{rank}(\mathbf{A}) < 2L_b$ is possible only if condition (35) of Lemma 1 is not satisfied. Now, by Lemma 2 we can bound this probability as

$$Pr(\mathcal{E}_r) \leq \kappa N^{1-2\gamma}. \quad (45)$$

Again, by selecting N large enough, we can make this as small as desired, which completes the proof. ■

V. Comparison to Other Strategies

In this section we present results on the achievable throughput of various other strategies. Subsection V-A deals with slotted ALOHA. The throughput of multipacket reception is analyzed in Subsection V-B. A brief discussion of other strategies is given in Subsection V-C. Finally, an upper bound to the achievable throughput is given in Subsection V-D. A numerical evaluation of all these results is given in the next section.

A. Slotted ALOHA

In slotted ALOHA users transmit at the maximum achievable rate, where the maximum is under the condition that there is only a single user in the system, *i.e.*,

$$\frac{L_c \log_2 q}{B} = \frac{1}{2}\log_2(1 + a^{-1}P). \quad (46)$$

The receiver is only able to decode if there is a single active user.

Theorem 3. *Slotted ALOHA achieves throughput*

$$T = a(1-a)\log_2\left(1 + a^{-1}P\right). \quad (47)$$

Proof: The probability that a particular user is the only active user is $a(1-a)$. ■

B. Multipacket reception with adaptive rate

In this section we consider multipacket reception together with adaptive rates [14], [15]. The strategy consists of choosing the rate such that the packets of all active users can be decoded, *i.e.*, both users transmit at rate

$$\frac{L_c \log_2 q}{B} = \frac{1}{4}\log_2\left[1 + 2a^{-1}P\right]. \quad (48)$$

Theorem 4 ([14])**.** *Multipacket reception with adaptive rate achieves throughput*

$$T = \sum_{i=1}^{2}\binom{2}{i}a^i(1-a)^{2-i}\frac{i}{4}\log_2\left[1 + 2a^{-1}P\right]. \quad (49)$$

C. Other strategies

Other strategies have been proposed in the literature. In [14], for instance, a superposition strategy is proposed in which different layers are used. As commented on in [14] the best known strategy for AWGN channels is in fact to not use layers, but to follow either a slotted ALOHA or a multipacket reception with rate adaption strategy.

D. Upper bound

Finally, we provide an upper bound on the achievable throughput.

Theorem 5. *If throughput T is achievable, then*

$$T \leq a(1-a)\log_2\left[1+a^{-1}P\right]+\frac{a^2}{2}\log_2\left(1+2a^{-1}P\right). \quad (50)$$

Proof: We construct an upper bound by considering the case that users have complete knowledge of the state of all users, *i.e.*, we consider a block-fading multiple access channel with full channel state information. A well-known result, *cf.* [12], is that in this case the sum rate is upper bounded as

$$T \leq \mathbb{E}\left[\frac{1}{2}\log_2\left(1+\sum_{i=1}^{2} S_i a^{-1} P\right)\right], \quad (51)$$

where the expectation is over the user states. The result follows directly from the above expression. ■

VI. Numerical evaluation

In this section we provide a numerical evaluation of the performance of the various strategies and the upper bound. In Figures 4 and 5 we have plotted the throughput T as a function of the access probability a for $P = 10^2$ and 10^6 respectively.

The figures clearly demonstrate the well-known fact that ALOHA does not perform well for high access probability. Note, also, that for $a = 1$, the model reduces to a classical multiple-access channel, *i.e.*, a channel without states. For such a channel multipacket reception is optimal. This is reflected in the figures, where at $a = 1$, multipacket reception achieves the upper bound.

For moderate values of a the physical-layer network coding strategy performs significantly better than the other schemes. Moreover, the difference between the performance of the physical-layer network coding scheme and the upper bound is decreasing in the transmitter power.

VII. Discussion

We have presented an approach to random access that is based on physical-layer network coding. The gist of this strategy is that whenever packets collide, the receiver decodes a linear combination of these packets. The throughput that is achieved by this approach is significantly better than that of other approaches.

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