Open network of $M|M|1$ queues – 1

- Customer leaving queue $j$ can route to any of the queues $1, \ldots, J$, or may leave the network.
- $p_{ij}$ fraction of customers from queue $i$ to queue $j$, $p_{i0}$ fraction leaving the network.
- Arrival process is Poisson process with rate $\mu_0$.
- Fraction $p_{0j}$ of these customers is routed to queue $j$.
- The service rate at queue $j$ is $\mu_j$.
- Arrival rate $\lambda_j$ of customers to queue $j$ is obtained from the traffic equations

$$
\lambda_j = \mu_0 p_{0j} + \sum_{i=1}^{J} \lambda_i p_{ij}, \quad j = 1, \ldots, J,
$$
Open network of $M|M|1$ queues – 2

- Evolution number of customers in the queues recorded by Markov chain $\{N(t) = (N_1(t), \ldots, N_J(t)), \ t \in \mathbb{R}\}$
- State space $S \subseteq \mathbb{N}_0^J$, states $n = (n_1, \ldots, n_J)$.
- If $\{N(t)\}$ is in state $n$ and a customer routes from queue $i$ to queue $j$ then the next state is $n - e_i + e_j$, $i, j = 0, \ldots, J$.
- Queue 0 is introduced to represent the outside.
- If a customer routes from queue $i$ to queue 0 then this customer leaves the network.
- If a customer routes from queue 0 to queue $j$ then this customer enters the network at queue $j$, $j = 1, \ldots, J$.
- The transition rates of $\{N(t)\}$ for an open network are, for $n \neq n'$, $n, n' \in S$,

$$q(n, n') = \begin{cases} \mu_i p_{ij}, & \text{if } n' = n - e_i + e_j, \ i, j = 0, \ldots, J, \\ 0, & \text{otherwise.} \end{cases}$$
Open network of $M|M|1$ queues – 4

Theorem (3.1.4 Equilibrium distribution)

Consider the Markov chain $\{N(t)\}$ at state space $S = \mathbb{N}_0^J$ for the open network of $M|M|1$ queues. Assume the routing matrix $P = (p_{ij})$ is irreducible and let $\{\lambda_j\}$ be the unique solution of the traffic equations. If $\rho_j := \lambda_j/\mu_j < 1$, $j = 1, \ldots, J$, then $\{N(t)\}$ has unique product-form equilibrium distribution

$$
\pi(n) = \prod_{j=1}^{J} (1 - \rho_j) \rho_j^{n_j} = \prod_{j=1}^{J} \pi_j(n_j), \quad n \in S.
$$

Moreover, the equilibrium distribution satisfies partial balance, for all $n \in S$, $i = 0, \ldots, J$,

$$
\sum_{j=0}^{J} \{\pi(n)q(n, n - e_i + e_j) - \pi(n - e_i + e_j)q(n - e_i + e_j, n)\} = 0.
$$
Proof of Theorem 3.1.4

\[
\sum_{j=0}^{J} \left\{ m(n)q(n, n - e_i + e_j) - m(n - e_i + e_j)q(n - e_i + e_j, n) \right\} \\
= \sum_{j=0}^{J} \left\{ \prod_{k=1}^{J} \rho_k^{n_k} \mu_i p_{ij} 1(n - e_i \in \mathbb{N}_0^J) - \prod_{k=1}^{J} \rho_k^{n_k - \delta_{ki} + \delta_{kj}} \mu_j p_{ji} 1(n - e_i \in \mathbb{N}_0^J) \right\}
\]

\[
\sum_{j=0}^{J} \left\{ m(n)q(n, n - e_i + e_j) - m(n - e_i + e_j)q(n - e_i + e_j, n) \right\} 1(i = 0)
= \left\{ \mu_0 - \sum_{j=1}^{J} \lambda_j \rho_{j0} \right\} \prod_{k=1}^{J} \rho_k^{n_k} 1(n \in \mathbb{N}_0^J) = 0,
\]

\[
\sum_{j=0}^{J} \left\{ m(n)q(n, n - e_i + e_j) - m(n - e_i + e_j)q(n - e_i + e_j, n) \right\} 1(i \neq 0)
= \left\{ \sum_{j=0}^{J} \lambda_i \rho_{ij} - \mu_0 \rho_{0i} - \sum_{j=1}^{J} \lambda_j \rho_{ji} \right\} \prod_{k=1}^{J} \rho_k^{n_k - \delta_{ki}} 1(n - e_i \in \mathbb{N}_0^J) 1(i \neq 0) = 0.
\]
Moreover, the equilibrium distribution satisfies partial balance, for all \( n \in S, \ i = 0, \ldots, J, \)

\[
\sum_{j=0}^{J} \left\{ \pi(n)q(n, n-e_i+e_j) - \pi(n-e_i+e_j)q(n-e_i+e_j, n) \right\} = 0.
\]
Networks of Queues

Lecture 3
Richard J. Boucherie
Stochastic Operations Research
\[
\sum_{j=0}^{J} \left\{ m(n)q(n, n - e_i + e_j) - m(n - e_i + e_j)q(n - e_i + e_j, n) \right\} \\
= \sum_{j=0}^{J} \left\{ \prod_{k=1}^{J} \rho^{n_k}_{k} \mu_{i} \rho_{ij} \mathbb{1}(n - e_i \in \mathbb{N}_0^{J}) - \prod_{k=1}^{J} \rho^{n_k - \delta_{ki} + \delta_{kj}}_{k} \mu_{j} \rho_{ji} \mathbb{1}(n - e_i \in \mathbb{N}_0^{J}) \right\} \\
\sum_{j=0}^{J} \left\{ m(n)q(n, n - e_i + e_j) - m(n - e_i + e_j)q(n - e_i + e_j, n) \right\} \mathbb{1}(i \neq 0) \\
= \left\{ \sum_{j=0}^{J} \lambda_{i} \rho_{ij} - \mu_{0} \rho_{0j} - \sum_{j=1}^{J} \lambda_{j} \rho_{ji} \right\} \prod_{k=1}^{J} \rho^{n_k - \delta_{ki}}_{k} \mathbb{1}(n - e_i \in \mathbb{N}_0^{J}) \mathbb{1}(i \neq 0) = 0.
\]
Kelly-Whittle networks – 2

\[
\sum_{j=0}^{J} \left\{ m(n)q(n, n - e_i + e_j) - m(n - e_i + e_j)q(n - e_i + e_j, n) \right\}
\]

state dependent sojourn time in state \( n \): \( q(n) \rightarrow \frac{q(n)}{\phi(n)} \)

\[
\neq \sum_{j=0}^{J} \left\{ m(n) \frac{q(n, n - e_i + e_j)}{\phi(n)} - m(n - e_i + e_j) \frac{q(n - e_i + e_j, n)}{\phi(n - e_i + e_j)} \right\}
\]
Kelly-Whittle networks – 3

\[ \sum_{j=0}^{J} \left\{ m(n)q(n, n - e_i + e_j) - m(n - e_i + e_j)q(n - e_i + e_j, n) \right\} \]

state dependent sojourn time in state \( n \): \( q(n) \rightarrow \frac{q(n)}{\phi(n)} \)

also scale \( m(n) \): fraction of time spent in state \( n \)

\[ = \sum_{j=0}^{J} \left\{ \phi(n)m(n) \frac{q(n, n - e_i + e_j)}{\phi(n)} - \phi(n - e_i + e_j)m(n - e_i + e_j) \frac{q(n - e_i + e_j, n)}{\phi(n - e_i + e_j)} \right\} \]
Kelly-Whittle networks – 4

\[ \sum_{j=0}^{J} \{m(n)q(n, n - e_i + e_j) - m(n - e_i + e_j)q(n - e_i + e_j, n)\} \]

State dependent sojourn time in state \( n \): \( q(n) \rightarrow \frac{q(n)}{\phi(n)} \)

Also scale \( m(n) \): fraction of time spent in state \( n \)

And add a function \( \psi(n - e_i) \) to the rates

\[ = \sum_{j=0}^{J} \left\{ \phi(n)m(n)\psi(n - e_i) \frac{q(n, n - e_i + e_j)}{\phi(n)} - \phi(n - e_i + e_j)m(n - e_i + e_j)\psi(n - e_i) \frac{q(n - e_i + e_j, n)}{\phi(n - e_i + e_j)} \right\} \]
Markov chain \( \{ N(t) \} \) at state space \( S \subseteq \mathbb{N}_0^J \) with transition rates, for \( n' \neq n \),

\[
q(n, n') = \begin{cases} 
\frac{\psi(n - e_i)}{\phi(n)} \mu_i \rho_{ij}, & \text{if } n' = n - e_i + e_j, \ i, j = 0, \ldots, J, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \psi : \mathbb{N}_0^J \rightarrow [0, \infty) \) and \( \phi : \mathbb{N}_0^J \rightarrow (0, \infty) \).

We will consider closed networks as special case of open networks with \( \mu_0 = 0 \) and \( p_{i0} = 0 \), \( i = 1, \ldots, J \).
Theorem (Equilibrium distribution: Kelly-Whittle network)

Consider the Kelly-Whittle network \( \{ N(t) \} \) at state space \( S \subseteq \mathbb{N}_0^J \). Assume the routing matrix \( P = (p_{ij}) \) is irreducible and let \( \{ \lambda_j \} \) be solution of traffic equations. Let \( \rho_j = \lambda_j / \mu_j \). Assume

\[
G_{KW}^{-1} = \sum_{n \in S} \phi(n) \prod_{j=1}^{J} \rho_j^{n_j} < \infty,
\]

and that \( \{ N(t) \} \) is irreducible. Then

\[
\pi(n) = G_{KW} \phi(n) \prod_{j=1}^{J} \rho_j^{n_j}, \quad n \in S.
\]

Moreover, \( \pi \) satisfies partial balance, for all \( n \in S, i = 0, \ldots, J, \)

\[
\sum_{j=0}^{J} \{ \pi(n) q(n, n - e_i + e_j) - \pi(n - e_i + e_j) q(n - e_i + e_j, n) \} = 0.
\]
Kelly-Whittle networks – 7

- Poisson arrivals
- Independent queues:

\[
q(n, n - e_i + e_j) = \begin{cases} 
\kappa_i(n_i) \mu_i p_{ij}, & i, j = 1, \ldots, J, \\
\kappa_i(n_i) \mu_i p_{i0}, & i = 1, \ldots, J, \\
\mu_0 p_{0j}, & j = 1, \ldots, J,
\end{cases}
\]

for \( \kappa_i : \mathbb{N}_0 \rightarrow (0, \infty), i = 1, \ldots, J \).

Typical examples are, for \( n \in \mathbb{N}, i = 1, \ldots, J \),

\[
\kappa_i(n) = \begin{cases} 
1, & \text{single server queue}, \\
\min(n, s), & \text{s server queue}, \\
n, & \text{infinite server queue}.
\end{cases}
\]
Kelly-Whittle networks – 8

- Poisson arrivals
- Independent queues:

\[
q(n, n - e_i + e_j) = \begin{cases} 
\kappa_i(n_i) \mu_i p_{ij}, & i, j = 1, \ldots, J, \\
\kappa_i(n_i) \mu_i p_{i0}, & i = 1, \ldots, J, \\
\mu_0 p_{0j}, & j = 1, \ldots, J,
\end{cases}
\]

for \( \kappa_i : \mathbb{N}_0 \rightarrow (0, \infty), \ i = 1, \ldots, J. \)

Let \( \eta_i : \mathbb{N}_0 \rightarrow (0, \infty), \ i = 1, \ldots, J, \)

\[
\eta_i(n)^{-1} = \prod_{r=1}^{n} \kappa_i(r), \quad n \in \mathbb{N}_0, \ i = 1, \ldots, J.
\]

Then

\[
\kappa_i(n) = \frac{\eta_i(n - 1)}{\eta_i(n)}, \quad n \in \mathbb{N}, \ i = 1, \ldots, J,
\]
Partial balance – 2

Transition balance:
\[ q(n, n - e_i + e_j) = q(n - e_i + e_j, n), \quad i, j = 0, \ldots, J, \]

Detailed balance:
\[ \pi(n)q(n, n - e_i + e_j) = \pi(n - e_i + e_j)q(n - e_i + e_j, n), \quad i, j = 0, \ldots, J, \]

Partial balance:
\[ \sum_{j=0}^{J} \pi(n)q(n, n - e_i + e_j) = \sum_{j=0}^{J} \pi(n - e_i + e_j)q(n - e_i + e_j, n), \quad i = 0, \ldots, J, \]

Global balance:
\[ \sum_{i,j=0}^{J} \pi(n)q(n, n - e_i + e_j) = \sum_{i,j=0}^{J} \pi(n - e_i + e_j)q(n - e_i + e_j, n). \]
Moreover, the equilibrium distribution satisfies **partial balance**, for all $n \in S$, $i = 0, \ldots, J$,

$$
\sum_{j=0}^{J} \left\{ \pi(n)q(n, n - e_i + e_j) - \pi(n - e_i + e_j)q(n - e_i + e_j, n) \right\} = 0.
$$
Interpretation of the traffic equations

- Average number of customers moving from queue $i$ to queue $j$ is (see reader for proper definition)

$$\lambda_{ij} = \mathbb{E} q(N, N - e_i + e_j) = \sum_{n \in S} \pi(n) q(n, n - e_i + e_j).$$

- For network with Poisson arrivals $\psi(n) = \phi(n)$, $n \in \mathbb{N}_0^J$

- Then, with $\lambda_0 = \mu_0$,

$$\lambda_{ij} = \sum_{n \in S} G_{KW} \phi(n) \prod_{j=1}^J \rho_j^n \frac{\phi(n - e_i)}{\phi(n)} \mu_i \rho_{ij}$$

$$= \lambda_i \rho_{ij} \sum_{n \in S, n_i > 0} G_{KW} \phi(n - e_i) \prod_{j=1}^J \rho_j^{n_j - \delta_{ij}} = \lambda_i \rho_{ij}.$$

- We may interpret the solution $\lambda_j, j = 1, \ldots, J$, of traffic equations as the arrival rate of customers.
Networks of Queues

Lecture 3
Richard J. Boucherie
Stochastic Operations Research
A Kelly-Whittle network with state-dependent routing is a Markov chain \( \{N(t)\} \) at state space \( S \subseteq \mathbb{N}_0^J \) with transition rates, for \( n' \neq n \),

\[
q(n, n') = \begin{cases} 
\frac{\psi(n - e_i) \theta_i(n - e_i)}{\phi(n)} \mu_i b_{ij}(n - e_i), & n' = n - e_i + e_j, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \phi : S \to (0, \infty) \) and \( \psi, \theta_i, b_{ij} : S^b \to [0, \infty) \), and \( S^b \) is the set of base states:

\[
S^b = \{ m \in \mathbb{N}_0^J : \exists i, j \in \{0, \ldots, J\}, i \neq j \text{ s.t. } m + e_i \text{ and } m + e_j \in S \}.
\]

Without loss of generality \( \sum_{j=0}^J b_{ij}(m) = 1 \).
Theorem (3.4.1 Equilibrium distribution)

Consider the Kelly-Whittle network with state-dependent routing \( \{N(t)\} \) at state space \( S \subseteq \mathbb{N}_0^J \). Assume a solution \( H : S \to [0, \infty) \) exists of the state-dependent traffic equations, for \( n \in S, i = 0, \ldots, J \):

\[
\sum_{j=0}^{J} H(n) \theta_i(n - e_i) b_{ij}(n - e_i) = H(n - e_i) \theta_0(n - e_i) \mu_0 b_{0i}(n - e_i) \\
+ \sum_{j=1}^{J} H(n - e_i + e_j) \theta_j(n - e_i) b_{ji}(n - e_i).
\]

Assume that \( G^{-1} = \sum_{n \in S} \phi(n) \prod_{j=1}^{J} \left( \frac{1}{\mu_j} \right)^{n_j} H(n) < \infty \), and that \( \{N(t)\} \) is irreducible. Then

\[
\pi(n) = G\phi(n) \prod_{j=1}^{J} \left( \frac{1}{\mu_j} \right)^{n_j} H(n), \quad n \in S.
\]
State-dependent routing; blocking protocols – 3

- Product-form
  \[ \pi(n) = G\phi(n) \prod_{j=1}^{J} \left( \frac{1}{\mu_j} \right)^{n_j} H(n), \quad n \in S. \]

- State-dependent traffic equations just as difficult to solve as the partial balance equations.

- In applications, often Markov routing probabilities \( p_{ij}, i, j = 0, \ldots, J \), and a function of the base state: \( b_{ij}(m) = p_{ij}f(m), \quad m \in S^b, \)
  for some \( f : S^b \to [0, \infty) \).

- With \( \lambda_j, j = 1, \ldots, J \), solution of the traffic equations, solution of the state-dependent traffic equations:
  \[ H(n) = \prod_{j=1}^{J} \lambda_j^{n_j}, \quad n \in S. \]
Product-form?

- Why product-form useful?
- Capacity constraints: no product-form
  Tandem of 2 queues. Queue 1 has capacity restriction $c_1$. If $n_1 = c_1$ customers arriving customer discarded.

```
(0, n_2)      (c_1, n_2)
```

```
(0, 0)        (c_1, 0)
```

(network_diagram)
Blocking protocols: Stop-protocol – 1

- If queue $i$ in a Kelly-Whittle network with finite capacity constraints becomes saturated ($n_i = c_i$) then stop service at all other queues $j = 1, \ldots, J, j \neq i$, and stop the arrival process to the network.

![Diagram of two queues with finite capacity](image-url)
Blocking protocols: Stop-protocol – 2

If queue $i$ in a Kelly-Whittle network with finite capacity constraints becomes saturated ($n_i = c_i$) then stop service at all other queues, and stop the arrival process to the network. For the open network the state space is

$$S_{c,o} = \{ n \in \mathbb{N}_0^J : n_j \leq c_j, \ n_i + n_j < c_i + c_j, \ i \neq j, \ i, j = 1, \ldots, J \}.$$ 

Transition rates

$$q(n, n') = \begin{cases} \frac{\psi(n - e_i)\theta_i(n - e_i)}{\phi(n)} \mu_i b_{ij}(n - e_i), & n' = n - e_i + e_j, \\ 0, & \text{otherwise,} \end{cases}$$

with

$$\theta_i(m) = 1, \quad i = 0, \ldots, J, \ m \in S_{c,o}^b,$$

$$f(m) = 1 (m_j < c_j, \ j = 1, \ldots, J), \quad m \in S_{c,o}^b,$$

$$b_{ij}(m) = p_{ij} f(m), \quad i, j = 0, \ldots, J, \ m \in S_{c,o}^b,$$

and

$$S_{c,o}^b = \{ m \in \mathbb{N}_0^J : 0 \leq m_j \leq c_j - 1 \}.$$
The state-dependent traffic equations now reduce to the traffic equations, and

\[
H(n) = \prod_{j=1}^{J} \lambda_{j}^{n_j}, \quad n \in S_{c,o},
\]

satisfies the state-dependent traffic equations. Assume that

\[
G_{c,o}^{-1} = \sum_{n \in S_{c,o}} \phi(n) \prod_{j=1}^{J} \rho_{j}^{n_j} < \infty,
\]

and that \(\{N(t)\}\) is irreducible. Then \(\{N(t)\}\) has unique equilibrium distribution

\[
\pi(n) = G_{c,o} \phi(n) \prod_{j=1}^{J} \rho_{j}^{n_j}, \quad n \in S_{c,o}.
\]
Blocking protocols: Jump-over-protocol – 1

- **Jump-over-blocking** If queue $i$ in a Kelly-Whittle network with finite capacity constraints becomes saturated ($n_i = c_i$) then a customer arriving to queue $i$ will immediately select a new station $j$ with probability $p_{ij}, j = 0, \ldots, J, i = 1, \ldots, J$.

- **Generalised jump-over-blocking** A customer arriving at station $i$ when $n_i$ customers are present will be accepted with probability $a_i(n_i)$, and will jump over the station with probability $1 - a_i(n_i)$. A rejected customer selects a new station $j$ with probability $p_{ij}, j = 0, \ldots, J, i = 1, \ldots, J$. 
Blocking protocols: Jump-over-protocol – 2

- **Generalised jump-over-blocking** A customer arriving at station \(i\) when \(n_i\) customers are present will be accepted with probability \(a_i(n_i)\), and will jump over the station with probability \(1 - a_i(n_i)\). A rejected customer selects a new station \(j\) with probability \(p_{ij}\), \(j = 0, \ldots, J\), \(i = 1, \ldots, J\).
- Let \(c_j = \inf\{k : a_j(k) = 0, \ k = 0, 1, 2, \ldots\}\), \(j = 1, \ldots, J\).
- For the open network the state space is
  \[
  S_{jo,c} = \{n \in \mathbb{N}_0^J : 0 \leq n_j \leq c_j, \ i = 1, \ldots, J\}
  \]
- Let \(P(m) = (p_{ij}a_j(m_j), \ i, j = 0, \ldots, J)\), and \(P_*(m) = (p_{ij}(1 - a_j(m_j)), \ i, j = 0, \ldots, J)\).
- Transition rates, \(m \in S^b_c\),
  \[
  b_{ij}(m) = p_{ij}a_j(m_j) + (P_*(m)P(m))_{ij} + (P_*^2(m)P(m))_{ij} + \cdots
  = \sum_{k=0}^{\infty} (P_*^k(m)P(m))_{ij}.
  \]
Blocking protocols: Jump-over-protocol – 3

For $a_j(m_j) = 1(m_j \leq c_j)$

$$H(n) = \prod_{j=1}^{J} \lambda_j^{n_j}, \quad n \in S_{jo,c},$$

satisfies the state-dependent traffic equations. Assume that

$$G_{jo,c}^{-1} = \sum_{n \in S_{jo,c}} \phi(n) \prod_{j=1}^{J} \rho_j^{n_j} < \infty,$$

and that $\{N(t)\}$ is irreducible. Then $\{N(t)\}$ has unique equilibrium distribution

$$\pi(n) = G_{jo,c} \phi(n) \prod_{j=1}^{J} \rho_j^{n_j}, \quad n \in S_{jo,c}.$$
Networks of Queues

Lecture 3
Richard J. Boucherie
Stochastic Operations Research