

MQSN – 2016 - Second set for Part Boucherie

3. Consider a system consisting of a CPU with $J - 1$ clients that send jobs to the CPU. The CPU equally divides its computing time over the jobs present at the CPU. Each client has a Poisson arrival process of new jobs with rate λ_i for client i , $i = 1, \dots, J - 1$. A job that arrives at a client first needs to be handled by the client (e.g. obtains a time stamp). To this end, the client has a single server handling the jobs, and jobs wait for their turn in a queue of unlimited capacity (are treated First In First Out). The service time of a job at the client is exponential with rate ν_i for client i , $i = 1, \dots, J - 1$. Upon completion of this activity, the job is sent to the CPU. Here the required amount of service is exponential with rate μ . Upon completion of the job at the CPU, the job returns to the client it originated from to be handled a second time. To this end, it again joins the tail of the queue, waits for its turn (FIFO) and has again exponential service time with rate ν_i to e.g. obtain a second time stamp. Upon this second service completion the job leaves the system.
- (a) Model this system as a network of queues and give a complete description of the Markov chain that records the number of customers in each queue for the tandem network, i.e., state space, transition rates.
 - (b) Give the stability condition for this network.
 - (c) Give the equilibrium distribution for this network and prove correctness.
 - (d) Give the mean sojourn time in the system of a job that arrives at client 1.
 - (e) Relate the network of this exercise to that of exercise 1. Elaborate on the difference in the models and behaviour of these networks.
4. Consider an open network of J quasi-reversible nodes¹ linked via Markov routing, that is a customer that leaves queue i moves to queue j with probability p_{ij} , $i, j = 0, \dots, J$.
- (a) Give a complete description of the Markov chain that records the number of customers in the nodes, i.e., state space, transition rates.

¹At this point, node refers to the abstract definition as used in lecture 3 for a network of quasi-reversible nodes, i.e., a Markov chain with transitions rates $q_i(x_i, x'_i)$.

- (b) Give the equilibrium distribution and prove correctness of the equilibrium distribution.
- (c) Now consider node 1 in isolation (i.e. the network consisting of node 1 only), and assume that node 1 is a symmetric queue. Prove that for phase type service distribution, the distribution of the number of customers in this queue is independent of the phase type distribution except for its mean. You may assume that all customers are of the same type.
- (d) Now return to considering the network of J nodes. Assume that each node is a symmetric queue. Prove that for phase type service distribution at each queue, the joint distribution of the number of customers in the queues is independent of the phase type distributions at the queues except for their means. You may assume that all customers are of the same type.

Now let each node be a general quasi reversible node.

- (f) We may aim to aggregate each quasi reversible node into a single queue, such that the aggregated queue is first order equivalent to the node, see slides lecture 4 for details on this aggregation. Prove or dispute the following statement: *A quasi reversible node can be aggregated into a first order equivalent queue if and only if the node is insensitive to the distribution of the sojourn time except for its mean at the node.* [You may restrict to phase type distributions]

5. Consider a Kelly Whittle network of J queues with state-dependent Markov routing. The transition rates are

$$q(n, n - e_i + e_j) = \frac{\psi(n - e_i)}{\phi(n)} p_{ij} b_{ij}(n - e_i), \quad (1)$$

where $\psi() \geq 0$, $\phi() > 0$. Assume that queue i has finite capacity N_i , $i = 1, \dots, J$, i.e., $n_i \leq N_i$, $i = 1, \dots, J$.

- (a) Assume that the network operates under the *stop-protocol*, that is when queue i , say, reaches its capacity limit, then service at all queues $j \neq i$, $j = 0, \dots, J$, is stopped, see lecture 4 for more details. Give a complete description of the Markov chain that records the number of customers in the nodes, i.e., state space, transition rates, including a specification of the $b_{ij}(n - e_i)$.
- (b) For the network operating under the stop-protocol, give the equilibrium distribution and prove correctness of the equilibrium distribution via partial balance.
- (c) For the network operating under the stop-protocol, give (i) the distribution of the number of customers seen by a customer arriving to queue i from outside the network, and (ii) the distribution of the number of customers seen by a customer arriving to queue i from another queue in the network, and (iii) the distribution of the number of customers seen by a customer arriving to queue i . In each case, prove correctness.
- (d) Now assume that the network operates under the *jump-over-protocol*, where customers route among the queues according to the routing probabilities p_{ij} , and when a customer reaches queue k , say, that has reached its capacity limit, the customer jumps over the queue, and continues its route according to the routing probabilities p_{kj} . Give a complete description of the Markov chain that records the number of customers in the nodes, i.e., state space and transition rates. Note that the structure of the transition rates under the jump-over-protocol differs from that in (1).
- (e) For the network operating under the jump-over-protocol, give the equilibrium distribution and prove correctness of the equilibrium distribution. You may restrict your proof to the case with $\psi(n) = \phi(n) = 1$ for all n .²

²You may NOT use the argument that the network under the jump-over-protocol is obtained from the stop-protocol by increasing the service rates, as this results in a Markov chain with infinite transition rates.

6. Consider a $J \times I$ grid (the points $\{(1, 1), (1, 2), \dots, (1, J), (2, 1), \dots, (I, J)\}$). Each node in the grid is a queue. Customers arrive according to a Poisson process to the nodes on the edge of the grid (source nodes), and follow a path through the grid to a node on the edge of the grid (sink nodes), e.g., from node $(1, 2)$ to node $(J - 1, J)$. The customer-type uniquely defines its source and sink nodes. In a path, a customer may route to the horizontal or vertical neighbouring nodes, only, i.e., from node (i, j) to $(i - 1, j), (i + 1, j), (i, j - 1), (i, j + 1)$. Let $1/\mu(i, j)$ denote the mean service time of a customer at node (i, j) and assume that the service times of all customers are exponentially distributed. Let the service discipline of the node (i, j) be described by three functions $\phi_{(i,j)}(n) = 1(n > 0)$, $\gamma_{(i,j)}(k, n)$, and $\delta_{(i,j)}(k, n)$ as given in the slides of lecture 3.
- (a) Give an optimisation problem to optimally select routes for each customer type such that the total sojourn time for all types is minimised.
- (b) Consider the 6×6 grid with the following source to destination pairs: $(1, 3)$ to $(3, 6)$; $(1, 4)$ to $(6, 4)$; $(1, 5)$ to $(5, 1)$; $(2, 6)$ to $(6, 1)$; $(4, 6)$ to $(4, 1)$; $(5, 6)$ to $(1, 2)$. Solve the optimisation problem in the case that all queues are single server FIFO queues with $\lambda_t = \lambda$ for all t , $t = 1, \dots, 6$, and $\mu(i, j) = \mu$ for all i, j , $i, j = 1, \dots, 6$. Draw conclusion on the structure of the optimal routes for a general grid with source and destination nodes.