Random Walks in Dependent Random Environments

Werner Scheinhardt   Dirk Kroese

Workshop on Stochastic Processes
at the occasion of retirement Erik van Doorn
September 26, 2014

Outline

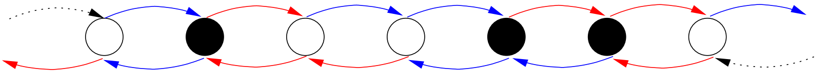
- Introduction
- General behavior
- Evaluating the drift for "swap model"
  - iid environment: trapping phenomenon
  - Markov environment
  - 2-dependent environment
  - Moving average environment
- Conclusions

Introduction

(1-dimensional) Random Walk in Random Environment (RWRE):

1. "Underlying" environment **U** is random but fixed (Assume **U** stationary and ergodic w.r.t. location)
2. Random walk $\{X_n, n = 0, 1, \ldots\}$ with conditional transition probabilities, determined by the environment:

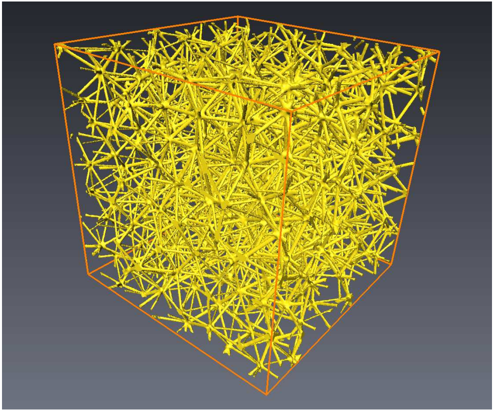
$$\mathbb{P}(X_{n+1} = i + 1 \,|\, X_n = i, \mathbf{U} = \mathbf{u}) = \alpha_i(\mathbf{u})$$
$$\mathbb{P}(X_{n+1} = i - 1 \,|\, X_n = i, \mathbf{U} = \mathbf{u}) = \beta_i(\mathbf{u}) = 1 - \alpha_i(\mathbf{u}).$$



RWREs exhibit interesting and unusual behavior, not seen in ordinary random walks.

Introduction—Motivation

RWREs (d-dimensional) used in physics to model motion through disorganized (random) media.



Introduction—Goal

Theoretical behavior of *drift* in 1-dimensional RWRE well understood, but

- numerical evaluation?
- influence of dependent environments?
- d-dimensional case??

Our goal: develop theory and methods to compute the drift of the random walk for *dependent* environments.

Current results: for 1-dimensional case

General behavior

Drift behavior of $\{X_n\}$ follows, in principle, from Solomon 1975, Alili 1999

Key quantities:

$$\sigma_i = \sigma_i(\mathbf{u}) = \frac{\beta_i(\mathbf{u})}{\alpha_i(\mathbf{u})}$$

$$S = 1 + \sigma_1 + \sigma_1\,\sigma_2 + \sigma_1\,\sigma_2\,\sigma_3 + \cdots$$

and

$$F = 1 + \frac{1}{\sigma_{-1}} + \frac{1}{\sigma_{-1}\,\sigma_{-2}} + \frac{1}{\sigma_{-1}\,\sigma_{-2}\,\sigma_{-3}} + \cdots$$

General behavior—Transience/Recurrence

Will $\{X_n\}$ ultimately move to $+\infty$ or $-\infty$, or not?

Theorem 1 (Solomon–Alili)

1. *If* $\mathbb{E}[\ln \sigma_0] < 0$, *then almost surely* $\lim_{n\to\infty} X_n = \infty$.
2. *If* $\mathbb{E}[\ln \sigma_0] > 0$, *then almost surely* $\lim_{n\to\infty} X_n = -\infty$.
3. *If* $\mathbb{E}[\ln \sigma_0] = 0$, *then almost surely* $\liminf_{n\to\infty} X_n = -\infty$ *and* $\limsup_{n\to\infty} X_n = \infty$.

General behavior—Law of Large Numbers

If $\{X_n\}$ ultimately moves to $+\infty$ or $-\infty$, how fast?

Theorem 2 (Solomon–Alili)

1. *If* $\mathbb{E}[S] < \infty$, *then almost surely* $\displaystyle\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{2\mathbb{E}[S]-1}$.
2. *If* $\mathbb{E}[F] < \infty$, *then almost surely* $\displaystyle\lim_{n\to\infty} \frac{X_n}{n} = \frac{-1}{2\mathbb{E}[F]-1}$.
3. *If* $\mathbb{E}[S] = \infty$ *and* $\mathbb{E}[F] = \infty$, *then almost surely* $\displaystyle\lim_{n\to\infty} \frac{X_n}{n} = 0$.

$\lim_{n\to\infty} X_n/n$ is the **drift** of the process $\{X_n\}$, denote by V.

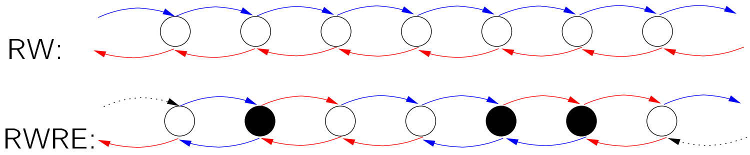
Swap model

- Simple but versatile model with rich behavior
- $\alpha_i(\mathbf{u})$ and $\beta_i(\mathbf{u})$ only depend on $\mathbf{u}$ via u_i
- Each U_i can only take value -1 (swap) or $+1$ (no swap).
- For some fixed value p in $(0, 1)$, let $\beta_i(\mathbf{u}) = 1 - \alpha_i(\mathbf{u})$ and

$$\alpha_i(\mathbf{u}) = \begin{cases} p & \text{if } u_i = 1 \\ 1 - p & \text{if } u_i = -1 \end{cases}$$

- As a result we have

$$\sigma_i = \frac{p}{1-p}\mathbb{P}(U_i = -1) + \frac{1-p}{p}\mathbb{P}(U_i = 1) = \left(\frac{1-p}{p}\right)^{U_i}$$



Dependency structure

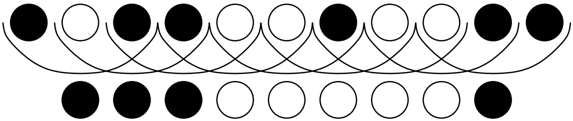
Standard iid case (Sinai 1983):

- Assumes the $\{U_i\}$ are iid with, for some $0 < \alpha < 1$,

$$\mathbb{P}(U_i = 1) = \alpha, \qquad \mathbb{P}(U_i = -1) = 1 - \alpha.$$

Other possibilities for the environment states $\{U_i\}$:

- Markov environment
- k-dependent process
- Moving average of an iid environment:



Iid case

Parameters α and p.

$$\mathbb{E}[\ln \sigma_0] = (1-2\alpha)\ln\left(\frac{1-p}{p}\right)$$

Hence $X_n \to +\infty$ a.s. if and only if
either $\alpha > 1/2$ and $p > 1/2$, or $\alpha < 1/2$ and $p < 1/2$.

$$\mathbb{E}[S] = \sum_{n=0}^{\infty} \mathbb{E}\left[\sigma_1 \cdots \sigma_n\right] = \sum_{n=0}^{\infty} \left(\mathbb{E}\left[\left(\frac{1-p}{p}\right)^{U_1}\right]\right)^n = \frac{1}{1-\mathbb{E}\left[\left(\frac{1-p}{p}\right)^{U_1}\right]}$$

$$\text{iff} \quad \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{U_1}\right] = \frac{p}{1-p}(1-\alpha) + \frac{1-p}{p}\alpha \ < \ 1.$$

Hence when $\alpha > 1/2$ and $p \in (1/2, \alpha)$, or $\alpha < 1/2$ and $p \in (\alpha, 1/2)$, the drift is

$$V = \frac{1}{2\mathbb{E}[S]-1} = (2p-1)\frac{\alpha - p}{\alpha(1-p)+(1-\alpha)p} > 0\ .$$

lid case

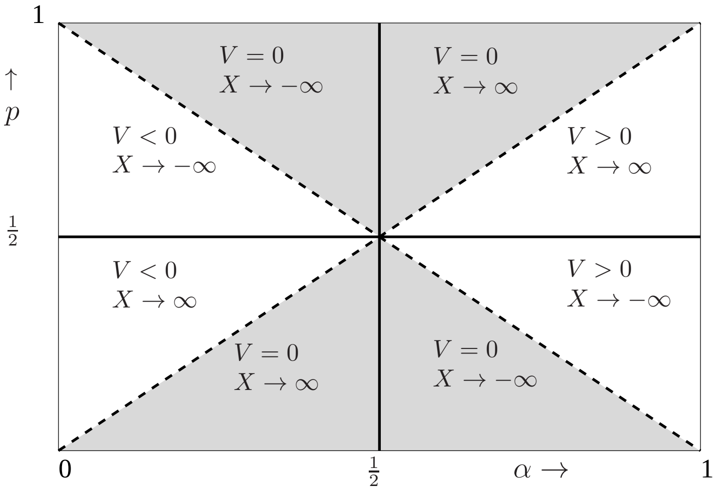


Figure 1: Drift regimes as a function of p and α.
Grey areas: process transient, but drift is zero!

lid case—Trapping phenomenon

How can both $X_n \to \infty$ and $V = 0$ be true?
(e.g. when $p > \alpha > 1/2$)

lid case—Trapping phenomenon

How can both $X_n \to \infty$ and $V = 0$ be true?
(e.g. when $p > \alpha > 1/2$)

- Consider $\alpha = 3/4$, i.e. 25% of locations are black (swapped)
- When $1/2 < p < 3/4$, drift to the right is 'obvious'
- When $3/4 < p < 1$,
  - still (strong) push to the right in majority of locations (hence $X_n \to \infty$)
  - but in remaining 25% of locations strong (!) push to the left

Iid case—Trapping phenomenon

How can both $X_n \to \infty$ and $V = 0$ be true?
(e.g. when $p > \alpha > 1/2$)

- Consider $\alpha = 3/4$, i.e. 25% of locations are black (swapped)
- When $1/2 < p < 3/4$, drift to the right is 'obvious'
- When $3/4 < p < 1$,
  - still (strong) push to the right in majority of locations (hence $X_n \to \infty$)
  - but in remaining 25% of locations strong (!) push to the left
- Hence $V = 0$ in latter case:
  difficult to cross barrier of many adjacent black locations !
- Apparently the "cut-off" value for p is equal to α …
- … in the iid case!

Markov environment

Let $\{\mathbf{U}_i\}$ be a stationary discrete-time Markov chain on $\{-1, 1\}$, with one-step transition matrix P given by

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

Parameters a, b and p

$$\mathbb{E}[\ln \sigma_0] = \frac{a-b}{a+b} \ln \frac{1-p}{p}.$$

so transience/recurrence behaviour is same as for iid case (with $a/(a+b)$ playing the role of α).

Again the drift is determined by:

$$\mathbb{E}[S] = \sum_{n=0}^{\infty} \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{\sum_{i=1}^{n} U_i}\right].$$

Let, with $\sigma = (1-p)/p$,

$$G_u^{(n)}(\sigma) = \mathbb{E}\left[\sigma^{\sum_{i=1}^{n} U_i} \mid U_0 = u\right], \quad u \in \{-1, 1\}.$$

Markov environment (cont'd)

Conditioning on U_0 we have,

$$G_u^{(n+1)}(\sigma) = \mathbb{E}\left[\sigma^{\sum_{i=1}^{n+1} U_i} \mid U_0 = u\right] = \mathbb{E}\left[\sigma^{\sum_{i=2}^{n+1} U_i}\sigma^{U_1} \mid U_0 = u\right]$$
$$= \sum_{v\in\{-1,1\}} P_{u,v}\sigma^v G_v^{(n)}(\sigma) .$$

In matrix notation, with column vector $\mathbf{G}^{(n)}(\sigma)$,

$$\mathbf{G}^{(n+1)}(\sigma) = PD\mathbf{G}^{(n)}(\sigma),$$

where $D = \mathrm{diag}(\sigma^{-1}, \sigma)$ so

$$PD = \begin{bmatrix} (1-a)\sigma^{-1} & a\sigma \\ b\sigma^{-1} & (1-b)\sigma \end{bmatrix}$$

Markov environment (cont'd)

Also using $G_u^{(0)}(\sigma) = 1$, we find

$$\mathbf{G}^{(n)}(\sigma) = (PD)^n \mathbf{G}^{(0)}(\sigma) = (PD)^n \mathbf{1},$$

where $\mathbf{1} = (1, 1)^\top$, and hence

$$\mathbb{E}[S] = \sum_{n=0}^{\infty} \boldsymbol{\pi} \mathbf{G}^{(n)}(\sigma) \; = \; \boldsymbol{\pi} \sum_{n=0}^{\infty} (PD)^n \mathbf{1},$$

where $\boldsymbol{\pi}$ is stationary distribution vector of $\{U_i\}$.
Hence,

$$V = \begin{cases} 1/\left[2\boldsymbol{\pi}(I - PD)^{-1}\mathbf{1} - 1\right] & \text{if } \mathrm{Sp}(PD) < 1 \\ 0 & \text{else.} \end{cases}$$

Markov environment (cont'd)

Working out details:

- $\lim_{n\to\infty} X_n = \infty \quad a.s.$ if either
  - $a > b$ and $p \in \left(\frac{1}{2}, \frac{1-b}{(1-a)+(1-b)}\right)$, or
  - $a < b$ and $p \in \left(\frac{1-b}{(1-a)+(1-b)}, \frac{1}{2}\right)$.
- and

$$V = (2p-1)\frac{(1-b)(1-p)-(1-a)p}{\left(b+\frac{a-b}{a+b}\right)(1-p)+\left(a-\frac{a-b}{a+b}\right)p} > 0 \ .$$

Comparison with iid case

To study impact of the (Markovian) dependence, reparameterize a and b via

$$\alpha = \mathbb{P}(U_0 = 1) = a/(a+b)$$

and

$$\varrho = \frac{\mathrm{Cov}(U_0, U_1)}{\mathrm{Var}(U_0)} = 1 - a - b,$$

yielding

$$V = (2p-1)\frac{\alpha - p + \varrho(1-\alpha-p)}{\big(\alpha(1-p) + (1-\alpha)p\big)(1+\varrho) - \varrho}.$$

(Taking $\varrho = 0$ gives iid case)

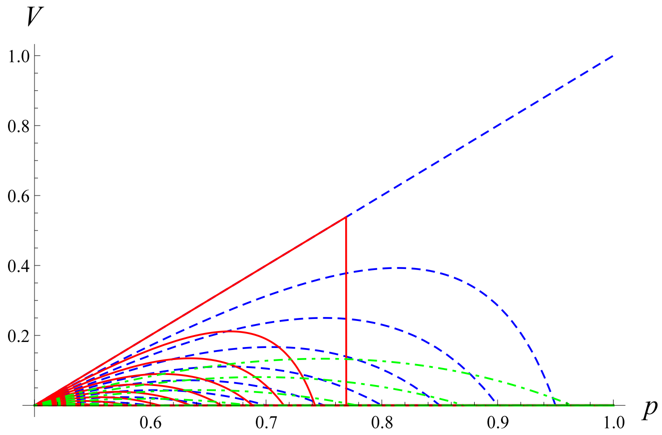


Figure 2: Drift for $\varrho = 0$ (blue, dashed), $\varrho = 0.3$ (red,solid), and $\varrho = -0.3$ (green,dotdashed) as a function of p. From highest to lowest curves for $\alpha = 1, 0.95, \ldots, 0.55$ (for $\varrho = 0$ and $\varrho = 0.3$), and for $\alpha = 0.75, 0.70, \ldots, 0.55$ (for $\varrho = -0.3$).

Other types of dependence model

Setup:

- Let $\{Y_i, i \in \mathbb{Z}\}$ be auxiliary Markov chain on $\{1, \ldots, m\}$ for some m (stationary and ergodic)
- Let $U_i = g(Y_i)$ for some given function $g : \{0, \ldots, m\} \to \{-1, 1\}$

Covers (a.o.):

- Markov environment (U_i and Y_i coincide)
- 2-dependent environment (4×4 transition matrix)
- Moving average environment (8×8 transition matrix)

Markov analysis easily extended, but now need to study larger matrix PD.

Using Perron–Frobenius theory and the implicit function theorem we identify where drift is 0

Moving average environment

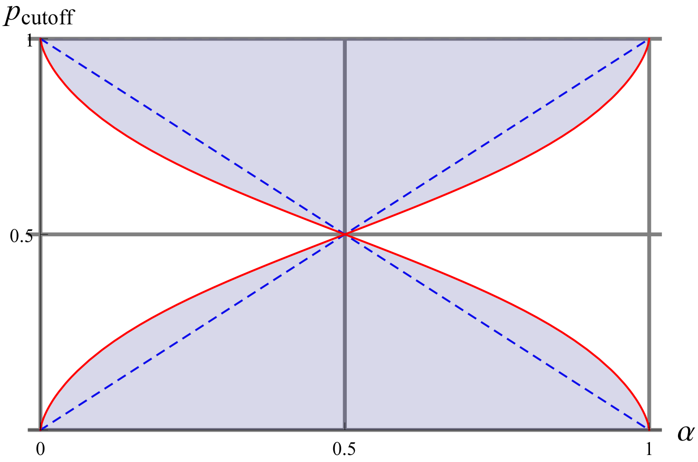


Figure 3: Cutoff values for the moving average process as function of α

Moving average environment

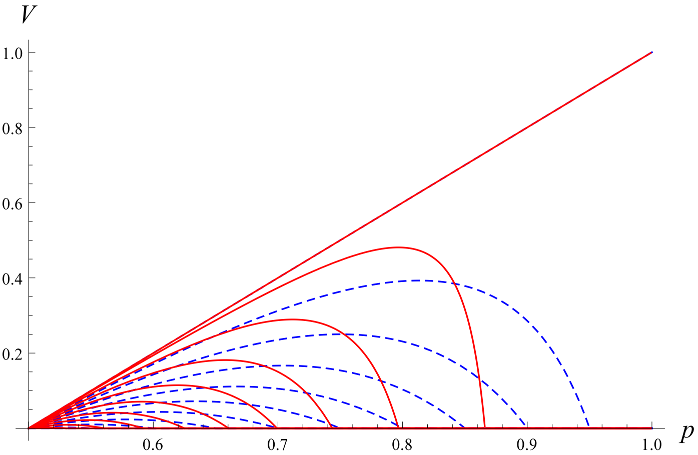


Figure 4: Red: Drift for the moving average environment as a function of p for $\alpha = 1, 0.95, \ldots, 0.55$ (from highest to lowest curves). Blue: comparison with the independent case.

Conclusions

- RWREs are interesting (trapping phenomenon)
- Construction of swap models with (a.o.) k-dependent and moving average environments, using auxilliary Markov chain
- Dependent environments affect the drift
- Cut-off values for p where drift becomes zero are determined via Perron–Frobenius theory

Conclusions

- RWREs are interesting (trapping phenomenon)
- Construction of swap models with (a.o.) k-dependent and moving average environments, using auxilliary Markov chain
- Dependent environments affect the drift
- Cut-off values for p where drift becomes zero are determined via Perron–Frobenius theory

Thanks for your attention!