Math notation: convention or convenience?

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Lifestyle mathematics

- Math as a way of life
- (Math as a) language: communication and thought
- Semantics of languages: based on convention
- Semantics of math: also, + aim for convenience

Three examples of conventional/convenient notation

- Representation of numbers
- Triangle trig
- 2 pi or not 2 pi?

Representation of numbers

Convention was 'Roman' numerals, e.g. *MMXX*

- For many centuries
- Uniqueness? (~~*IIII*~~ or *IV*? ~~*MIM*~~ or *MCMXCIX*?)
- Cumbersome for calculations

Convenient is 'Arabic' numerals, e.g. 2020

- Only since 500 AD
- Positional system
- Any integer $x \in \mathbb{N}$ has unique representation:

$$x = \sum_{k=0}^{\infty} x_k \cdot 10^k \quad \text{with} \quad x_k = \left\lfloor \frac{x}{10^k} \right\rfloor \bmod 10 \in \{0, \ldots, 9\}$$

- (Can be generalized to $\mathbb{R}^+$ and to base $B \neq 10$)

Representation of numbers

Product:

```
   123
    24  x
  ----
   492          ( =        400 +  80 + 12)
  2460  +       ( = 2000 + 400 +  60     )  +
  ----          ---------------------------
  2952          ( = 2000 + 800 + 140 + 12)
```

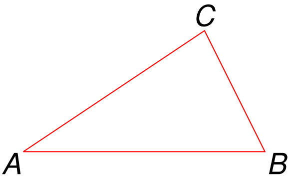
$$z = xy = \Big(\sum_i x_i 10^i\Big)\Big(\sum_j y_j 10^j\Big) = \sum_i \sum_j x_i y_j 10^{i+j}$$

$$\overset{j=k\text{-}i}{=} \sum_k \underbrace{\Big(\textstyle\sum_i x_i y_{k-i}\Big)}_{\hat{z}_k} 10^k = \sum_k z_k 10^k,$$

where $z_k = \tilde{z}_k \bmod 10$ and $\tilde{z}_k = \hat{z}_k + \left\lfloor \frac{\tilde{z}_{k-1}}{10} \right\rfloor$.

- Today, we take this revolutionary new notation for granted
- Lesson #1: (highly) conventional notation can be replaced

Triangle trig



- Convention was (once):
  sides AB, AC and BC, and angles BAC, ABC and ACB.
- Convenient is:
  sides a, b and c, and angles α, β and γ.

$$\frac{\sin(BAC)}{BC} = \frac{\sin(ABC)}{AC} = \frac{\sin(ACB)}{AB}$$

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

$$BC^2 = AC^2 + AB^2 - 2 \cdot AC \cdot AB \cdot \cos(BAC)$$
$$AC^2 = BC^2 + AB^2 - 2 \cdot BC \cdot AB \cdot \cos(ABC)$$
$$AB^2 = BC^2 + AC^2 - 2 \cdot BC \cdot AC \cdot \cos(ACB)$$

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$
$$b^2 = a^2 + c^2 - 2ac\cos(\beta)$$
$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

Triangle trig

Redundant, but convenient:

- *a*, *b*, *c*, α, β and γ
- Tangent:

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$$

- Secant, Cosecant and Cotangent (convenient?):

$$\sec(\alpha) = \frac{1}{\cos(\alpha)}, \quad \csc(\alpha) = \frac{1}{\sin(\alpha)}, \quad \cot(\alpha) = \frac{1}{\tan(\alpha)}$$

- Lesson #2: use redundant notation when convenient

2 pi or not 2 pi?

2π is everywhere, more than π itself!
(transformations, transforms, normal distribution, etc., etc.)

$$\pi = \frac{\text{circumference}}{\text{diameter}} = 3.14\ldots \qquad 2\pi = \frac{\text{circumference}}{\text{radius}} = 6.28\ldots$$

2π is "more fundamental" than π, since...

- diameter versus radius
- half turn versus full turn
- $e^{\pi i} = -1$ versus $e^{2\pi i} = 1$

Define

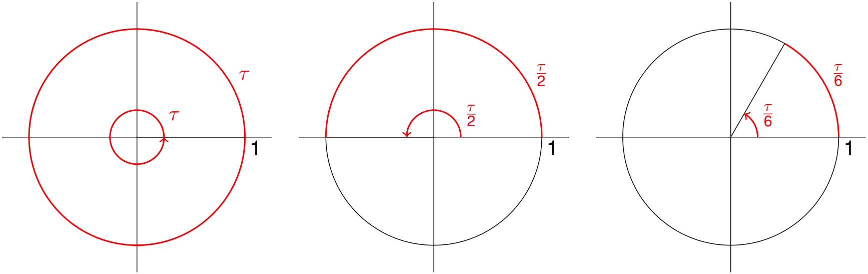
$$\tau = 2\pi = 6.28\ldots$$

NB Archimedes (Greece, 250 BC), al Kāshī (Persia, 1424), Euler (Basel, 1727)

2 pi or not 2 pi?

Working with τ is also "more convenient" than with π

Unit circle:



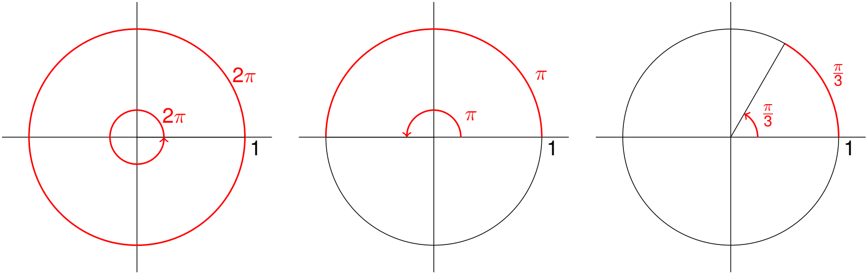
τ rad $= 360^\circ$ $\quad$ $\frac{\tau}{2}$ rad $= 180^\circ$ $\quad$ $\frac{\tau}{6}$ rad $= 60^\circ$

Expressed in τ: angles as fractions of a full turn (360°)

2 pi or not 2 pi?

Working with τ is also "more convenient" than with π

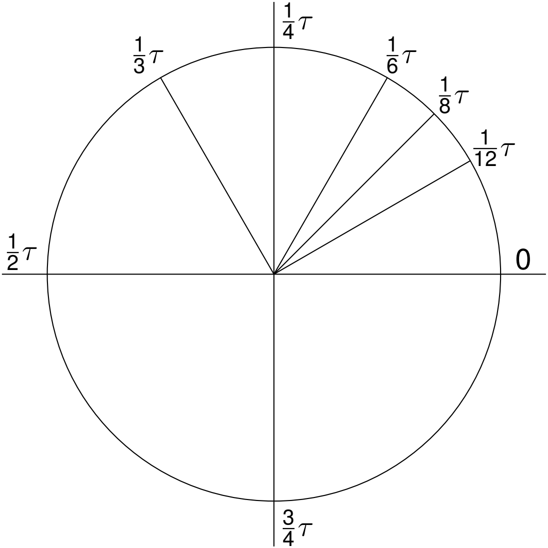
Unit circle:



2π rad $= 360^\circ$ π rad $= 180^\circ$ $\frac{\pi}{3}$ rad $= 60^\circ$

Expressed in π: angles as fractions of a **half** turn (180°)

2 pi or not 2 pi?



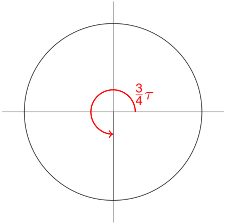
Expressed in τ, angles are where you "expect them to be"

2 pi or not 2 pi?

$$270^\circ = \tfrac{3}{4}\text{ turn} = \tfrac{3}{4}\tau\text{ rad}\ \left(= 1\tfrac{1}{2}\pi\text{ rad}\right)$$

Compare:

$$3\tfrac{3}{4}\tau\text{ rad} = 3\tfrac{3}{4}\text{ turns} = 3\text{ turns} + 270^\circ$$



$$7\tfrac{1}{2}\pi\text{ rad} = 7\tfrac{1}{2}\ \textbf{half}\text{-turns} = 3\text{ turns} + 1\tfrac{1}{2}\text{ half-turns} = 3\text{ turns} + 270^\circ$$

Periodic functions:

$\cos(x)$ has period τ (or: 2π)

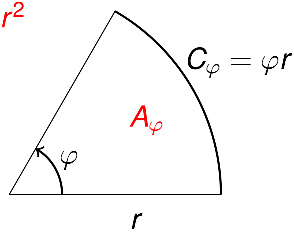
$\cos(5x)$ has period $\frac{\tau}{5}$ (or: $\frac{2}{5}\pi = \frac{2}{5}$th of **half**-period of $\cos(x)$)

2 pi or not 2 pi – Area

- Convention: Circumference $C = 2\pi r$ Area $A = \pi r^2$
- Convenient: Circumference $C = \tau r$ Area $A = \frac{1}{2}\tau r^2$

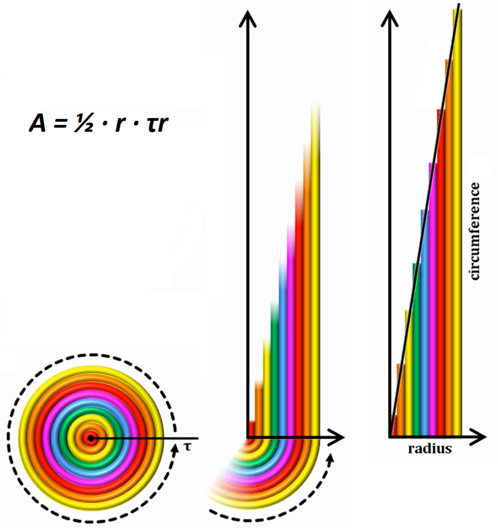
Why a factor $\frac{1}{2}$?

- Cf. many other quadratic forms
- Integrating a linear function ax gives $\frac{1}{2}ax^2$, so:
- Integrating $C = \tau r$ gives $A = \frac{1}{2}\tau r^2$
- Integrating $C_\varphi = \varphi r$ gives $A_\varphi = \frac{1}{2}\varphi r^2$



2 pi or not 2 pi – Area

$$A = \tfrac{1}{2} \cdot r \cdot \tau r$$



From wikimedia: https://commons.wikimedia.org/wiki/File:RolledUpTriangleInsideEveryCircle.ogv

Conclusions

- Notation is important part of 'our language'
- As part of a math lifestyle: choose notation wisely!

- Lesson #1: (highly) conventional notation can be replaced
- Lesson #2: use redundant notation when convenient

- 2 pi or not 2 pi? Something really wrong with π? No,...

... but use τ, and let

$$\pi = \tfrac{1}{2}\tau$$

... or don't use τ, and let

$$\pi = \tfrac{1}{2}\,(2\pi)$$

(Find out more about τ? See tauday.org)