Energy Consumption in Coded Queues for Wireless Information Exchange

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Abstract — **We show the close relation between network coding and queuing networks with negative and positive customers. We use this relation to obtain bounds on the energy consumption in a wireless information exchange setting using network coding.**

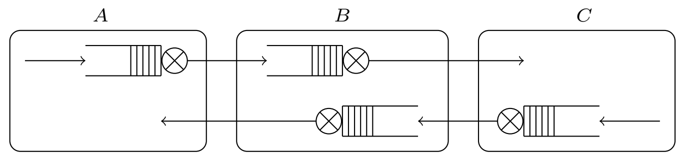

Figure 1: Wireless network in which nodes A and C need to exchange information.

I. Introduction

We consider the problem where two nodes in a wireless network need to exchange information with the help of a single relay node, as depicted in Figure 1. By making use of the relay, node A needs to deliver all its data to node C and vice versa. Node B, therefore, has a queue of packets that need to be delivered to C and another queue of packets that need to be delivered to A. In a traditional wireless network, node B divide its available resources by transmitting a packet from either one of the queues. If network coding [1, 2] is used, node B tranmits packets that are the bitwise exclusive or of two packets, one packet destined for A and one destined for C. Now, A knows the contribution to the exclusive or of the packet destined for C, since this is a packet it transmitted itself. Therefore, it can substract this information from the packet it receives and recover the packet originally transmitted by C. Similarly, also C can obtain the required packets. As a consequence, B no longer has to share its resources to serve A or C, but can simultaneously serve both.

The benefit of using network coding in the wireless information exchange setting was first demonstrated by Wu et al. [3]. It follows, for example, from the observations in [3] that using network coding can reduce the energy consumption in a network by a factor 2. The assumption used, however, is that all nodes always have packets of both types to transmit. If we take the stochastic arrival of packets into consideration, this assumption no longer holds.

In this work we will analyze the energy consumption of coded wireless networks under the scenario that the arrival times of packets at nodes are stochastic processes, in which case it is possible that one of the queues is empty. If there is an empty queue, there are different strategies that can be used. Two examples of strategies are 1) transmit an uncoded packet, and 2) wait for a packet to arrive and only then transmit a codes packet, *i.e.*, never transmit an uncoded packet. These two strategies are extreme cases. We will consider strategies in which the number of uncoded packets that is being transmitted depends on a system parameter. We will analyze the energy consumption at the relay node as a function of this parameter.

We model the system as a continuous time Markov chain and show that networks that employ network coding are strongly related to *queueing networks with negative and positive customers* [6]. From this it follows that for specific parameters, the stationary distribution of the system has a product form that can be explicitly computed. For the cases that the system does not have a product form distribution we use Markov reward techniques [9] to provide upper and lower bounds on the energy consumption.

There is work by Sagduyu and Ephremides [4, 5] in which stochastic arrivals are taken into account in analyzing coded wireless networks. The kind of coding strategies considerd in [4] and [5] are, however, different from the strategies considered in this work. The strategies considered by Sagduyu and Ephremides focus on deciding from which connections to transmit packets. No attention is given to deciding whether or not to transmit uncoded packets.

In Section II we define our model. In Section III we provide an overview of queueing networks with negative and positive customers and show the relation with network coding. Bounds on the energy consumption are presented in Section IV. A proof of the main result is given in Section V. A discussion of the work is presented in Section VI.

II. Model and Notation

We model the state of the queues at the relay node with a continuous-time Markov chain. The system has two queues, let N_i be the number of packets in queue $i \in \{1, 2\}$. We assume that packets arrive at queue i according to a homogeneous Poisson process with intensity λ_i. Note, that nodes A and C from Figure 1 are not part of the model. All packets have unit size, *i.e.*, unit service

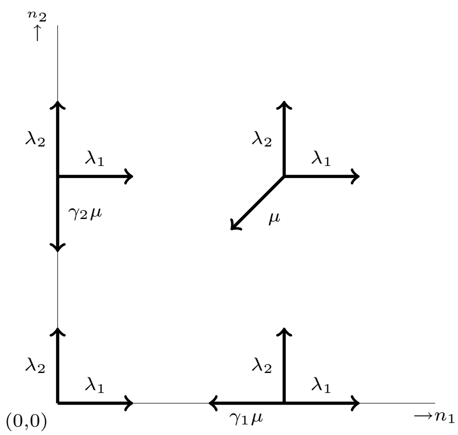

Figure 2: Transition diagram for Markov process of coded system.

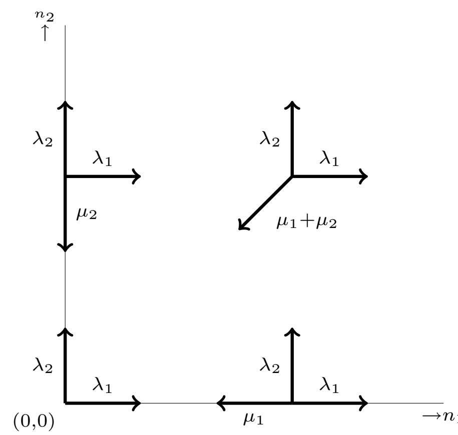

Figure 3: Transition diagram for queueing network with negative and positive customers. There are two queues, external arrival of positive customers at rates λ_1, λ_2, no external arrivals of negative customers, service rates μ_1, μ_2 and all customers leaving one queue enter the other queue as negative customers

requirement. The service rate is exponentially distributed with rate μ. If there is a packet in queue i, but not in queue $j \neq i$, the relay will start transmitting the packet with probability γ_i.

Definition 1. *γ_i is the probability that the relay will transmit an uncoded packet from queue i.*

The above leads to the Markov chain with transition diagram depicted in Figure 2. Let $\{S, Q\}$ denote the chain, where $S = \mathbb{N} \times \mathbb{N}$ is the state space and $Q = [q((n_1, n_2), (m_1, m_2))]$ is the transition rate matrix, where, for (n_1, n_2) and (m_1, m_2) from SS, $(n_1, n_2) \neq (m_1, m_2)$,

$$q((n_1, n_2), (m_1, m_2)) = \begin{cases} \lambda_1, & \text{if } (m_1, m_2) = (n_1 + 1, n_2), \\ \lambda_2, & \text{if } (m_1, m_2) = (n_1, n_2 + 1), \\ \gamma_1\mu, & \text{if } (m_1, 0) = (n_1 - 1, 0), \\ \gamma_2\mu, & \text{if } (0, m_2) = (0, n_2 - 1), \\ \mu, & \text{if } (m_1, m_2) = (n_1 - 1, n_2 - 1) \end{cases} \quad (1)$$

and zero otherwise, and $q(s_1, s_1) = -\sum_{s_2 \neq s_1} q(s_1, s_2)$.

Our interest is in the expected energy consumption per unit time in steady state, denoted by $\mathcal{E}$. We assume that the cost of providing service is μ per unit time. Therefore, we have $\mathcal{E} = \mathbb{E}[\mathbf{e}(N_1, N_2)]$, where

$$\mathbf{e}(n_1, n_2) = \gamma_1\mu\mathbf{1}_{\{n_1>0, n_2=0\}} + \gamma_2\mu\mathbf{1}_{\{n_1=0, n_2>0\}} + \mu\mathbf{1}_{\{n_1>0, n_2>0\}}. \quad (2)$$

Note, that for the uncoded system we have

$$\mathcal{E}_{\text{uncoded}} = \lambda_1 + \lambda_2. \quad (3)$$

III. Queueing Networks with Negative and Positive Customers

An appropriate model for the relay node in an uncoded system is that of having two queues with a single server. The available service rate of the server needs to be shared between the two queues. A useful interpretation of a coded system is the following. If the server is completing service for a packet from one of the queues and there is a packet in the other queue, that packet is also removed from the queue. This type of queueing network was first considered by Gelenbe [6]. The networks considered by Gelenbe in [6] are very similar to Jackson networks, with the additional feature that there are two types of customers, positive and negative, in the systems. Positive customers, upon arriving at a node, require service and are placed in the queue. Negative customers arriving at a node do not require service and remove a positive customer from the queue. There are three possible actions for a positive customer completing service: 1) it leaves the system, 2) it enters another queue in the system as a positive customer, or 3) it enters another queue in the system as a negative customer. It is shown in [6], that these networks have a product form stationary distribution. Applying this result to the network for which the transition diagram is depicted in Figure 3, *i.e.*, the system with two queues, external arrival of positive customers with rates λ_1, λ_2, no external arrivals of negative customers, service rates μ_1, μ_2 and customers leaving one queue entering the other queue as negative customers, gives the following.

Theorem 1 (Gelenbe [6])**.** *Consider the system with the transition diagram depicted in Figure 3. Let q_1 and q_2 be the solution of*

$$q_1 = \frac{\lambda_1}{\mu_1 + q_2\mu_2}, \quad q_2 = \frac{\lambda_2}{\mu_2 + q_1\mu_1}.$$

If $q_1 < 1$ and $q_2 < 1$, the stationary distribution $\pi(n_1, n_2)$

is given by

$$\pi(n_1,n_2) = (1-q_1)\, q_1^{n_1}\, (1-q_2)\, q_2^{n_2}.$$

Note, that if $\{S,Q\}$ is such that $\gamma_1+\gamma_2 = 1$, it is a network with negative and positive customers and its product form stationary distribution is given by Theorem 1.

The original work of Gelenbe was based on applications in neural networks. There has been follow-up work showing that there are also applications in, *e.g.*, distributed computing and database systems. Moreover, generalizations are possible, see, for instance, [7, 8]. These generalizations include networks in which a negative customer removes positive customers from multiple queues simultaneously.

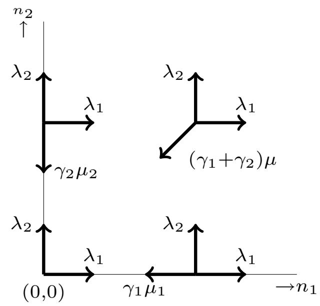

Figure 4: Transition diagram of $\{S,\bar{Q}\}$.

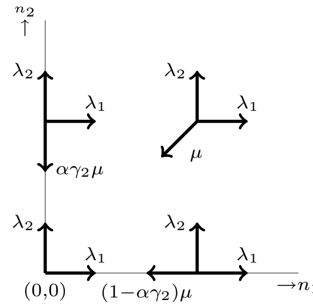

Figure 5: Transition diagram of $\{S,\hat{Q}_\alpha\}$, where $0 \le \alpha \le 1/\gamma_2$ is a parameter to be defined.

IV. Performance Bounds

The stationary distribution of the system $\{S,Q\}$ can, in special cases, be derived using Theorem 1. In general, however, no analytical form for the stationary distribution is known. In this section we provide bounds on the energy consumption of $\{S,Q\}$ by comparing it to systems for which we do have expressions for the stationary distribution. By making changes to $\{S,Q\}$ in a controlled way, we obtain a system that has a product distribution and provides a bound on system performance.

Upper and lower bounds will be given by the systems $\{S,\bar{Q}\}$ and $\{S,\hat{Q}_\alpha\}$ for which the transition diagrams are depicted in Figures 4 and 5 respectively. The transition rate matrix $\bar{Q} = [\bar{q}(s_1,s_2)]$ is given by

$$\bar{q}((n_1,n_2),(m_1,m_2)) = \begin{cases} (\gamma_1+\gamma_2)\mu, & \text{if } (m_1,m_2) = (n_1-1,n_2-1), \\ q((n_1,n_2),(m_1,m_2)), & \text{otherwise.} \end{cases}$$

The second system $\{S,\bar{Q}_\alpha\}$ is indexed by a parameter α and has transition rate matrix $\hat{Q}_\alpha = [\hat{q}_\alpha(s_1,s_2)]$, given by

$$\hat{q}_\alpha((n_1,n_2),(m_1,m_2)) = \begin{cases} \alpha\gamma_2\mu, & \text{if } (0,m_2) = (0,n_2-1), \\ (1-\alpha\gamma_2)\mu, & \text{if } (m_1,0) = (n_1-1,0), \\ q((n_1,n_2),(m_1,m_2)), & \text{otherwise.} \end{cases}$$

Let $\bar{\mathbb{E}}\left[\mathbf{e}(N_1,N_2)\right]$ and $\hat{\mathbb{E}}_\alpha\left[f(N_1,N_2)\right]$ denote the expected value of $\mathbf{e}(N_1,N_2)$ under the stationary distribution of $\{S,\bar{Q}\}$ and $\{S,\hat{Q}_\alpha\}$ respectively.

Intuitively, if $\gamma_1+\gamma_2<1$, $\bar{\mathbb{E}}\left[\mathbf{e}(N_1,N_2)\right]$ will be greater than $\mathcal{E}$. If, in addition, $1 \le \alpha \le (1-\gamma_1)/\gamma_2$, $\bar{\mathbb{E}}\left[\mathbf{e}(N_1,N_2)\right]$ will be smaller than $\mathcal{E}$. This is made rigorous in the following theorem. The proof is presented in Section V.

Theorem 2. *If* $\gamma_1+\gamma_2 \le 1$ *and* $1 \le \alpha \le (1-\gamma_1)/\gamma_2$,

$$\hat{\mathbb{E}}_\alpha\left[\mathbf{e}(N_1,N_2)\right] \le \mathcal{E} \le \bar{\mathbb{E}}\left[\mathbf{e}(N_1,N_2)\right].$$

If $\gamma_1+\gamma_2 \ge 1$ *and* $(1-\gamma_1)/\gamma_2 \le \alpha \le 1$,

$$\bar{\mathbb{E}}\left[\mathbf{e}(N_1,N_2)\right] \le \mathcal{E} \le \hat{\mathbb{E}}_\alpha\left[\mathbf{e}(N_1,N_2)\right].$$

Both $\{S,\bar{Q}\}$ and $\{S,\hat{Q}_\alpha\}$ correspond to queueing networks with negative and positive customers, similar to the system depicted in Figure 3. Hence, they have a product form stationary distribution that can be computed using Theorem 1. This leads to the following corollaries to Theorem 1.

Corollary 1. *Let* $\bar{q}_1$ *and* $\bar{q}_2$ *be the solution of*

$$\bar{q}_1 = \frac{\lambda_1}{\mu\,(\gamma_1+\bar{q}_2\gamma_2)}, \quad \bar{q}_2 = \frac{\lambda_2}{\mu\,(\gamma_2+\bar{q}_1\gamma_1)}. \tag{4}$$

Then

$$\bar{\mathbb{E}}\left[\mathbf{e}(N_1,N_2)\right] = \gamma_1\mu(1-\bar{q}_2)\bar{q}_1 + \gamma_2\mu(1-\bar{q}_1)\bar{q}_2 + \mu\bar{q}_1\bar{q}_2.$$

In order to get useful bounds from the chain $\{S,\hat{Q}_\alpha\}$ we need to carefully choose α. We provide example bounds for two specific values of α.

Corollary 2. *Let* $\hat{\alpha} = \frac{1-\gamma_1}{\gamma_2}$ *and* $\hat{q}_1$ *and* $\hat{q}_2$ *the solution of*

$$\hat{q}_1 = \frac{\lambda_1}{\mu\,(\gamma_1+(1-\gamma_1)\hat{q}_2)}, \quad \hat{q}_2 = \frac{\lambda_2}{\mu\,(1-\gamma_1+\gamma_1\hat{q}_1)}. \tag{5}$$

Then

$$\hat{\mathbb{E}}_{\hat{\alpha}}\left[\mathbf{e}(N_1,N_2)\right] = \gamma_1\mu(1-\hat{q}_2)\hat{q}_1 + \gamma_2\mu(1-\hat{q}_1)\hat{q}_2 + \mu\hat{q}_1\hat{q}_2,$$

Corollary 3. *Let* $\tilde{\alpha} = \frac{1}{2\gamma_2}$ *and* $\tilde{q}_1$ *and* $\tilde{q}_2$ *the solution of*

$$\tilde{q}_1 = \frac{2\lambda_1}{\mu\,(1+\tilde{q}_2)}, \quad \tilde{q}_2 = \frac{2\lambda_2}{\mu\,(1+\tilde{q}_1)}. \tag{6}$$

Then

$$\hat{\mathbb{E}}_{\tilde{\alpha}}\left[\mathbf{e}(N_1,N_2)\right] = \gamma_1\mu(1-\tilde{q}_2)\tilde{q}_1 + \gamma_2\mu(1-\tilde{q}_1)\tilde{q}_2 + \mu\tilde{q}_1\tilde{q}_2.$$

Note, that the results do not claim optimality of the choice of $\hat{\alpha}$ and $\tilde{\alpha}$. These choices, however, provide sufficiently strong bounds to allow some interesting observations.

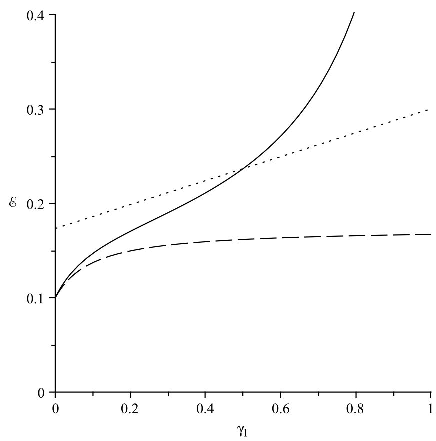

Figure 6: Bounds on energy consumption for $\mu = 1$, $\lambda_1 = 0.09$, $\lambda_2 = 0.1$, $\gamma_2 = 1$. The dashed line denotes $\bar{\mathbb{E}}\left[\mathbf{e}(N_1, N_2)\right]$ and forms a lower bound, upper bounds are given by the solid line, denoting $\hat{\mathbb{E}}_{\hat{\alpha}}\left[\mathbf{e}(N_1, N_2)\right]$, and the dotted line denoting $\hat{\mathbb{E}}_{\tilde{\alpha}}\left[\mathbf{e}(N_1, N_2)\right]$.

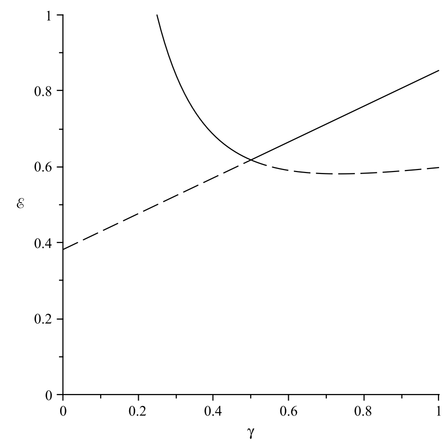

Figure 7: Bounds on energy consumption for $\mu = 1$, $\lambda_1 = \lambda_2 = 0.5$, $\gamma_1 = \gamma_2 = \gamma$. Bounds are obtained from $\{S, \bar{Q}\}$ and $\{S, \hat{Q}_{\tilde{\alpha}}\}$.

We provide some numerical examples. First consider the system with $\mu = 1$, $\lambda_1 = 0.09$, $\lambda_2 = 0.1$, $\gamma_2 = 1$. We will analyze the energy consumption of the system as a function of γ_1. Figure 6 presents the bounds from Corollaries 1–3 on $\mathcal{E}$ as a function of γ_1. Since $\gamma_1 + \gamma_2 \geq 1$, $\{S, \bar{Q}\}$ always gives a lower bound and $\{S, \hat{Q}_\alpha\}$ always gives an upper bound. The dashed line denotes $\bar{\mathbb{E}}\left[\mathbf{e}(N_1, N_2)\right]$, the solid one $\hat{\mathbb{E}}_{\hat{\alpha}}\left[\mathbf{e}(N_1, N_2)\right]$ and the dotted one $\hat{\mathbb{E}}_{\tilde{\alpha}}$.

From the figure we can conclude that making γ_1 sufficiently small is reducing the energy consumption compared to the $\gamma_1 = 1$ system. As mentioned in Section II, the energy consumption in an uncoded system is equal to $\lambda_1 + \lambda_2 = 0.19$. At $\gamma_1 = 1$ the energy consumption of the coded system is lower bounded by approximately 0.16. Therefore, the full potential of network coding, *i.e.*, reducing energy consumption by a factor 2, is not exploited in the system with $\gamma_1 = 1$. Also, observe that in the limit of γ_1 approaching 0, the maximum possible benefit is achieved.

Finally, note that our techniques do not provide useful bounds for all system parameters. Some bounds for $\mu = 1$, $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\gamma_1 = \gamma_2 = \gamma$ are presented in Figure 7. In this case $\gamma_1 + \gamma_2 = 1$ for $\gamma = 0.5$. Therefore, at $\gamma = 0.5$ the system is a queueing network with negative and positive customers and has a product form stationary distribution. As a consequence, at $\gamma = 0.5$ our bounds are tight. For the remaining values of γ, bounds obtained from $\{S, \bar{Q}\}$ and $\{S, \hat{Q}_\alpha\}$, are presented. It can be observed that not much can be said about the energy consumption of the system.

V. Proof of Theorem 2

We will use the Markov reward approach to prove Theorem 2. An accessible introduction to this technique applied to queueing networks is provided in [9].

We analyze the discrete-time Markov chains obtained by uniformization of $\{S, Q\}$, $\{S, \bar{Q}\}$ and $\{S, \hat{Q}_\alpha\}$ with the same time-interval h. In order for all chains to be uniformizable under h, let

$$h^{-1} \geq \lambda_1 + \lambda_2 + \max\{1, \gamma_1 + \gamma_2\}\mu. \tag{7}$$

. Let P be the resulting one step transition matrix after uniformization of $\{S, Q\}$, *i.e.*,

$$P = I + hQ. \tag{8}$$

Moreover, let $V^k(i)$ denote the expected cumulative energy consumption for the uniformized DTMC $\{S, P\}$ over k steps, each of length h, with one-step reward $h\mathbf{e}(j)$ per step whenever the system is in state j and when starting at state i at time 0, *i.e.*,

$$V^k(i) = \begin{cases} 0, & \text{if } k = 0, \\ h\mathbf{e}(i) + \sum_{j \in S} p(i,j)^k V^{k-1}(j), & \text{if } k > 0. \end{cases} \tag{9}$$

Now, since $\{S, Q\}$ is irreducible and ergodic,

$$\mathcal{E} = \lim_{k \to \infty} \frac{1}{hk} V^k(i), \tag{10}$$

for any $i \in S$. Similarly, since $\{S, \bar{Q}\}$ and $\{S, \hat{Q}_\alpha\}$ are also irreducible and ergodic, their performance can also be evaluated using the corresponding uniformized DTMCs. Now, we can use the following result from [9], which due to space constraints, we will not prove.

Theorem 3 (Van Dijk and Puterman [9]). *If for all $i \in S$ and $k \geq 0$*

$$\sum_{j \in S} \left(\bar{q}(i,j) - q(i,j)\right)\left[V^k(j) - V^k(i)\right] \geq 0, \tag{11}$$

then $\bar{\mathbb{E}}\left[\mathbf{e}(N_1, N_2)\right] \geq \mathcal{E}$.

The theorem also holds with reversed signs and with $\{S, \hat{Q}_\alpha\}$ instead of $\{S, \bar{Q}\}$.

For the proof of Theorem 2 we first show that

$$V^k(j) - V^k(i) \geq 0$$

for all $V^k(j) - V^k(i)$ appearing in (11). We use induction over k. For $k = 0$ we have the following.

Lemma 1. *For any $n_1 \geq 0$ and $n_2 \geq 0$*

$$0 \leq V^0(n_1 + 1, n_2) - V^0(n_1, n_2) \leq 1, \qquad (12)$$

$$0 \leq V^0(n_1, n_2 + 1) - V^0(n_1, n_2) \leq 1 \qquad (13)$$

Proof. By (9), $V^0(n_1, n_2)$ equals zero for any n_1 and n_2. □

The induction step is given in the following lemma.

Lemma 2. *Let $k \geq 0$. Suppose that for all $n_1' \geq 0$ and $n_2' \geq 0$,*

$$0 \leq V^k(n_1' + 1, n_2') - V^k(n_1', n_2') \leq 1, \qquad (14)$$

$$0 \leq V^k(n_1', n_2' + 1) - V^k(n_1', n_2') \leq 1. \qquad (15)$$

Then, for all $n_1 \geq 0$ and $n_2 \geq 0$

$$0 \leq V^{k+1}(n_1 + 1, n_2) - V^{k+1}(n_1, n_2) \leq 1, \qquad (16)$$

$$0 \leq V^{k+1}(n_1, n_2 + 1) - V^{k+1}(n_1, n_2) \leq 1 \qquad (17)$$

Proof. We prove (17). First, assume that $n_1 > 0$ and $n_2 > 0$. We have

$$\begin{aligned}
&V^{k+1}(n_1, n_2 + 1) - V^{k+1}(n_1, n_2) \\
&\quad = h\mathbf{e}(n_1, n_2 + 1) - h\mathbf{e}(n_1, n_2) + \\
&\qquad + h\lambda_1 V^k(n_1 + 1, n_2 + 1) - h\lambda_1 V^k(n_1 + 1, n_2) \\
&\qquad + h\lambda_2 V^k(n_1, n_2 + 1) - h\lambda_2 V^k(n_1, n_2 + 1) \\
&\qquad + h\mu V^k(n_1 - 1, n_2) - h\mu V^k(n_1 - 1, n_2 - 1) \\
&\qquad + (1 - h\lambda_1 - h\lambda_2 - h\mu) V^k(n_1, n_2 + 1) \\
&\qquad - (1 - h\lambda_1 - h\lambda_2 - h\mu) V^k(n_1, n_2).
\end{aligned}$$

Note, that $h\mathbf{e}(n_1, n_2 + 1) = h\mathbf{e}(n_1, n_2)$. Now, (17) follows directly from the assumptions and the fact that from (7) we have $1 - h\lambda_1 - h\lambda_2 - h\mu \geq 0$. The proof for $n_1 = 0$ and $n_2 > 0$ follows in similar fashion.

If $n_1 > 0$ and $n_2 = 0$ we have

$$\begin{aligned}
&V^{k+1}(n_1, 1) - V^{k+1}(n_1, 0) \\
&\quad = h\mathbf{e}(n_1, 1) - h\mathbf{e}(n_1, 0) + \\
&\qquad + h\lambda_1 V^k(n_1 + 1, 1) - h\lambda_1 V^k(n_1 + 1, 0) \\
&\qquad + h\lambda_2 V^k(n_1, 2) - h\lambda_2 V^k(n_1, 1) \\
&\qquad + h\mu V^k(n_1 - 1, 0) - h\gamma_1 \mu V^k(n_1 - 1, 0) \\
&\qquad + (1 - h\lambda_1 - h\lambda_2 - h\mu) V^k(n_1, 1) \\
&\qquad - (1 - h\lambda_1 - h\lambda_2 - h\gamma_1 \mu) V^k(n_1, 0),
\end{aligned}$$

Note, that $h\mathbf{e}(n_1, 1) - h\mathbf{e}(n_1, 0) = h\mu(1 - \gamma_1)$. The lower bound follows by

$$\begin{aligned}
&V^{k+1}(n_1, 1) - V^{k+1}(n_1, 0) \\
&\quad \geq h\mu(1 - \gamma_1) \\
&\qquad + h\mu V^k(n_1 - 1, 0) - h\gamma_1 \mu V^k(n_1 - 1, 0) \\
&\qquad + (1 - h\lambda_1 - h\lambda_2 - h\mu) V^k(n_1, 0) \\
&\qquad - (1 - h\lambda_1 - h\lambda_2 - h\gamma_1 \mu) V^k(n_1, 0) \\
&\quad = h\mu(1 - \gamma_1) + h\mu(1 - \gamma_1) V^k(n_1 - 1, 0) \\
&\qquad - h\mu(1 - \gamma_1) V^k(n_1, 0) \\
&\quad = h\mu(1 - \gamma_1)\left[1 - \left(V^k(n_1, 0) - V^k(n_1 - 1, 0)\right)\right] \\
&\quad \geq 0.
\end{aligned}$$

The upper bound follows from

$$\begin{aligned}
&V^{k+1}(n_1, 1) - V^{k+1}(n_1, 0) \\
&\quad \leq h\mu(1 - \gamma_1) + h\lambda_1 + h\lambda_2 \\
&\qquad + h\mu V^k(n_1 - 1, 0) - h\gamma_1 \mu V^k(n_1 - 1, 0) \\
&\qquad + h\gamma_1 \mu V^k(n_1, 0) - h\mu V^k(n_1, 0) \\
&\qquad + (1 - h\lambda_1 - h\lambda_2 - h\mu) V^k(n_1, 1) \\
&\qquad - (1 - h\lambda_1 - h\lambda_2 - h\mu) V^k(n_1, 0) \\
&\quad \leq 1 - h\gamma_1 \mu \\
&\qquad + h\mu V^k(n_1 - 1, 0) - h\gamma_1 \mu V^k(n_1 - 1, 0) \\
&\qquad - h\mu V^k(n_1, 0) + h\mu\gamma_1 V^k(n_1, 0) \\
&\quad = 1 - h\gamma_1 \mu \\
&\qquad + h\mu(1 - \gamma_1)\left[V^k(n_1 - 1, 0) - V^k(n_1, 0)\right] \\
&\quad \leq 1.
\end{aligned}$$

If $n_1 = 0$ and $n_2 = 0$,

$$\begin{aligned}
&V^{k+1}(0, 1) - V^{k+1}(0, 0) \\
&\quad = h\mathbf{e}(1, 0) - h\mathbf{e}(0, 0) + \\
&\qquad + h\lambda_1 V^k(1, 1) - h\lambda_1 V^k(1, 0) \\
&\qquad + h\lambda_2 V^k(0, 2) - h\lambda_2 V^k(0, 1) \\
&\qquad + h\gamma_2 \mu V^k(0, 0) \\
&\qquad + (1 - h\lambda_1 - h\lambda_2 - h\gamma_2 \mu) V^k(0, 1) \\
&\qquad - (1 - h\lambda_1 - h\lambda_2) V^k(0, 0).
\end{aligned}$$

Hence the lower bound is given by

$$\begin{aligned}
&V^{k+1}(0, 1) - V^{k+1}(0, 0) \\
&\quad \geq h\gamma_2 \mu + \\
&\qquad + h\gamma_2 \mu V^k(0, 0) \\
&\qquad + (1 - h\lambda_1 - h\lambda_2 - h\gamma_2 \mu) V^k(0, 0) \\
&\qquad - (1 - h\lambda_1 - h\lambda_2) V^k(0, 0) \\
&\quad = h\gamma_2 \mu + \\
&\qquad + h\gamma_2 \mu V^k(0, 0) - h\gamma_2 \mu V^k(0, 0) \\
&\qquad + (1 - h\lambda_1 - h\lambda_2) V^k(0, 0)
\end{aligned}$$

$$
\begin{aligned}
&\quad - (1 - h\lambda_1 - h\lambda_2) V^k(0,0) \\
&\geq 0
\end{aligned}
$$

and the lower bound by

$$
\begin{aligned}
&V^{k+1}(0,1) - V^{k+1}(0,0) \\
&\quad \leq h\gamma_2\mu + h\lambda_1 + h\lambda_2 \\
&\qquad + h\gamma_2\mu V^k(0,0) - h\gamma_2\mu V^k(0,1) \\
&\qquad + (1 - h\lambda_1 - h\lambda_2 - h\gamma_2\mu) V^k(0,1) \\
&\qquad - (1 - h\lambda_1 - h\lambda_2 - h\gamma_2\mu) V^k(0,0) \\
&\quad \leq 1 + h\gamma_2\mu \left[V^k(0,0) - V^k(0,1)\right] \\
&\quad \leq 1.
\end{aligned}
$$

This concludes the proof of (17). The proof of (16) follows from symmetry of the system and the reward function $\mathbf{e}$. □

Proof of Theorem 2. First consider $\{S, \bar{Q}\}$. We have

$$
\bar{q}((n_1,n_2),(m_1,m_2)) - q((n_1,n_2),(m_1,m_2)) = \begin{cases} (\gamma_1 + \gamma_2 - 1)\mu, & \text{if } (m_1,m_2) = (n_1 - 1, n_2 - 1), \\ 0, & \text{otherwise.} \end{cases} \tag{18}
$$

Therefore, (11) will be either zero or of the form

$$
(\gamma_1 + \gamma_2 - 1)\mu \left[V^k(n_1 - 1, n_2 - 1) - V^k(n_1,n_2)\right], \tag{19}
$$

where by Lemmas 1 and 2, $V^k(n_1 - 1, n_2 - 1) - V^k(n_1,n_2) \leq 0$. Now, if $\gamma_1 + \gamma_2 \leq 1$, (19) is non-negative and $\mathbb{E} \leq \bar{\mathbb{E}}\left[\mathbf{e}(N_1,N_2)\right]$ by Theorem 3. Similarly, for $\gamma_1 + \gamma_2 \geq 1$, $\mathbb{E} \geq \bar{\mathbb{E}}\left[\mathbf{e}(N_1,N_2)\right]$.

For $\{S, \hat{Q}_\alpha\}$ we have

$$
\hat{q}_\alpha((n_1,n_2),(m_1,m_2)) - q((n_1,n_2),(m_1,m_2)) = \begin{cases} (1 - \alpha\gamma_2 - \gamma_1)\mu, & \text{if } (m_1,0) = (n_1 - 1, 0), \\ (\alpha\gamma_2 - \gamma_2)\mu, & \text{if } (0,m_2) = (0, n_2 - 1), \\ 0, & \text{otherwise.} \end{cases} \tag{20}
$$

Therefore, (11) will be either zero or of one of the forms

$$
(1 - \alpha\gamma_2 - \gamma_1)\mu \left[V^k(n_1 - 1, 0) - V^k(n_1,0)\right], \tag{21}
$$

$$
(\alpha\gamma_2 - \gamma_2)\mu \left[V^k(0, n_2 - 1) - V^k(0,n_2)\right]. \tag{22}
$$

For $\gamma_1 + \gamma_2 \leq 1$ and $1 \leq \alpha \leq (1-\gamma_1)/\gamma_2$, (21) and (22) are non-positive and, therefore, $\mathbb{E} \geq \hat{\mathbb{E}}_\alpha\left[\mathbf{e}(N_1,N_2)\right]$. Also, for $\gamma_1 + \gamma_2 \geq 1$ and $(1-\gamma_1)/\gamma_2 \leq \alpha \leq 1$, $\mathbb{E} \leq \hat{\mathbb{E}}_\alpha\left[\mathbf{e}(N_1,N_2)\right]$. □

VI. Discussion

We have demonstrated the close relationship between network coding and queuing networks with negative and positive customers. This allowed us to obtain explicit expressions for the stationary distributions for specific system parameters. In addition, for the cases that no analytical result for the stationary distribution is known, we have used to the relation with queuing networks with negative and positive customers to obtain bounds on the system performance.

In future work we will generalize our results to larger networks. Moreover, we will be considering other performance measures besides energy consumption. One of the interesting performance measures is the expected waiting times, *i.e.*, delay.

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