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Markovian Queues and Stochastic Networks

Lecture 5

Richard J. Boucherie

Stochastic Operations Research

Time-reversed process and Kelly's Lemma – 2

Theorem (4.1.3 Kelly's lemma)

Let $\{N(t),\ t \in \mathbb{R}\}$ *be a stationary Markov chain with transition rates* $q(\mathbf{n}, \mathbf{n}')$, $\mathbf{n}, \mathbf{n}' \in S$. *If we can find a collection of numbers* $q^r(\mathbf{n}, \mathbf{n}')$, $\mathbf{n}, \mathbf{n}' \in S$, *such that*

$$\sum_{\mathbf{n}' \neq \mathbf{n}} q(\mathbf{n}, \mathbf{n}') = \sum_{\mathbf{n}' \neq \mathbf{n}} q^r(\mathbf{n}, \mathbf{n}'), \quad \mathbf{n} \in S,$$

and a distribution $\pi = (\pi(\mathbf{n}),\ \mathbf{n} \in S)$ *such that*

$$\pi(\mathbf{n}) q^r(\mathbf{n}, \mathbf{n}') = \pi(\mathbf{n}') q(\mathbf{n}', \mathbf{n}), \quad \mathbf{n}, \mathbf{n}' \in S,$$

then $q^r(\mathbf{n}, \mathbf{n}')$, $\mathbf{n}, \mathbf{n}' \in S$, *are the transition rates of the time-reversed Markov chain* $\{N(\tau - t),\ t \in \mathbb{R}\}$ *and* $\pi(\mathbf{n})$, $\mathbf{n} \in S$, *is the equilibrium distribution of both Markov chains.*

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Networks: customer types and fixed routes – 1

- Network of J queues.
- Customers of types $u = 1, \ldots, U$, arrive to a according to a Poisson process with rate $\mu_0(u)$, $u = 1, \ldots, U$.
- Customer type uniquely determines route through the network along the sequence of queues

$$r(u, 1), r(u, 2), \ldots, r(u, L(u)).$$

- Customer may visit the same queue at multiple stages.
- Queue j operates according to the $(\kappa_j, \gamma_j, \delta_j)$-protocol.
- Let $c_j(\ell) = (u_j(\ell),\ s_j(\ell))$, with $u_j(\ell)$ the type and $s_j(\ell)$ the stage of the customer in position ℓ in queue j.
- State of queue j is $\mathbf{c}_j = (c_j(1), \ldots, c_j(n_j))$.
- State of the network is $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_J)$.

Networks: customer types and fixed routes – 2

- Let $\{N(t)\}$ record state of Markov chain at state space $S = \{\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_J)\}$.
- For $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_J)$, let

  $C^{(u,s)}_{(\ell,j),(\ell',k)}\mathbf{c}$ denote state $\mathbf{c}'$ obtained from state $\mathbf{c}$ by removing customer of type u in stage s in position ℓ from queue j and adding that customer in position ℓ' to queue k.

- Transition rates (more precise in reader)

$$q(\mathbf{c}, \mathbf{c}') =$$

$$\begin{cases} \mu_0(u)\delta_k(\bar{\ell}', n_k + 1), & \text{if } \mathbf{c}' = C^{(u,0)}_{(0,0),(\ell',k)}\mathbf{c}, \\ \mu_j(u)\kappa_j(n_j)\gamma_j(\bar{\ell}, n_j)\delta_k(\bar{\ell}'_k, n_k + 1), & \text{if } \mathbf{c}' = C^{(u,s)}_{(\ell,j),(\ell',k)}\mathbf{c}, \\ \mu_j(u)\kappa_j(n_j)\gamma_j(\bar{\ell}, n_j), & \text{if } \mathbf{c}' = C^{(u,L(u))}_{(\ell,j),(0,0)}\mathbf{c}. \end{cases}$$

Networks: customer types and fixed routes – 4

Theorem (4.3.1 Network with fixed routes)
Let queue j operate according to the $(\kappa_j, \gamma_j, \delta_j)$*-protocol. Negative-exponential(1) service requirements for all customers at all queues. Let*

$$\pi_j(\mathbf{c}_j) = G_j \prod_{\ell=1}^{n} \frac{\rho_j(c_j(\ell))}{\kappa_j(\ell)}, \quad G_j = \left[\sum_{n=0}^{\infty} \prod_{\ell=1}^{n} \frac{\rho_j}{\kappa_j(\ell)}\right]^{-1} < \infty,$$

Then

$$\pi(\mathbf{c}) = \prod_{j=1}^{J} \pi_j(\mathbf{c}_j), \quad \mathbf{c} \in S.$$

Networks: customer types and fixed routes – 5

Proof. Natural guess for the reversed process:

- customers of type u arrive according to a Poisson process with rate $\mu_0(u)$ to queue $L(u)$
- and follow the reversed route $r(u, L(u)), \ldots, r(u, 1)$,
- and that the transition rates have the role of γ and δ reversed:

$$q^r(\mathbf{c}', \mathbf{c}) =$$

$$\begin{cases} \kappa_k(n_k + 1)\delta_k(\bar{\ell}', n_k + 1), & \text{if } \mathbf{c}' = C^{(u,0)}_{(0,0),(\ell',k)}\mathbf{c}, \\ \kappa_k(n_k + 1)\delta_k(\bar{\ell}'_k, n_k + 1)\gamma_j(\bar{\ell}, n_j), & \text{if } \mathbf{c}' = C^{(u,s)}_{(\ell,j),(\ell',k)}\mathbf{c}, \\ \mu_0(u)\gamma_j(\bar{\ell}, n_j), & \text{if } \mathbf{c}' = C^{(u,L(u))}_{(\ell,j),(0,0)}\mathbf{c}. \end{cases}$$

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Markovian Queues and Stochastic Networks

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Burke's theorem and feedforward networks –1,2

Theorem (2.5.1 Burke's theorem)

Let $\{N(t)\}$ *record the number of customers in the* $M|M|1$ *queue with arrival rate* λ *and service rate* μ, $\lambda < \mu$. *Let* $\{D(t)\}$ *record the customers' departure process from the queue. In equilibrium the departure process* $\{D(t)\}$ *is a Poisson process with rate* λ, *and* $N(t)$ *is independent of* $\{D(s),\ s < t\}$.

- Tandem network of two $M|M|1$ queues
- Poisson λ arrival process to queue 1, service rates μ_i.
- Provided $\rho_i = \lambda/\mu_i < 1$, $\pi_i(n_i) = (1 - \rho_i)\rho_i^{n_i}$, $n_i \in \mathbb{N}_0$.
- Burke's theorem: departure process from queue 1 before t^* and $N_1(t^*)$, are independent.
- Hence, in equilibrium, the at time t^* the random variables $N_1(t^*)$ and $N_2(t^*)$ are independent:
  $\pi(\mathbf{n}) = \prod_{i=1}^{2} \pi_i(n_i), \quad \mathbf{n} \in S = \mathbb{N}_0^2$.

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Markovian Queues and Stochastic Networks

Lecture 5

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Quasi-reversibility – 1

- Burke's theorem: output process from a reversible queue before t, the input process after t and the state at t independent.
- Quasi-reversibility formalises this independence property.
- $\{N(t),\ t \in \mathbb{R}\}$ Markov process, state space S, states $\mathbf{n} \in S$, transition rates $q(\mathbf{n}, \mathbf{n}')$, equilibrium distribution $\pi(\mathbf{n})$.
- Let $S(c, \mathbf{n}) \subset S$ denote the set of states that may be obtained from state $\mathbf{n}$ when a customer of class c arrives to the queue.
- Let $\{A_c(t),\ t \in \mathbb{R}\}$ and $\{D_c(t),\ t \in \mathbb{R}\}$ record the arrival and departure processes of customers of class c.

Quasi-reversibility – 2

Definition (4.4.1 Quasi-reversibility)

The stationary Markov chain $\{N(t)\}$ is quasi-reversible if for all $t \in \mathbb{R}$ the state at time t, $N(t)$, is independent of $\{A_c(s),\ s > t\}$, the arrival process of class c customers after time t, and independent of $\{D_c(s),\ s < t\}$, the departure process of class c customers prior to time t, $c = 1, \ldots, C$.

Theorem (4.4.2)

If $\{N(t)\}$ is a quasi-reversible Markov chain, then

(i) *the arrival processes $\{A_c(t),\ t \in \mathbb{R}\}$, $c = 1, \ldots, C$, form independent Poisson processes;*

(ii) *the departure processes $\{D_c(t),\ t \in \mathbb{R}\}$, $c = 1, \ldots, C$, form independent Poisson processes.*

Quasi-reversibility – 3

Algebraic characterisation of quasi-reversibility:

$$\begin{aligned}\lambda(c) &= \sum_{\mathbf{n}' \in S(c,\mathbf{n})} q(\mathbf{n}, \mathbf{n}'), \\ \lambda(c) &= \sum_{\mathbf{n}' \in S(c,\mathbf{n})} q^r(\mathbf{n}, \mathbf{n}'),\end{aligned}$$

so that

$$\sum_{\mathbf{n}' \in S(c,\mathbf{n})} \pi(\mathbf{n}) q(\mathbf{n}, \mathbf{n}') = \sum_{\mathbf{n}' \in S(c,\mathbf{n})} \pi(\mathbf{n}') q(\mathbf{n}', \mathbf{n}).$$

- In equilibrium the flow out of state **n** due to a customer of type *c* arriving to the queue balances with the probability flow into state **n** due to a customer of type *c* departing from the queue.

Symmetric queue – 1

Definition (4.2.6 Symmetric queue)

A queue that operates under the (κ, γ, δ)-protocol is called symmetric if

$$\gamma(\ell, n) = \delta(\ell, n), \quad \ell = 1, \ldots, n, \; n \in \mathbb{N}.$$

Theorem (4.4.6)

Let $\{N(t)\}$ record the state of a symmetric queue to which customers of class c arrive according to independent Poisson processes with rate $\lambda(c)$, $c = 1, \ldots, C$. Then $\{N(t)\}$ is quasi-reversible.

Symmetric queue – 2

Proof.

- Transition rates, for $\mathbf{c} = (c(1), \ldots, c(n))$, $\mathbf{c}' \neq \mathbf{c}$,

$$q(\mathbf{c}, \mathbf{c}') = \begin{cases} \lambda(c)\gamma(\ell, n+1), & \text{if } \mathbf{c}' = (c(1), \ldots, c(\ell), c, c(\ell+1), \ldots, c(n)), \\ \mu_{c(\ell)}\kappa(n)\gamma(\ell, n), & \text{if } \mathbf{c}' = c(1), \ldots, c(\ell-1), c(\ell+1), \ldots, c(n)). \end{cases}$$

- Arrivals of class c customers independent Poisson processes $\Rightarrow N(t)$ independent of $\{A_c(s),\ s > t\}$.
- Transition rates of time-reversed queue: $q^r = q$.
- Arrival process to the time-reversed queue is Poisson process.
- Arrivals in the time-reversed process coincide with departures of $\{N(t)\} \Rightarrow N(t)$ is independent of $\{D_c(s),\ s < t\}$.

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Quasi-reversible queues and fixed routes – 1

- Network of J quasi-reversible queues.
- Customers of types $u = 1, \dots, U$, arrive to a according to a Poisson process with rate $\mu_0(u)$, $u = 1, \dots, U$.
- Customer type uniquely determines route along the sequence of queues $r(u,1), r(u,2), \dots, r(u, L(u))$.
- State of queue j: $\{N_j(t)\}$, state space S_j, transition rates $q_j(\mathbf{c}_j, \mathbf{c}'_j)$, customers of class (u,s) arrive according to Poisson process with rate

$$\lambda_j(u,s) \sum_{\mathbf{c}'_j \in S_j((u,s),\mathbf{c}_j)} q_j(\mathbf{c}_j, \mathbf{c}'_j),$$

- Equilibrium distribution $\pi_j = (\pi_j(\mathbf{c}_j),\ \mathbf{c}_j \in S_j)$ satisfies

$$\sum_{\mathbf{c}'_j \in S_j(c,\mathbf{c}_j)} \pi_j(\mathbf{c}_j) q_j(\mathbf{c}_j, \mathbf{c}'_j) = \sum_{\mathbf{c}'_j \in S_j(c,\mathbf{c}_j)} \pi_j(\mathbf{c}'_j) q_j(\mathbf{c}'_j, \mathbf{c}_j).$$

Quasi-reversible queues and fixed routes – 2

- For $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_J)$, and $j, k = 0, \ldots, J$, let

  $C_{j,k}^{(u,s)}\mathbf{c}$ denote the set of states $\mathbf{c}'$ obtained from state $\mathbf{c}$ by removing the customer of type u in stage s from queue j and adding that customer in stage $s+1$ to queue k:

$$(C_{j,k}^{(u,s)}\mathbf{c})_i = \begin{cases} \{\mathbf{c}_i\}, & \text{if } i \neq j, k, \\ S_k((u, s+1), \mathbf{c}_k), & \text{if } i = k, \\ \{\mathbf{c}'_j \text{ s.t. } \mathbf{c}_j \in S_j((u,s), \mathbf{c}'_j)\}, & \text{if } i = j, \end{cases}$$

Quasi-reversible queues and fixed routes – 3

- Transition rates, for $u = 1, \dots, U$, $\mathbf{c} \neq \mathbf{c}'$, $\mathbf{c}, \mathbf{c}' \in S$,

$q(\mathbf{c}, \mathbf{c}') =$

$$\begin{cases} q_k(\mathbf{c}_k, \mathbf{c}_k'), & \text{if } \mathbf{c}' \in C_{0,k}^{(u,1)}\mathbf{c}, & \text{(arrival)} \\ q_j(\mathbf{c}_j, \mathbf{c}_j') \dfrac{q_k(\mathbf{c}_k, \mathbf{c}_k')}{\sum_{\mathbf{c}_k' \in S_k((u,s+1),\mathbf{c}_k)} q_k(\mathbf{c}_k, \mathbf{c}_k')}, & \text{if } \mathbf{c}' \in C_{j,k}^{(u,s)}\mathbf{c}, & \text{(routing)} \\ q_j(\mathbf{c}_j, \mathbf{c}_j'), & \text{if } \mathbf{c}' \in C_{j,0}^{(u,L(u))}\mathbf{c}, & \text{(departure)} \\ q_j(\mathbf{c}_j, \mathbf{c}_j'), & \text{if } \mathbf{c}_j, \mathbf{c}_j' \in S_j,\ \mathbf{c}_i' = \mathbf{c}_i,\ i \neq j, & \text{(internal)} \end{cases}$$

- Quasi-reversibility implies that

$$\frac{q_k(\mathbf{c}_k, \mathbf{c}_k')}{\sum_{\mathbf{c}_k' \in S_k((u,s+1),\mathbf{c}_k)} q_k(\mathbf{c}_k, \mathbf{c}_k')} = \frac{q_k(\mathbf{c}_k, \mathbf{c}_k')}{\lambda_k(u, s+1)}.$$

Quasi-reversible queues and fixed routes – 3

Theorem (4.5.1)

Let $\{N(t)\} = \{(N_1(t), \ldots, N_J(t))\}$ *record the state of a network of J quasi-reversible queues to which customers of types* $u = 1, \ldots, U$ *arrive according to independent Poisson processes with rates* $\mu_0(u)$ *to follow a fixed route* $r(u,1), r(u,2), \ldots, r(u, L(u))$*,* $u = 1, \ldots, U$*. Let* S_j*,* q_j*, and* π_j *denote the state space, transition rates and equilibrium distribution of queue j,* $j = 1, \ldots, J$*. Then* $\{N(t)\}$ *has equilibrium distribution*

$$\pi(\mathbf{c}_1, \ldots, \mathbf{c}_J) = \prod_{j=1}^{J} \pi_j(\mathbf{c}_j), \quad (\mathbf{c}_1, \ldots, \mathbf{c}_J) \in S = S_1 \times \cdots \times S_J.$$

Quasi-reversible queues and fixed routes – 4

Proof. Natural guess for time-reversed process:

- customers of types $u = 1, \ldots, U$ arrive according to a Poisson process with rate $\mu_0(u)$,
- route through the network along the sequence of queues in reversed order $r(u, L(u)), \ldots, r(u, 1)$
- each queue operates according to its time-reversed transition rates: for $u = 1, \ldots, U$, $\mathbf{c} \neq \mathbf{c}'$, $\mathbf{c}, \mathbf{c}' \in S$,

$$q^r(\mathbf{c}', \mathbf{c}) =$$

$$\begin{cases} q_k^r(\mathbf{c}_k', \mathbf{c}_k), & \text{if } \mathbf{c}' \in C_{0,k}^{(u,1)}\mathbf{c}, & \text{(departure)} \\ q_k^r(\mathbf{c}_k', \mathbf{c}_k)\dfrac{q_j^r(\mathbf{c}_j', \mathbf{c}_j)}{\lambda_j(u,s)}, & \text{if } \mathbf{c}' \in C_{j,k}^{(u,s)}\mathbf{c}, & \text{(routing)} \\ q_j^r(\mathbf{c}_j', \mathbf{c}_j), & \text{if } \mathbf{c}' \in C_{j,0}^{(u,L(u))}\mathbf{c}, & \text{(arrival)} \\ q_j^r(\mathbf{c}_j', \mathbf{c}_j), & \text{if } \mathbf{c}_j, \mathbf{c}_j' \in S_j,\ \mathbf{c}_i' = \mathbf{c}_i,\ i \neq j. & \text{(internal)} \end{cases}$$

Quasi-reversible queues and fixed routes – 5

- For a routing transition from queue $j = r(u, s)$ to queue $k = r(u, s+1)$ it must be that $\lambda_j(u, s) = \lambda_k(u, s+1)$, which implies that

$$\pi_j(\mathbf{c}_j)\pi_k(\mathbf{c}_k)q_j(\mathbf{c}_j, \mathbf{c}'_j)\frac{q_k(\mathbf{c}_k, \mathbf{c}'_k)}{\lambda_k(u, s+1)} = \pi^r(\mathbf{c}'_j)\pi^r_k(\mathbf{c}'_k)q^r_k(\mathbf{c}'_k, \mathbf{c}_k)\frac{q^r_j(\mathbf{c}'_j, \mathbf{c}_j)}{\lambda_j(u, s)}.$$

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