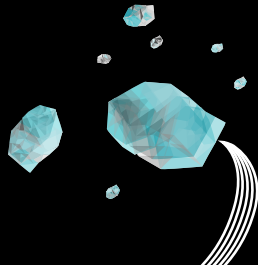
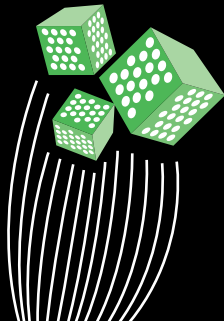


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Time-reversed process and Kelly's Lemma – 2

Theorem (4.1.3 Kelly's lemma)

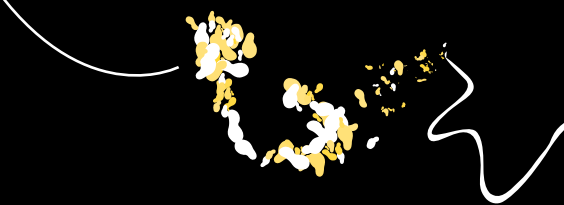
Let $\{N(t), t \in \mathbb{R}\}$ be a stationary Markov chain with transition rates $q(\mathbf{n}, \mathbf{n}')$, $\mathbf{n}, \mathbf{n}' \in S$. If we can find a collection of numbers $q^r(\mathbf{n}, \mathbf{n}')$, $\mathbf{n}, \mathbf{n}' \in S$, such that

$$\sum_{\mathbf{n}' \neq \mathbf{n}} q(\mathbf{n}, \mathbf{n}') = \sum_{\mathbf{n}' \neq \mathbf{n}} q^r(\mathbf{n}, \mathbf{n}'), \quad \mathbf{n} \in S,$$

and a distribution $\pi = (\pi(\mathbf{n}), \mathbf{n} \in S)$ such that

$$\pi(\mathbf{n})q^r(\mathbf{n}, \mathbf{n}') = \pi(\mathbf{n}')q(\mathbf{n}', \mathbf{n}), \quad \mathbf{n}, \mathbf{n}' \in S,$$

then $q^r(\mathbf{n}, \mathbf{n}')$, $\mathbf{n}, \mathbf{n}' \in S$, are the transition rates of the time-reversed Markov chain $\{N(\tau - t), t \in \mathbb{R}\}$ and $\pi(\mathbf{n}), \mathbf{n} \in S$, is the equilibrium distribution of both Markov chains.

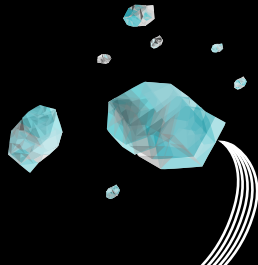
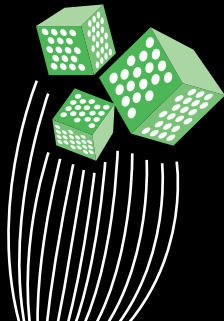


Markovian Queues and Stochastic Networks

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Networks: customer types and fixed routes – 1

- ▶ Network of J queues.
- ▶ Customers of types $u = 1, \dots, U$, arrive to a according to a Poisson process with rate $\mu_0(u)$, $u = 1, \dots, U$.
- ▶ Customer type uniquely determines route through the network along the sequence of queues

$$r(u, 1), r(u, 2), \dots, r(u, L(u)).$$

- ▶ Customer may visit the same queue at multiple stages.
- ▶ Queue j operates according to the $(\kappa_j, \gamma_j, \delta_j)$ -protocol.
- ▶ Let $\mathbf{c}_j(\ell) = (u_j(\ell), s_j(\ell))$, with $u_j(\ell)$ the type and $s_j(\ell)$ the stage of the customer in position ℓ in queue j .
- ▶ State of queue j is $\mathbf{c}_j = (\mathbf{c}_j(1), \dots, \mathbf{c}_j(n_j))$.
- ▶ State of the network is $\mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_J)$.

Networks: customer types and fixed routes – 2

- ▶ Let $\{N(t)\}$ record state of Markov chain at state space $S = \{\mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_J)\}$.

- ▶ For $\mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_J)$, let

$C_{(\ell,j),(\ell',k)}^{(u,s)} \mathbf{c}$ denote state \mathbf{c}' obtained from state \mathbf{c} by removing customer of type u in stage s in position ℓ from queue j and adding that customer in position ℓ' to queue k .

- ▶ Transition rates (more precise in reader)

$$q(\mathbf{c}, \mathbf{c}') =$$

$$\begin{cases} \mu_0(u) \delta_k(\bar{\ell}', n_k + 1), & \text{if } \mathbf{c}' = C_{(0,0),(\ell',k)}^{(u,0)} \mathbf{c}, \\ \mu_j(u) \kappa_j(n_j) \gamma_j(\bar{\ell}, n_j) \delta_k(\bar{\ell}', n_k + 1), & \text{if } \mathbf{c}' = C_{(\ell,j),(\ell',k)}^{(u,s)} \mathbf{c}, \\ \mu_j(u) \kappa_j(n_j) \gamma_j(\bar{\ell}, n_j), & \text{if } \mathbf{c}' = C_{(\ell,j),(0,0)}^{(u,L(u))} \mathbf{c}. \end{cases}$$

Networks: customer types and fixed routes – 4

Theorem (4.3.1 Network with fixed routes)

Let queue j operate according to the $(\kappa_j, \gamma_j, \delta_j)$ -protocol. Negative-exponential(1) service requirements for all customers at all queues. Let

$$\pi_j(\mathbf{c}_j) = G_j \prod_{\ell=1}^n \frac{\rho_j(\mathbf{c}_j(\ell))}{\kappa_j(\ell)}, \quad G_j = \left[\sum_{n=0}^{\infty} \prod_{\ell=1}^n \frac{\rho_j}{\kappa_j(\ell)} \right]^{-1} < \infty,$$

Then

$$\pi(\mathbf{c}) = \prod_{j=1}^J \pi_j(\mathbf{c}_j), \quad \mathbf{c} \in \mathcal{S}.$$

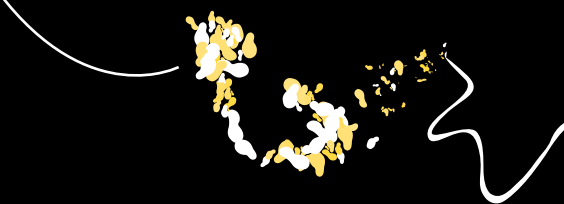
Networks: customer types and fixed routes – 5

Proof. Natural guess for the reversed process:

- ▶ customers of type u arrive according to a Poisson process with rate $\mu_0(u)$ to queue $L(u)$
- ▶ and follow the reversed route $r(u, L(u)), \dots, r(u, 1)$,
- ▶ and that the transition rates have the role of γ and δ reversed:

$$q^r(\mathbf{c}', \mathbf{c}) =$$

$$\begin{cases} \kappa_k(n_k + 1)\delta_k(\bar{\ell}', n_k + 1), & \text{if } \mathbf{c}' = C_{(0,0),(\ell',k)}^{(u,0)} \mathbf{c}, \\ \kappa_k(n_k + 1)\delta_k(\bar{\ell}'_k, n_k + 1)\gamma_j(\bar{\ell}, n_j), & \text{if } \mathbf{c}' = C_{(\ell,j),(\ell',k)}^{(u,s)} \mathbf{c}, \\ \mu_0(u)\gamma_j(\bar{\ell}, n_j), & \text{if } \mathbf{c}' = C_{(\ell,j),(0,0)}^{(u,L(u))} \mathbf{c}. \end{cases}$$

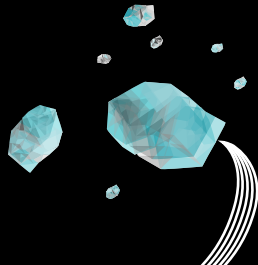
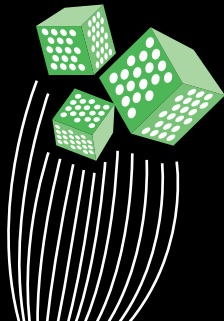


Markovian Queues and Stochastic Networks

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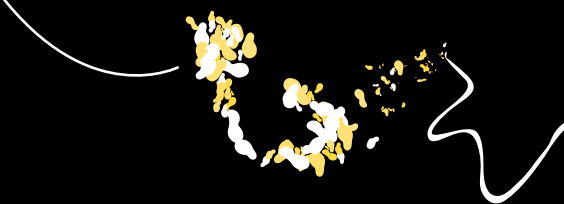


Burke's theorem and feedforward networks –1,2

Theorem (2.5.1 Burke's theorem)

Let $\{N(t)\}$ record the number of customers in the $M|M|1$ queue with arrival rate λ and service rate μ , $\lambda < \mu$. Let $\{D(t)\}$ record the customers' departure process from the queue. In equilibrium the departure process $\{D(t)\}$ is a Poisson process with rate λ , and $N(t)$ is independent of $\{D(s), s < t\}$.

- ▶ **Tandem network** of two $M|M|1$ queues
- ▶ Poisson λ arrival process to queue 1, service rates μ_i .
- ▶ Provided $\rho_i = \lambda/\mu_i < 1$, $\pi_i(n_i) = (1 - \rho_i)\rho_i^{n_i}$, $n_i \in \mathbb{N}_0$.
- ▶ Burke's theorem: departure process from queue 1 before t^* and $N_1(t^*)$, are independent.
- ▶ Hence, in equilibrium, the at time t^* the random variables $N_1(t^*)$ and $N_2(t^*)$ are independent:
$$\pi(\mathbf{n}) = \prod_{i=1}^2 \pi_i(n_i), \quad \mathbf{n} \in \mathcal{S} = \mathbb{N}_0^2.$$

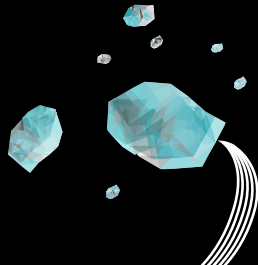
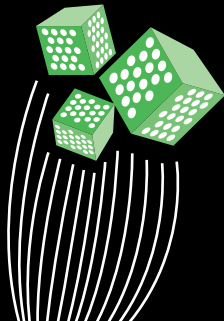


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Quasi-reversibility – 1

- ▶ Burke's theorem: output process from a reversible queue before t , the input process after t and the state at t independent.
- ▶ Quasi-reversibility formalises this independence property.
- ▶ $\{N(t), t \in \mathbb{R}\}$ Markov process, state space S , states $\mathbf{n} \in S$, transition rates $q(\mathbf{n}, \mathbf{n}')$, equilibrium distribution $\pi(\mathbf{n})$.
- ▶ Let $S(c, \mathbf{n}) \subset S$ denote the set of states that may be obtained from state \mathbf{n} when a customer of class c arrives to the queue.
- ▶ Let $\{A_c(t), t \in \mathbb{R}\}$ and $\{D_c(t), t \in \mathbb{R}\}$ record the arrival and departure processes of customers of class c .

Quasi-reversibility – 2

Definition (4.4.1 Quasi-reversibility)

The stationary Markov chain $\{N(t)\}$ is quasi-reversible if for all $t \in \mathbb{R}$ the state at time t , $N(t)$, is independent of $\{A_c(s), s > t\}$, the arrival process of class c customers after time t , and independent of $\{D_c(s), s < t\}$, the departure process of class c customers prior to time t , $c = 1, \dots, C$.

Theorem (4.4.2)

If $\{N(t)\}$ is a quasi-reversible Markov chain, then

- (i) the arrival processes $\{A_c(t), t \in \mathbb{R}\}$, $c = 1, \dots, C$, form independent Poisson processes;*
- (ii) the departure processes $\{D_c(t), t \in \mathbb{R}\}$, $c = 1, \dots, C$, form independent Poisson processes.*

Quasi-reversibility – 3

Algebraic characterisation of quasi-reversibility:

$$\lambda(c) = \sum_{\mathbf{n}' \in \mathcal{S}(c, \mathbf{n})} q(\mathbf{n}, \mathbf{n}'),$$

$$\lambda(c) = \sum_{\mathbf{n}' \in \mathcal{S}(c, \mathbf{n})} q'(\mathbf{n}, \mathbf{n}'),$$

so that

$$\sum_{\mathbf{n}' \in \mathcal{S}(c, \mathbf{n})} \pi(\mathbf{n}) q(\mathbf{n}, \mathbf{n}') = \sum_{\mathbf{n}' \in \mathcal{S}(c, \mathbf{n})} \pi(\mathbf{n}') q(\mathbf{n}', \mathbf{n}).$$

- ▶ In equilibrium the flow out of state \mathbf{n} due to a customer of type c arriving to the queue balances with the probability flow into state \mathbf{n} due to a customer of type c departing from the queue.

Symmetric queue – 1

Definition (4.2.6 Symmetric queue)

A queue that operates under the (κ, γ, δ) -protocol is called symmetric if

$$\gamma(\ell, n) = \delta(\ell, n), \quad \ell = 1, \dots, n, \quad n \in \mathbb{N}.$$

Theorem (4.4.6)

Let $\{N(t)\}$ record the state of a symmetric queue to which customers of class c arrive according to independent Poisson processes with rate $\lambda(c)$, $c = 1, \dots, C$. Then $\{N(t)\}$ is quasi-reversible.

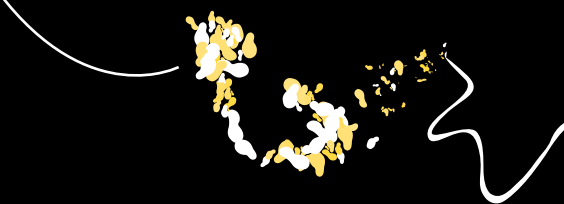
Symmetric queue – 2

Proof.

- ▶ Transition rates, for $\mathbf{c} = (c(1), \dots, c(n))$, $\mathbf{c}' \neq \mathbf{c}$,

$$q(\mathbf{c}, \mathbf{c}') = \begin{cases} \lambda(\mathbf{c})\gamma(\ell, n+1), & \text{if } \mathbf{c}' = (c(1), \dots, c(\ell), c, c(\ell+1), \dots, c(n)), \\ \mu_{c(\ell)}\kappa(n)\gamma(\ell, n), & \text{if } \mathbf{c}' = c(1), \dots, c(\ell-1), c(\ell+1), \dots, c(n). \end{cases}$$

- ▶ Arrivals of class c customers independent Poisson processes $\Rightarrow N(t)$ independent of $\{A_c(s), s > t\}$.
- ▶ Transition rates of time-reversed queue: $q^r = q$.
- ▶ Arrival process to the time-reversed queue is Poisson process.
- ▶ Arrivals in the time-reversed process coincide with departures of $\{N(t)\} \Rightarrow N(t)$ is independent of $\{D_c(s), s < t\}$.

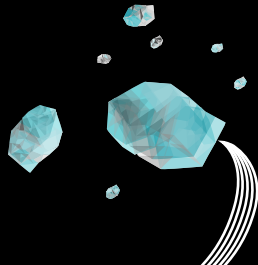
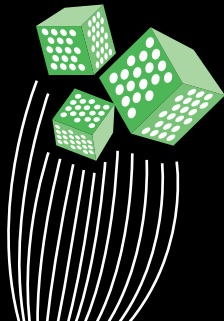


Markovian Queues and Stochastic Networks

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Quasi-reversible queues and fixed routes – 1

- ▶ Network of J quasi-reversible queues.
- ▶ Customers of types $u = 1, \dots, U$, arrive to a according to a Poisson process with rate $\mu_0(u)$, $u = 1, \dots, U$.
- ▶ Customer type uniquely determines route along the sequence of queues $r(u, 1), r(u, 2), \dots, r(u, L(u))$.
- ▶ State of queue j : $\{N_j(t)\}$, state space S_j , transition rates $q_j(\mathbf{c}_j, \mathbf{c}'_j)$, customers of class (u, s) arrive according to Poisson process with rate

$$\lambda_j(u, s) \sum_{\mathbf{c}'_j \in S_j((u,s), \mathbf{c}_j)} q_j(\mathbf{c}_j, \mathbf{c}'_j),$$

- ▶ Equilibrium distribution $\pi_j = (\pi_j(\mathbf{c}_j), \mathbf{c}_j \in S_j)$ satisfies

$$\sum_{\mathbf{c}'_j \in S_j(\mathbf{c}, \mathbf{c}_j)} \pi_j(\mathbf{c}_j) q_j(\mathbf{c}_j, \mathbf{c}'_j) = \sum_{\mathbf{c}'_j \in S_j(\mathbf{c}, \mathbf{c}_j)} \pi_j(\mathbf{c}'_j) q_j(\mathbf{c}'_j, \mathbf{c}_j).$$

Quasi-reversible queues and fixed routes – 2

- ▶ For $\mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_J)$, and $j, k = 0, \dots, J$, let

$C_{j,k}^{(u,s)} \mathbf{c}$ denote the set of states \mathbf{c}' obtained from state \mathbf{c} by removing the customer of type u in stage s from queue j and adding that customer in stage $s + 1$ to queue k :

$$(C_{j,k}^{(u,s)} \mathbf{c})_i = \begin{cases} \{\mathbf{c}_i\}, & \text{if } i \neq j, k, \\ S_k((u, s + 1), \mathbf{c}_k), & \text{if } i = k, \\ \{\mathbf{c}'_j \text{ s.t. } \mathbf{c}_j \in S_j((u, s), \mathbf{c}'_j)\}, & \text{if } i = j, \end{cases}$$

Quasi-reversible queues and fixed routes – 3

- ▶ Transition rates, for $u = 1, \dots, U$, $\mathbf{c} \neq \mathbf{c}'$, $\mathbf{c}, \mathbf{c}' \in S$,

$q(\mathbf{c}, \mathbf{c}') =$

$$\left\{ \begin{array}{ll} q_k(\mathbf{c}_k, \mathbf{c}'_k), & \text{if } \mathbf{c}' \in C_{0,k}^{(u,1)} \mathbf{c}, \quad (\text{arrival}) \\ q_j(\mathbf{c}_j, \mathbf{c}'_j) \frac{q_k(\mathbf{c}_k, \mathbf{c}'_k)}{\sum_{\mathbf{c}'_k \in S_k((u,s+1), \mathbf{c}_k)} q_k(\mathbf{c}_k, \mathbf{c}'_k)}, & \text{if } \mathbf{c}' \in C_{j,k}^{(u,s)} \mathbf{c}, \quad (\text{routing}) \\ q_j(\mathbf{c}_j, \mathbf{c}'_j), & \text{if } \mathbf{c}' \in C_{j,0}^{(u,L(u))} \mathbf{c}, \quad (\text{departure}) \\ q_j(\mathbf{c}_j, \mathbf{c}'_j), & \text{if } \mathbf{c}_j, \mathbf{c}'_j \in S_j, \mathbf{c}'_i = \mathbf{c}_i, i \neq j, \quad (\text{internal}) \end{array} \right.$$

- ▶ Quasi-reversibility implies that

$$\frac{q_k(\mathbf{c}_k, \mathbf{c}'_k)}{\sum_{\mathbf{c}'_k \in S_k((u,s+1), \mathbf{c}_k)} q_k(\mathbf{c}_k, \mathbf{c}'_k)} = \frac{q_k(\mathbf{c}_k, \mathbf{c}'_k)}{\lambda_k(u, s+1)}.$$

Quasi-reversible queues and fixed routes – 3

Theorem (4.5.1)

Let $\{N(t)\} = \{(N_1(t), \dots, N_J(t))\}$ record the state of a network of J quasi-reversible queues to which customers of types $u = 1, \dots, U$ arrive according to independent Poisson processes with rates $\mu_0(u)$ to follow a fixed route $r(u, 1), r(u, 2), \dots, r(u, L(u))$, $u = 1, \dots, U$. Let S_j , q_j , and π_j denote the state space, transition rates and equilibrium distribution of queue j , $j = 1, \dots, J$. Then $\{N(t)\}$ has equilibrium distribution

$$\pi(\mathbf{c}_1, \dots, \mathbf{c}_J) = \prod_{j=1}^J \pi_j(\mathbf{c}_j), \quad (\mathbf{c}_1, \dots, \mathbf{c}_J) \in \mathcal{S} = \mathcal{S}_1 \times \dots \times \mathcal{S}_J.$$

Quasi-reversible queues and fixed routes – 4

Proof. Natural guess for time-reversed process:

- ▶ customers of types $u = 1, \dots, U$ arrive according to a Poisson process with rate $\mu_0(u)$,
- ▶ route through the network along the sequence of queues in reversed order $r(u, L(u)), \dots, r(u, 1)$
- ▶ each queue operates according to its time-reversed transition rates: for $u = 1, \dots, U, \mathbf{c} \neq \mathbf{c}', \mathbf{c}, \mathbf{c}' \in S$,

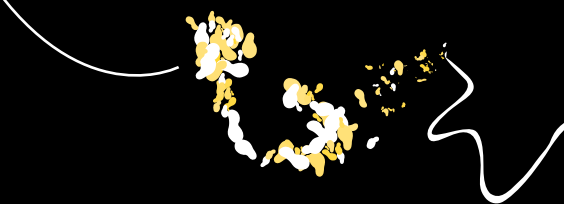
$$q^r(\mathbf{c}', \mathbf{c}) =$$

$$\left\{ \begin{array}{ll} q_k^r(\mathbf{c}'_k, \mathbf{c}_k), & \text{if } \mathbf{c}' \in C_{0,k}^{(u,1)} \mathbf{c}, \quad (\text{departure}) \\ q_k^r(\mathbf{c}'_k, \mathbf{c}_k) \frac{q_j^r(\mathbf{c}'_j, \mathbf{c}_j)}{\lambda_j(u, s)}, & \text{if } \mathbf{c}' \in C_{j,k}^{(u,s)} \mathbf{c}, \quad (\text{routing}) \\ q_j^r(\mathbf{c}'_j, \mathbf{c}_j), & \text{if } \mathbf{c}' \in C_{j,0}^{(u,L(u))} \mathbf{c}, \quad (\text{arrival}) \\ q_j^r(\mathbf{c}'_j, \mathbf{c}_j), & \text{if } \mathbf{c}_j, \mathbf{c}'_j \in S_j, \mathbf{c}'_i = \mathbf{c}_i, i \neq j. \quad (\text{internal}) \end{array} \right.$$

Quasi-reversible queues and fixed routes – 5

- ▶ For a routing transition from queue $j = r(u, s)$ to queue $k = r(u, s + 1)$ it must be that $\lambda_j(u, s) = \lambda_k(u, s + 1)$, which implies that

$$\pi_j(\mathbf{c}_j)\pi_k(\mathbf{c}_k)q_j(\mathbf{c}_j, \mathbf{c}'_j)\frac{q_k(\mathbf{c}_k, \mathbf{c}'_k)}{\lambda_k(u, s + 1)} = \pi^r(\mathbf{c}'_j)\pi_k^r(\mathbf{c}'_k)q_k^r(\mathbf{c}'_k, \mathbf{c}_k)\frac{q_j^r(\mathbf{c}'_j, \mathbf{c}_j)}{\lambda_j(u, s)}.$$



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