Markovian Queues and Stochastic Networks

Lecture 1
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Overview MQSN

- Background on Markov chains
- Reversibility, output theorem, tandem networks, feedforward networks
- Partial balance, Markovian routing, Kelly-Whittle networks
- Kelly’s lemma, time-reversed process, networks with fixed routes
- Advanced topics
Literature

- F.P. Kelly, Reversibility and stochastic networks, 1979, chapters 1–4
  
  www.statslab.cam.ac.uk/~frank/BOOKS/kelly_book.html
- Reader: R.J. Boucherie, Markovian queueing networks, 2018 (work in progress)
Internet of Things: optimal route in Jackson network

- Jobs arrive at outside nodes with given destination
- Each node single server queue minimize sojourn time
- Optimal route selection
- Inform jobs in neighbouring node
- Alternative route
Internet of Things: optimal route in Jackson network

- Tandem of $M|M|1$ queues
- Sojourn time
- Average sojourn time queue $i$:
  \[ \mathbb{E}S_i = (\mu_i - \lambda_i)^{-1} \]
- On route
  \[ \mathbb{E}S = \sum_i (\mu_i - \lambda_i)^{-1} \]
Internet of Things: optimal route in Jackson network

- For fixed routes via set of queues
- On route $r$

$$\mathbb{E}S_r = \sum_i (\mu_i - \lambda_i)^{-1} \mathbb{I}(i \text{ on } r)$$
Challenge

▶ Grid \( N \times N \)
▶ On each side \( k \) flows arrive from sources at randomly selected (but fixed) nodes with destination a randomly selected (but fixed) node on one of the 4 sides
▶ At each gridpoint a single server queue handles and forwards packets
▶ Packets select their route from source to destination to minimize their travelling time (no travelling time on link)
▶ Packets may communicate with neighbours to avoid congestions and change their route accordingly
▶ Poisson arrivals of packets; general processing time at nodes; one destination on each side
▶ Develop decentralized routing algorithm to minimize mean travelling times and demonstrate that it outperforms shortest (and fixed) route selection
Continuous-time Markov chain

- Stochastic process \( \{N(t), \ t \in T\} \) records evolution of random variable, \( T = \mathbb{R} \)
- State space \( S \subseteq \mathbb{N}_0^J \), state \( n = (n_1, \ldots, n_J) \)
- Stationary process if \( (N(t_1), N(t_2), \ldots, N(t_k)) \) has the same distribution as \( (N(t_1 + \tau), N(t_2 + \tau), \ldots, N(t_k + \tau)) \) for all \( k \in \mathbb{N}, t_1, t_2, \ldots, t_k \in T, \tau \in T \)
- Markov process satisfies the Markov property: for every \( k \geq 1, 0 \leq t_1 < \cdots < t_k < t_{k+1} \), and any \( n_1, \ldots, n_{k+1} \) in \( S \), the joint distribution of \( (N(t_1), \ldots, N(t_{k+1})) \) is such that

\[
\mathbb{P} \left\{ N(t_{k+1}) = n_{k+1} \mid N(t_1) = n_1, \ldots, N(t_k) = n_k \right\}
= \mathbb{P} \left\{ N(t_{k+1}) = n_{k+1} \mid N(t_k) = n_k \right\},
\]

whenever the conditioning event \( (N(t_1) = n_1, \ldots, N(t_k) = n_k) \) has positive probability.
A Markov process is time-homogeneous if the conditional probability \( \mathbb{P}\{N(t+s) = n' | N(t) = n\} \) is independent of \( t \) for all \( s > 0, \, n, n' \in S \).

For a time-homogeneous Markov process the transition probability from state \( n \) to state \( n' \) in time \( s \) is defined as

\[
P(n, n'; s) = \mathbb{P}\{N(t+s) = n' | N(t) = n\}, \quad s > 0.
\]
The transition matrix $P(t) = (P(n, n'; t), \ n, n' \in S)$ has non-negative entries (1) and row sums equal to one (2).

The Markov property implies that the transition probabilities satisfy the Chapman-Kolmogorov equations (3). Assume that the transition matrix is standard (4). For all $n, n' \in S, s, t \in T,$ a standard transition matrix satisfies:

\begin{align*}
& P(n, n'; t) \geq 0; \quad (1) \\
& \sum_{n' \in S} P(n, n'; t) = 1; \quad (2) \\
& P(n, n''; t + s) = \sum_{n' \in S} P(n, n'; t) P(n', n''; s); \quad (3) \\
& \lim_{t \downarrow 0} P(n, n'; t) = \delta_{n,n'}. \quad (4)
\end{align*}
Continuous-time Markov chain – 4

- For a standard transition matrix the transition rate from state $n$ to state $n'$ can be defined as

$$q(n, n') = \lim_{h \downarrow 0} \frac{P(n, n'; h) - \delta_{n,n'}}{h}.$$ 

- For all $n, n' \in S$ this limit exists.
- Markov process is called continuous-time Markov chain if for all $n, n' \in S$ the limit exists and is finite (5).
- Assume that the rate matrix $Q = (q(n, n'), \ n, n' \in S)$ is stable (6), and conservative (7)

$$0 \leq q(n, n') < \infty, \ n' \neq n; \quad (5)$$

$$0 \leq q(n) := -q(n, n) < \infty; \quad (6)$$

$$\sum_{n' \in S} q(n, n') = 0. \quad (7)$$
Continuous-time Markov chain – 5

If the rate matrix is stable the transition probabilities can be expressed in the transition rates: for \( n, n' \in S \),

\[
P(n, n'; h) = \delta_{n,n'} + q(n, n')h + o(h) \quad \text{for } h \downarrow 0,
\]

where \( o(h) \) denotes a function \( g(h) \) with the property that \( g(h)/h \to 0 \) as \( h \to 0 \).

For small positive values of \( h \), for \( n' \neq n \), \( q(n, n')h \) may be interpreted as the conditional probability that the Markov chain \( \{N(t)\} \) makes a transition to state \( n' \) during \( (t, t+h) \) given that the process is in state \( n \) at time \( t \).
Continuous-time Markov chain – 6

- For every initial state \( N(0) = n \), \( \{N(t), \ t \in T\} \) is a pure-jump process: the process jumps from state to state and remains in each state a strictly positive sojourn-time with probability 1.
- Markov chain remains in state \( n \) for a negative-exponential sojourn-time with mean \( q(n)^{-1} \).
- Conditional on the process departing from state \( n \) it jumps to state \( n' \) with probability \( p(n, n') = q(n, n')/q(n) \).
- The Markov chain represented via the holding times \( q(n) \) and transition probabilities \( p(n, n'), n, n' \in S \), is referred to as the Markov jump chain.
- The Markov chain with transition rates \( q(n, n') \) is obtained from the Markov jump chain with holding times with mean \( q(n)^{-1} \) and transition probabilities \( p(n, n') \) as \( q(n, n') = q(n)p(n, n') \), \( n, n' \in S \).
Continuous-time Markov chain – 7

- From the Chapman-Kolmogorov equations

\[ P(n, n''; t + s) = \sum_{n' \in S} P(n, n'; t)P(n', n''; s) \]

two systems of differential equations for the transition probabilities can be obtained:

- Conditioning on the first jump of the Markov chain in \((0, t]\)
yields the so-called **Kolmogorov backward equations** (9),
whereas conditioning on the last jump in \((0, t]\)
gives the **Kolmogorov forward equations** (10), for \(n, n' \in S, t \geq 0,\)

\[ \frac{dP(n, n'; t)}{dt} = \sum_{n'' \in S} q(n, n'')P(n'', n'; t), \quad (9) \]

\[ \frac{dP(n, n'; t)}{dt} = \sum_{n'' \in S} P(n, n''; t)q(n'', n'). \quad (10) \]
Continuous-time Markov chain – 8

- Derivation Kolmogorov forward equations (regular)

\[
P(n, n'; t + h) = \sum_{n''} P(n, n''; t) P(n'', n'; h)
\]

\[
P(n, n'; t + h) - P(n, n'; t) = \sum_{n'' \neq n'} P(n, n''; t) P(n'', n'; h) + P(n, n'; t) [P(n', n'; h) - 1]
\]

\[
= \sum_{n'' \neq n'} \{ P(n, n''; t) P(n'', n'; h) - P(n, n'; t) P(n', n''; h) \}
\]

\[
\frac{dP(n, n'; t)}{dt} = \sum_{n'' \neq n'} \{ P(n, n''; t) q(n'', n') - P(n, n'; t) q(n', n'') \}
\]
Explosion in a pure birth process

- Consider the Markov chain at state space $S = \mathbb{N}_0$ with transition rates

$$q(n, n') = \begin{cases} q(n), & \text{if } n' = n + 1, \\ -q(n), & \text{if } n' = n, \\ 0, & \text{otherwise}, \end{cases}$$

with initial distribution $P(N(0) = n) = \delta(n, 0)$.

- Let $\xi(n)$ denote the time spent in state $n$; $\xi = \sum_{n=0}^{\infty} \xi(n)$

- Let $q(n) = 2^n$, then

$$E\{\xi\} = \sum_{n=0}^{\infty} E\{\xi(n)\} = \sum_{n=0}^{\infty} 2^{-n} = 2$$

As $E\{\xi\} < \infty$ it must be that $P(\xi < \infty) = 1$ and therefore $\{N(t)\}$ is explosive (diverges to infinity in finite time).
Theorem (1.1.2)

For a conservative, stable, regular, continuous-time Markov chain the forward equations (10) and the backward equations (9) have the same unique solution \( \{ P(n, n'; t), \ n, n' \in S, \ t \geq 0 \} \). Moreover, this unique solution is the transition matrix of the Markov chain.

- The transient distribution \( P(n, t) = \mathbb{P} \{ N(t) = n \} \) can be obtained from the Kolmogorov forward equations for \( n \in S, \ t \geq 0, \)

\[
\begin{align*}
\frac{dP(n, t)}{dt} &= \sum_{n' \neq n} \left\{ P(n', t)q(n', n) - P(n, t)q(n, n') \right\}, \\
P(n, 0) &= P_0(n).
\end{align*}
\]
A measure \( m = (m(n), \ n \in S) \) such that \( 0 \leq m(n) < \infty \) for all \( n \in S \) and \( m(n) > 0 \) for some \( n \in S \) is called a stationary measure if for all \( n \in S, \ t \geq 0, \)

\[
m(n) = \sum_{n' \in S} m(n')P(n', n; t),
\]

and is called an invariant measure if for all \( n \in S, \)

\[
\sum_{n' \neq n} \{ m(n)q(n, n') - m(n')q(n', n) \} = 0.
\]

\( \{N(t)\} \) is ergodic if it is positive-recurrent with stationary measure having finite mass

Global balance; interpretation
Theorem (1.1.4 Equilibrium distribution)

Let \( \{N(t), \ t \geq 0\} \) be a conservative, stable, regular, irreducible continuous-time Markov chain.

(i) If a positive finite mass invariant measure \( m \) exists then the Markov chain is positive-recurrent (ergodic). In this case \( \pi(n) = m(n) \left[ \sum_{n \in S} m(n) \right]^{-1}, n \in S, \) is the unique stationary distribution and \( \pi \) is the equilibrium distribution, i.e., for all \( n, n' \in S, \)

\[
\lim_{t \to \infty} P(n', n; t) = \pi(n).
\]

(ii) If a positive finite mass invariant measure does not exist then for all \( n, n' \in S, \)

\[
\lim_{t \to \infty} P(n', n; t) = 0.
\]
Markovian Queues and Stochastic Networks

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Stochastic Operations Research
The birth-death process – 1

A birth-death process is a Markov chain \( \{ N(t), \ t \in T \} \), \( T = [0, \infty) \), or \( T = \mathbb{R} \), with state space \( S \subseteq \mathbb{N}_0 \) and transition rates for \( \lambda, \mu : S \rightarrow [0, \infty) \)

\[
q(n, n') = \begin{cases} 
\lambda(n), & \text{if } n' = n + 1, \quad \text{(birth rate)} \\
\mu(n) \mathbb{1}(n > 0), & \text{if } n' = n - 1, \quad \text{(death rate)} \\
-\lambda(n) - \mu(n), & \text{if } n' = n, \ n > 0, \\
-\lambda(n), & \text{if } n = 0.
\end{cases}
\]

Kolmogorov forward equations

\[
\frac{dP(n, t)}{dt} = P(n - 1, t)\lambda(n - 1) - P(n, t)[\lambda(n) + \mu(n)] + P(n + 1, t)\mu(n + 1), \quad n > 0,
\]

\[
\frac{dP(n, t)}{dt} = -P(n, t)\lambda(n) + P(n + 1, t)\mu(n + 1), \quad n = 0.
\]
The birth-death process – 2

**Global balance equations**

\[
\begin{align*}
0 &= \pi(n-1)\lambda(n-1) - \pi(n)[\lambda(n) + \mu(n)] + \pi(n+1)\mu(n+1), \quad n > 0, \\
0 &= -\pi(0)\lambda(0) + \pi(1)\mu(1).
\end{align*}
\]

**π satisfies the detailed balance equations**

\[
\pi(n)\lambda(n) = \pi(n+1)\mu(n+1), \quad n \in S.
\]

**Theorem (2.1.1)**

Let \(\{N(t)\}\) be a birth-death process with state space \(S = \mathbb{N}_0\), birth rates \(\lambda(n)\) and death rates \(\mu(n)\). If

\[
\pi(0) := \left[\sum_{n=0}^{\infty} \prod_{r=0}^{n-1} \frac{\lambda(r)}{\mu(r+1)} \right]^{-1} > 0,
\]

then the equilibrium distribution is

\[
\pi(n) = \pi(0) \prod_{r=0}^{n-1} \frac{q(r, r+1)}{q(r+1, r)} = \pi(0) \prod_{r=0}^{n-1} \frac{\lambda(r)}{\mu(r+1)}, \quad n \in S.
\]
Example: The $M|M|1$ queue

- Customers arrive to a queue according to a Poisson process (the arrival process) with rate $\lambda$.
- A single server serves the customers in order of arrival.
- Customers’ service times have a negative-exponential distribution with mean $\mu^{-1}$ and are independent of each other and of the arrival process.
- $\{N(t), \ t \in T\}, \ T = [0, \infty)$ recording number of customers in the queue is a birth-death process at $S = \mathbb{N}_0$ with

$$q(n, n') = \begin{cases} 
\lambda(n) = \lambda & \text{if } n' = n + 1, \quad \text{(birth rate)} \\
\mu(n) = \mu \mathbb{1}(n > 0), & \text{if } n' = n - 1, \quad \text{(death rate)} 
\end{cases}$$

and equilibrium distribution

$$\pi(n) = (1 - \rho)\rho^n, \quad n \in S,$$

provided that the queue is stable: $\rho := \frac{\lambda}{\mu} < 1$. 

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Example: The $M|M|1|c$ queue

- $M|M|1$ queue, but now with finite waiting room that may contain at most $c - 1$ customers.
- $\{N(t), t \in T\}, \ T = [0, \infty)$ recording the number of customers in the queue is a birth-death process at $S = \{0, 1, 2, \ldots, c\}$ with

$$q(n, n') = \begin{cases} 
\lambda(n) = \lambda 1(n < c) & \text{if } n' = n + 1, \\ 
\mu(n) = \mu 1(n > 0), & \text{if } n' = n - 1,
\end{cases}$$

(birth rate)

(death rate).

- Detailed balance equations are truncated at state $c$
- The equilibrium distribution is truncated to $S$:

$$\pi(n) = \pi(0) \rho^n, \quad n \in \{0, 1, \ldots, c\},$$

with

$$\pi(0) = \left[ \sum_{n=0}^{c} \rho^n \right]^{-1} = \frac{1 - \rho}{1 - \rho^{c+1}}.$$
Detailed balance – 1

Definition (2.2.1 Detailed balance)
A Markov chain \( \{N(t)\} \) at state space \( S \) with transition rates \( q(n, n') \), \( n, n' \in S \), satisfies detailed balance if a distribution \( \pi = (\pi(n), \ n \in S) \) exists that satisfies for all \( n, n' \in S \) the detailed balance equations:

\[
\pi(n)q(n, n') - \pi(n')q(n', n) = 0.
\]

Theorem (2.2.2)
If the distribution \( \pi \) satisfies the detailed balance equations then \( \pi \) is the equilibrium distribution.

- The detailed balance equations state that the probability flow between each pair of states is balanced.
Detailed balance – 2

Lemma (2.2.3, 2.2.4 Kolmogorov’s criterion)

\( \{N(t)\} \) satisfies detailed balance if and only if for all \( r \in \mathbb{N} \) and any finite sequence of states \( n_1, n_2, \ldots, n_r \in S, n_r = n_1, \)

\[
\prod_{i=1}^{r-1} q(n_i, n_{i+1}) = \prod_{i=1}^{r-1} q(n_{r-i+1}, n_{r-i}).
\]

If \( \{N(t)\} \) satisfies detailed balance, then

\[
\pi(n) = \pi(n') \frac{q(n_1, n_2)q(n_2, n_3) \cdots q(n_{r-1}, n_r)}{q(n_2, n_1)q(n_3, n_2) \cdots q(n_r, n_{r-1})},
\]

for arbitrary \( n' \in S \) for all \( r \in \mathbb{N} \) and any path \( n_1, n_2, \ldots, n_r \in S \) such that \( n_1 = n', n_r = n \) for which the denominator is positive.

- Direct generalisation of the result for birth-death process.
Theorem (2.2.5 Truncation)

Consider \( \{N(t)\} \) at state space \( S \) with transition rates \( q(n, n') \), \( n, n' \in S \), and equilibrium distribution \( \pi \). Let \( V \subset S \).

Let \( r > 0 \). If the transition rates are altered from \( q(n, n') \) to \( rq(n, n') \) for \( n \in V, n' \in S \setminus V \), then the resulting Markov chain \( \{N_r(t)\} \) satisfies detailed balance and has equilibrium distribution (\( G \) is the normalizing constant)

\[
\pi_r(n) = \begin{cases} 
G\pi(n), & n \in V, \\
Gr\pi(n), & n \in S \setminus V,
\end{cases}
\]

If \( r = 0 \) then the Markov chain is truncated to \( V \) and

\[
\pi_0(n) = \pi(n) \left[ \sum_{n \in V} \pi(n) \right]^{-1}, \quad n \in V.
\]
Example: Network of parallel $M|\!|M|1$ queues – 1

- Network of two $M|\!|M|1$ queues in parallel.
- Queue $j$ has arrival rate $\lambda_j$ and service rate $\mu_j$, $j = 1, 2$.
- $\{N_j(t)\}$, $j = 1, 2$, are assumed independent.
- $\{N(t) = (N_1(t), N_2(t))\}$, state space $S = \mathbb{N}_0^2$, $n = (n_1, n_2)$,
- Transition rates, for $n, n' \in S$, $n' \not= n$,

$$q(n, n') = \begin{cases} \lambda_j & \text{if } n' = n + e_j, \quad j = 1, 2, \\ \mu_j & \text{if } n' = n - e_j, \quad j = 1, 2. \end{cases}$$

- Random variables $N_j := N_j(\infty)$ recording the equilibrium number of customers in queue $j$ are independent.

$$\pi(n) = \prod_{j=1}^{2} \pi_j(n_j), \quad n \in S,$$

$$\pi_j(n_j) = (1 - \rho_j)\rho_j^{n_j}, \quad n_j \in \mathbb{N}_0, \text{ provided } \rho_j := \frac{\lambda_j}{\mu_j} < 1.$$
Example: Network of parallel $M|M|1$ queues – 2

- Common capacity restriction $n_1 + n_2 \leq c$.
- Customers arriving to the network with $c$ customers present are discarded.
- The Markov chain $\{N(t) = (N_1(t), N_2(t))\}$ has state space $S_c = \{(n_1, n_2) : n_j \geq 0, j = 1, 2, n_1 + n_2 \leq c\}$ and transition rates truncated to $S_c$.
- Invoking Truncation Theorem:

$$
\pi(n) = G \prod_{j=1}^{2} \rho_j^{n_j}, \quad n \in S_c,
$$

with normalising constant

$$
G = \left[ \sum_{n_1=0}^{c} \sum_{n_2=0}^{c-n_1} \prod_{i=1}^{2} \rho_i^{n_i} \right]^{-1}.
$$
Markovian Queues and Stochastic Networks

Lecture 1
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Reversibility – 1

Definition (Stationary process)
A stochastic process \( \{N(t), \ t \in \mathbb{R}\} \) is stationary if 
\((N(t_1), N(t_2), \ldots, N(t_k)) \) has the same distribution as 
\((N(t_1 + \tau), N(t_2 + \tau), \ldots, N(t_k + \tau)) \) for all \( k \in \mathbb{N}, \ t_1, t_2, \ldots, t_k \in T, \tau \in T \)

Definition (2.4.1 Reversibility)
A stochastic process \( \{N(t), \ t \in \mathbb{R}\} \) is reversible if 
\((N(t_1), N(t_2), \ldots, N(t_k)) \) has the same distribution as 
\((N(\tau - t_1), N(\tau - t_2), \ldots, N(\tau - t_k)) \) for all \( k \in \mathbb{N}, \ t_1, t_2, \ldots, t_k \in \mathbb{R}, \tau \in \mathbb{R} \).

Theorem (2.4.2)
If \( \{N(t)\} \) is reversible then \( \{N(t)\} \) is stationary.
Theorem (2.4.3 Reversibility and detailed balance)

Let \( \{N(t), \ t \in \mathbb{R}\} \) be a stationary Markov chain with transition rates \( q(n, n'), n, n' \in S \). \( \{N(t)\} \) is reversible if and only if there exists a distribution \( \pi = (\pi(n), n \in S) \) that satisfies the detailed balance equations. When there exists such a distribution \( \pi \), then \( \pi \) is the equilibrium distribution of \( \{N(t)\} \).
Example: Departures from the $M|M|1$ queue

- Arrival process to the $M|M|1$ queue is a Poisson process with rate $\lambda$.
- If $\lambda < \mu$ departure process from $M|M|1$ queue has rate $\lambda$.

- $\{N(t)\}$ recording the number of customers in $M|M|1$ with arrival rate $\lambda$ and service rate $\mu$ satisfies detailed balance.
- Markov chain $\{N'(t)\}$ in reversed time has Poisson arrivals at rate $\lambda$ and service rate $\mu$.
- Therefore $\{N'(t)\}$ is the Markov chain of an $M|M|1$ queue with Poisson arrivals at rate $\lambda$ and negative-exponential service at rate $\mu$.
- Epochs of the arrival process for the reversed queue coincide with the epochs of the arrival process for the original queue, it must be that the departure process from the $M|M|1$ queue is a Poisson process with rate $\lambda$. 
Reversibility – 3

Theorem (2.4.3 Reversibility and detailed balance)
Let \( \{N(t), \ t \in \mathbb{R}\} \) be a stationary Markov chain with transition rates \( q(n, n'), n, n' \in S \). \( \{N(t)\} \) is reversible if and only if there exists a distribution \( \pi = (\pi(n), \ n \in S) \) that satisfies the detailed balance equations. When there exists such a distribution \( \pi \), then \( \pi \) is the equilibrium distribution of \( \{N(t)\} \).

Proof. If \( \{N(t)\} \) is reversible, then for all \( t, h \in \mathbb{R}, n, n' \in S \):
\[
P(N(t + h) = n', N(t) = n) = P(N(t) = n', N(t + h) = n).
\]
\( \{N(t), \ t \in \mathbb{R}\} \) is stationary. Let \( \pi(n) = P(N(t) = n), \ t \in \mathbb{R} \).
\[
\frac{P(N(t + h) = n'|N(t) = n)}{h} \pi(n) = \frac{P(N(t + h) = n|N(t) = n')}{h} \pi(n').
\]
Letting \( h \to 0 \) yields the detailed balance equations.
Proof continued

Assume $\pi = (\pi(n), n \in S)$ satisfies detailed balance. Consider $\{N(t)\}$ for $t \in [-H, H]$. Suppose $\{N(t)\}$ moves along the sequence of states $n_1, \ldots, n_k$ and has sojourn time $h_i$ in $n_i$, $i = 1, \ldots, k - 1$, and remains in $n_k$ for at least $h_k$ until time $H$. With probability $\pi(n_1) = P(N(-H) = n_1)$, $\{N(t)\}$ starts in $n_1$. Probability density with respect to $h_1, \ldots, h_k$ for this sequence

$$\pi(n_1)q(n_1)e^{-q(n_1)h_1}p(n_1, n_2) \cdots q(n_{k-1})e^{-q(n_{k-1})h_{k-1}}p(n_{k-1}, n_k)e^{-q(n_k)h_k},$$

Kolmogorov's criterion implies that

$$\pi(n_1)q(n_1, n_2) \cdots q(n_{k-1}, n_k) = \pi(n_k)q(n_k, n_{k-1}) \cdots q(n_2, n_1),$$

probability density equals the probability density for the reversed path that starts in $n_k$ at time $H$. Thus $(N(t_1), N(t_2), \ldots, N(t_k)) \sim (N(-t_1), N(t_2), \ldots, N(-t_k))$. Stationarity completes the proof.
Burke’s theorem and feedforward networks – 1

Theorem (2.5.1 Burke’s theorem)
Let \( \{N(t)\} \) record the number of customers in the \( M|M|1 \) queue with arrival rate \( \lambda \) and service rate \( \mu \), \( \lambda < \mu \). Let \( \{D(t)\} \) record the customers’ departure process from the queue. In equilibrium the departure process \( \{D(t)\} \) is a Poisson process with rate \( \lambda \), and \( N(t) \) is independent of \( \{D(s), s < t\} \).

**Proof.** \( M|M|1 \) reversible: epochs at which \( \{N(-t)\} \) jumps up form Poisson process with rate \( \lambda \).
If \( \{N(-t)\} \) jumps up at time \( t^* \) then \( \{N(t)\} \) jumps down at \( t^* \).
Departure process forms a Poisson process with rate \( \lambda \).
\( \{N(t)\} \) reversible: departure process up to \( t^* \) and \( N(t^*) \) have same distribution as arrival process after \( -t^* \) and \( N(-t^*) \).
Arrival process is Poisson process: arrival process after \( -t^* \) independent of \( N(-t^*) \).
Hence, the departure process up to \( t^* \) independent of \( N(t^*) \).
Burke’s theorem and feedforward networks – 2

- Tandem network of two $M|M|1$ queues
- Poisson $\lambda$ arrival process to queue 1, service rates $\mu_i$.
- Provided $\rho_i = \lambda/\mu_i < 1$, marginal distributions
  $\pi_i(n_i) = (1 - \rho_i)\rho_i^{n_i}$, $n_i \in \mathbb{N}_0$.
- Burke’s theorem: departure process from queue 1 before $t^*$ and $N_1(t^*)$, are independent.
- Hence, in equilibrium, the at time $t^*$ the random variables $N_1(t^*)$ and $N_2(t^*)$ are independent:

$$\pi(n) = \prod_{i=1}^{2} \pi_i(n_i), \quad n \in S = \mathbb{N}_0^2.$$
Burke’s theorem and **feedforward networks** – 3

- Customer leaving queue \( j \) can route to any of the queues \( j + 1, \ldots, J \), or may leave the network.
- \( p_{ij} \) fraction of customers from queue \( i \) to queue \( j \geq i \), \( p_{i0} \) fraction leaving the network.
- Arrival process is Poisson process with rate \( \mu_0 \).
- Fraction \( \rho_{0j} \) of these customers is routed to queue \( j \).
- The service rate at queue \( j \) is \( \mu_j \).
- Burke’s theorem implies that all flows of customers among the queues are Poisson flows.
- Arrival rate \( \lambda_j \) of customers to queue \( j \) is obtained from superposition and random splitting of Poisson processes:

\[
\lambda_j = \mu_0 \rho_{0j} + \sum_{i=1}^{j-1} \lambda_i p_{ij}, \quad j = 1, \ldots, J,
\]

- **traffic equations**: the mean flow of customers.
Burke’s theorem and feedforward networks – 4

Theorem (2.5.4 Equilibrium distribution)
Let \( \{ N(t) = (N_1(t), \ldots, N_J(t)) \} \) at state space \( S = \mathbb{N}_0^J \), where \( n = (n_1, \ldots, n_J) \) and \( n_j \) the number of customers in queue \( j \), \( j = 1, \ldots, J \), record the number of customers in the feedforward network of \( J M|M|1 \) queues described above. If \( \rho_j = \lambda_j / \mu_j < 1 \), with \( \lambda_j \) the solution of the traffic equations, \( j = 1, \ldots, J \), then the equilibrium distribution is the product of the marginal distributions of the queues:

\[
\pi(n) = \prod_{j=1}^{J} (1 - \rho_j) \rho_j^{n_j}, \quad n_j \in \mathbb{N}_0, \ j = 1, \ldots, J. \tag{11}
\]

▶ Next time: networks of \( M|M|1 \) queues.
Markovian Queues and Stochastic Networks

Lecture 1
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