MQSN - First set

Deadline: October 7, 2024, 11:00

1. Consider Markov chain $\{N(t)\}$ at state space S with transition rates $q(\mathbf{n}, \mathbf{n}')$, $\mathbf{n}, \mathbf{n}' \in S$, and equilibrium distribution π. Let $V \subset S$. Let $0 \leq r < 1$ and suppose that the transition rates are altered from $q(\mathbf{n}, \mathbf{n}')$ to $rq(\mathbf{n}, \mathbf{n}')$ for $\mathbf{n} \in V$, $\mathbf{n}' \in S \setminus V$. Show that the resulting Markov chain $\{N_r(t)\}$ has equilibrium distribution (G is the normalizing constant)

$$\pi_r(\mathbf{n}) = \begin{cases} G\pi(\mathbf{n}), & \mathbf{n} \in V, \\ Gr\pi(\mathbf{n}), & \mathbf{n} \in S \setminus V, \end{cases}$$

if and only if π satisfies

$$\sum_{\mathbf{n}' \in S \setminus V} \pi(\mathbf{n}) q(\mathbf{n}, \mathbf{n}') = \sum_{\mathbf{n}' \in S \setminus V} \pi(\mathbf{n}') q(\mathbf{n}', \mathbf{n}), \quad \mathbf{n} \in V.$$

2. Consider a tandem network of J $M|M|1$ queues with Poisson arrivals with rate λ to queue 1 and negative-exponential service requirements with rate μ_j at queue j, $j = 1, \ldots, J$. Show that the time it takes a customer to pass through the network has mean $\sum_{j=1}^{J} (\mu_j - \lambda)^{-1}$ and variance $\sum_{j=1}^{J} (\mu_j - \lambda)^{-2}$, i.e., give a proof of the statement in Remark 2.5.4.

3. Consider a system consisting of a CPU with $J - 1$ clients that send jobs to the CPU. Each client has a Poisson arrival process of new jobs with rate λ_i for client i, $i = 1, \ldots, J - 1$. A job that arrives at a client first needs to be handled by the client (e.g. obtaines a time stamp). To this end, the client has a single server, and jobs wait for their turn in a queue of unlimited capacity. The service time of a job at the client is negative-exponential with rate ν_i for client i, $i = 1, \ldots, J - 1$. Upon completion of this activity, the job is sent to the CPU. Here the required amount of service is negative-exponential with rate μ. Upon completion of the job at the CPU, the job returns to the client it originated from to be handled again. Also at this second visit the job joins the tail of the queue, waits for its turn along with the jobs in their first visit, and has again negative-exponential service time with rate ν_i to e.g. obtain a second time stamp. Upon this second service completion the job leaves the system.

(a) Modify this system to model this system as a Jackson network of J single server queues and give a complete description of the Markov chain that records the number of customers in each queue for the tandem network, i.e., state space, routing probabilities, service rates and the resulting transition rates.

(b) Give the stability condition for this network.

(c) Give the equilibrium distribution for this network and explicitly prove correctness using partial balance.

(d) Give the mean sojourn time in the system of a job that arrives at client 1.