Networks of queues

Lectures 6--9:

- Norton's theorem
- Insensitivity
- Arrival theorem
- Optimal design of a Kelly Whittle network

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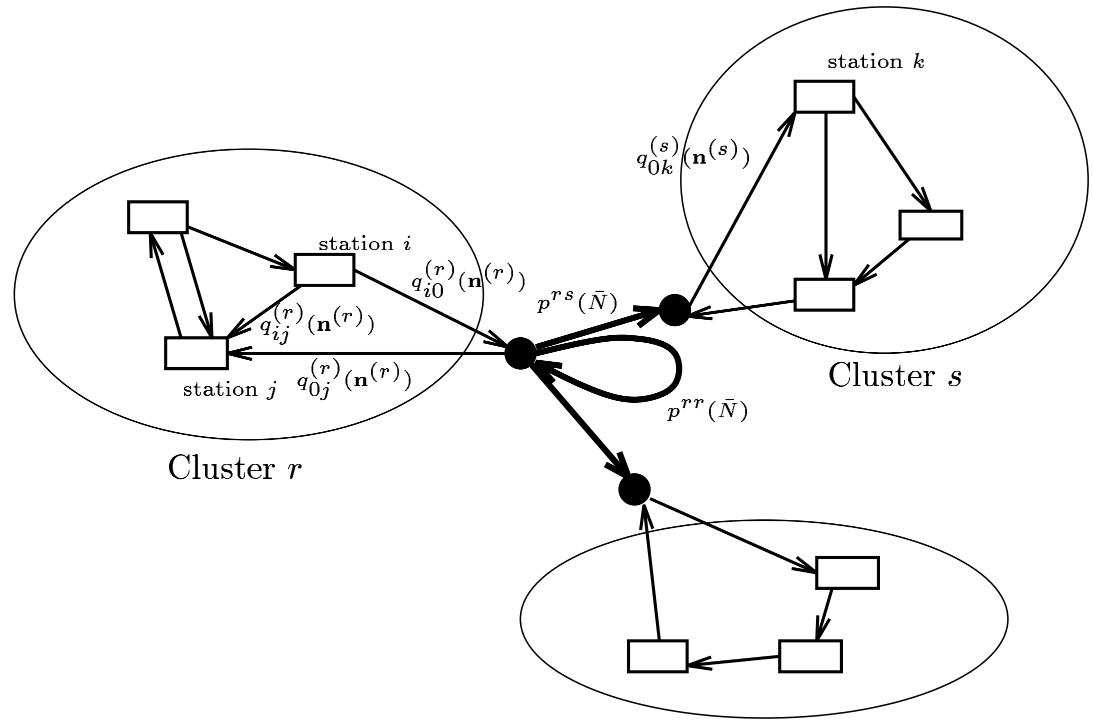
Norton's theorem

State n=($n_1,\ldots,n_N$)
Clusters $C_1,\ldots,C_R$
Stations i ε C_r
State of cluster $n^{(r)}$=(n_i,i ε C_r)
Global state $N_r=\Sigma_{i\ \varepsilon\ Cr} n_i$
N=($N_1,\ldots,N_R$)



$$\sum_{j\in C_r} q_{0j}^{(r)}(\mathbf{n}^{(r)}) = 1$$

$$q(\mathbf{n},\mathbf{n}-e_i+e_j) = \begin{cases} q_{i0}^{(r)}(\mathbf{n}^{(r)})q_R^{rs}(\bar{\mathrm{N}})q_{0j}^{(s)}(\mathbf{n}^{(s)}), & i\in C_r,\ j\in C_s, \\ q_{ij}^{(r)}(\mathbf{n}^{(r)})\mu^{(r)}(\bar{\mathrm{N}}) + q_{i0}^{(r)}(\mathbf{n}^{(r)})q_R^{rr}(\bar{\mathrm{N}})q_{0j}^{(r)}(\mathbf{n}^{(r)}-e_i^{(r)}), & i,j\in C_r, \\ q_{i0}^{(r)}(\mathbf{n}^{(r)})q_R^{r0}(\bar{\mathrm{N}}), & i\in C_r,\ j\in C_0, \\ q_R^{0s}(\bar{\mathrm{N}})q_{0j}^{(s)}(\mathbf{n}^{(s)}), & i\in C_0,\ j\in C_s, \end{cases}$$

$$q_R^{rs}(\bar{\mathrm{N}}) = \mu^{(r)}(\bar{\mathrm{N}})p^{rs}(\bar{\mathrm{N}}) \qquad (2.2)$$

Recap: Network of quasi reversible nodes

- Construct network by multiplying rates for individual queues
- Transition rates
- Arrival of type i causes queue k=r(i,1) to change at

$$q_k(x_k, x_k') \quad x_k' \in S_k(i,1,x_k)$$

- Departure type i from queue $j = r(i,S(i))$

$$q_j(x_j, x_j') \quad x_j \in S_j(i,S(i),x_j')$$

- Routing

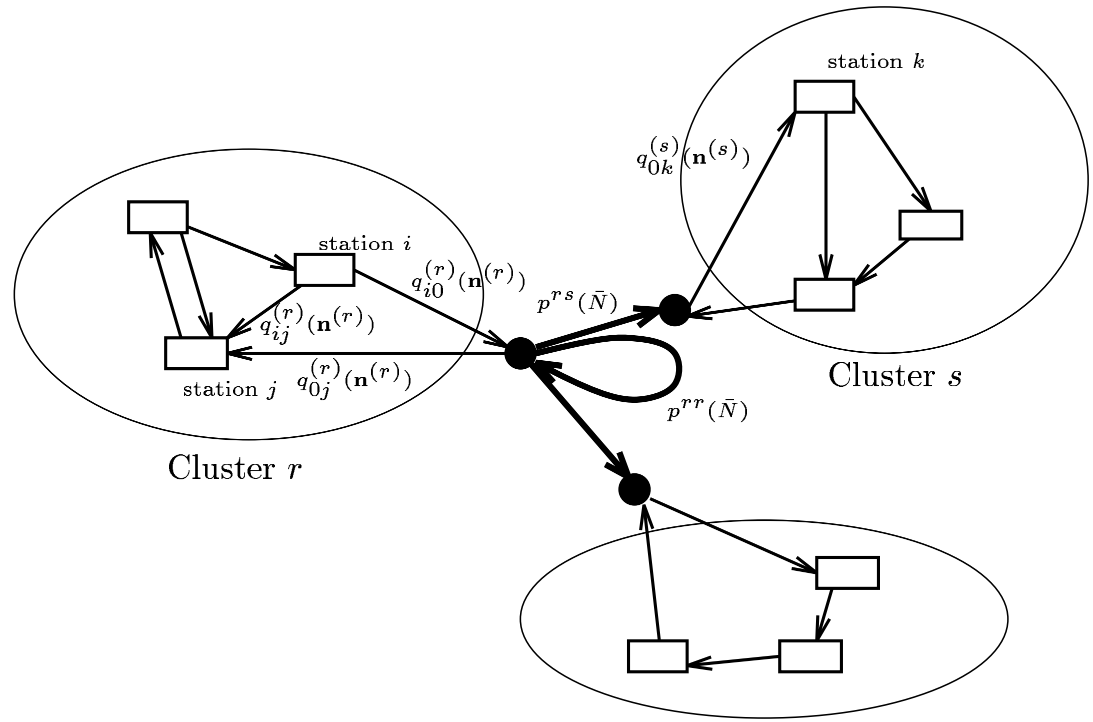
$$q_j(x_j, x_j')\frac{q_k(x_k, x_k')}{\sum_{x' \in S_k(i,s+1,x_k)} q_k(x_k, x')} = q_j(x_j, x_j')\frac{q_k(x_k, x_k')}{\alpha_k(i, s+1)}$$

$$x_k' \in S_k(i, s+1, x_k) \quad x_j \in S_j(i, s, x_j')$$

- Internal $\quad q_j(x_j, x_j')$

Norton's theorem

State $n=(n_1,\ldots,n_N)$
Clusters $C_1,\ldots,C_R$
Stations $i \in C_r$
State of cluster $n^{(r)}=(n_i, i \in C_r)$
Global state $N_r=\Sigma_{i \in Cr} n_i$
$N=(N_1,\ldots,N_R)$

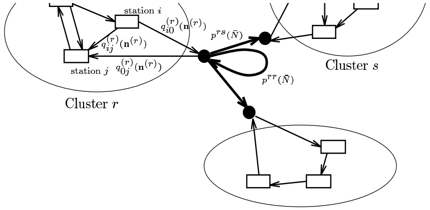


$$\sum_{j \in C_r} q_{0j}^{(r)}(\mathbf{n}^{(r)}) = 1$$

$$q(\mathbf{n}, \mathbf{n} - e_i + e_j) = \begin{cases} q_{i0}^{(r)}(\mathbf{n}^{(r)}) q_R^{rs}(\bar{\mathrm{N}}) q_{0j}^{(s)}(\mathbf{n}^{(s)}), & i \in C_r,\ j \in C_s, \\ q_{ij}^{(r)}(\mathbf{n}^{(r)}) \mu^{(r)}(\bar{\mathrm{N}}) + q_{i0}^{(r)}(\mathbf{n}^{(r)}) q_R^{rr}(\bar{\mathrm{N}}) q_{0j}^{(r)}(\mathbf{n}^{(r)} - e_i^{(r)}), & i, j \in C_r, \\ q_{i0}^{(r)}(\mathbf{n}^{(r)}) q_R^{r0}(\bar{\mathrm{N}}), & i \in C_r,\ j \in C_0, \\ q_R^{0s}(\bar{\mathrm{N}}) q_{0j}^{(s)}(\mathbf{n}^{(s)}), & i \in C_0,\ j \in C_s, \end{cases}$$

$$q_R^{rs}(\bar{\mathrm{N}}) = \mu^{(r)}(\bar{\mathrm{N}}) p^{rs}(\bar{\mathrm{N}}) \qquad (2.2)$$

Norton´s theorem (2)



$$q(\mathbf{n}, \mathbf{n} - e_i + e_j) = \begin{cases} q_{i0}^{(r)}(\mathbf{n}^{(r)}) q_R^{rs}(\bar{\mathrm{N}}) q_{0j}^{(s)}(\mathbf{n}^{(s)}), & i \in C_r,\ j \in C_s, \\ q_{ij}^{(r)}(\mathbf{n}^{(r)}) \mu^{(r)}(\bar{\mathrm{N}}) + q_{i0}^{(r)}(\mathbf{n}^{(r)}) q_R^{rr}(\bar{\mathrm{N}}) q_{0j}^{(r)}(\mathbf{n}^{(r)} - e_i^{(r)}), & i, j \in C_r, \\ q_{i0}^{(r)}(\mathbf{n}^{(r)}) q_R^{r0}(\bar{\mathrm{N}}), & i \in C_r,\ j \in C_0, \\ q_R^{0s}(\bar{\mathrm{N}}) q_{0j}^{(s)}(\mathbf{n}^{(s)}), & i \in C_0,\ j \in C_s, \end{cases} \tag{2.2}$$

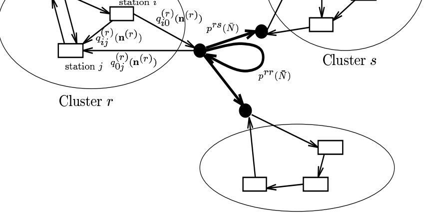
Stationary distributions: $$\sum_{i,j \in C_r \cup \{0\}} \{\pi^{(r)}(\mathbf{n}^{(r)}) q_{ij}^{(r)}(\mathbf{n}^{(r)}) - \pi^{(r)}(\mathbf{n}^{(r)} - e_i^{(r)} + e_j^{(r)}) q_{ji}^{(r)}(\mathbf{n}^{(r)} - e_i^{(r)} + e_j^{(r)})\} = 0$$

$$\sum_{s,r=0}^{R} \{\pi_R(\bar{\mathrm{N}}) q_R^{rs}(\bar{\mathrm{N}}) - \pi_R(\bar{\mathrm{N}} - \bar{\mathrm{E}}_r + \bar{\mathrm{E}}_s) q_R^{sr}(\bar{\mathrm{N}} - \bar{\mathrm{E}}_r + \bar{\mathrm{E}}_s)\} = 0.$$

Quasi-reversibility of clusters $$\sum_{j \in C_r} \{\pi^{(r)}(\mathbf{n}^{(r)}) q_{0j}^{(r)}(\mathbf{n}^{(r)}) - \pi^{(r)}(\mathbf{n}^{(r)} + e_j^{(r)}) q_{j0}^{(r)}(\mathbf{n}^{(r)} + e_j^{(r)})\} = 0$$

Partial balance global process $$\sum_{s=0}^{R} \{\pi_R(\bar{\mathrm{N}}) q_R^{rs}(\bar{\mathrm{N}}) - \pi_R(\bar{\mathrm{N}} - \bar{\mathrm{E}}_r + \bar{\mathrm{E}}_s) q_R^{sr}(\bar{\mathrm{N}} - \bar{\mathrm{E}}_r + \bar{\mathrm{E}}_s)\} = 0$$

Theorem: $$\pi(\mathbf{n}) = B \pi_R(\bar{\mathrm{N}}) \prod^{R} \pi^{(r)}(\mathbf{n}^{(r)})$$



Norton's theorem (3)

$$q(\mathbf{n},\mathbf{n}-e_i+e_j)=\begin{cases} q_{i0}^{(r)}(\mathbf{n}^{(r)})q_R^{rs}(\bar{\mathrm{N}})q_{0j}^{(s)}(\mathbf{n}^{(s)}), & i\in C_r,\ j\in C_s,\\ q_{ij}^{(r)}(\mathbf{n}^{(r)})\mu^{(r)}(\bar{\mathrm{N}})+q_{i0}^{(r)}(\mathbf{n}^{(r)})q_R^{rr}(\bar{\mathrm{N}})q_{0j}^{(r)}(\mathbf{n}^{(r)}-e_i^{(r)}), & i,j\in C_r,\\ q_{i0}^{(r)}(\mathbf{n}^{(r)})q_R^{r0}(\bar{\mathrm{N}}), & i\in C_r,\ j\in C_0,\\ q_R^{0s}(\bar{\mathrm{N}})q_{0j}^{(s)}(\mathbf{n}^{(s)}), & i\in C_0,\ j\in C_s,\end{cases} \tag{2.2}$$

Global process:

$$M^{(r)}(\bar{\mathrm{N}})=\mu^{(r)}(\bar{\mathrm{N}})\frac{\pi^{(r)}(\mathrm{N}_r-1)}{\pi^{(r)}(\mathrm{N}_r)},\quad r=0,\ldots,R$$

$$Q(\bar{\mathrm{N}},\bar{\mathrm{N}}-\bar{\mathrm{E}}_r+\bar{\mathrm{E}}_s)=M^{(r)}(\bar{\mathrm{N}})p^{rs}(\bar{\mathrm{N}})$$

First order equivalent:

$$\Pi(\bar{\mathrm{N}})=\sum_{\mathbf{n}:\sum_{i\in C_r}n_i=\mathrm{N}_i,\ r=1,\ldots,R}\pi(\mathbf{n})$$

$$\Pi(\bar{\mathrm{N}})Q(\bar{\mathrm{N}},\bar{\mathrm{N}}-\bar{\mathrm{E}}_r+\bar{\mathrm{E}}_s)=\sum_{\mathbf{n}:\sum_{i\in C_r}n_i=\mathrm{N}_r,\ r=1,\ldots,R}\ \sum_{i\in C_r,\ j\in C_s}\pi(\mathbf{n})q(\mathbf{n},\mathbf{n}-e_i+e_j)$$

Theorem: global process is first order equivalent, and

$$\Pi(\bar{\mathrm{N}})=B_R\pi_R(\bar{\mathrm{N}})\prod^{R}\pi^{(r)}(\mathrm{N}_r)\qquad \pi(\mathbf{n}|\bar{\mathrm{N}})=\prod^{R}\pi^{(r)}(\mathbf{n}^{(r)}|N_r)$$

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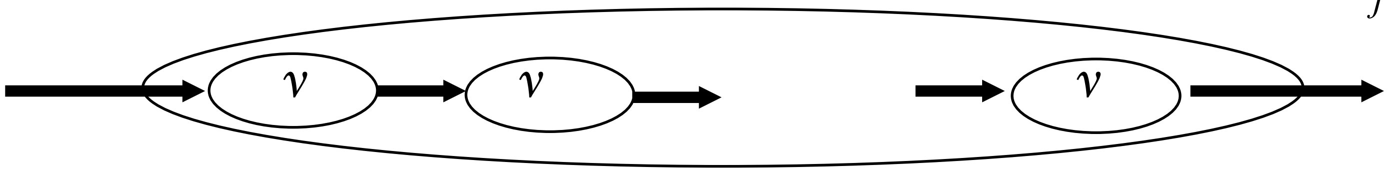
General distribution

- Erlang(k, ν)

$$Erl(k,\nu)(x) = 1 - \sum_{j=0}^{k-1} \frac{(\nu x)^j}{j!} e^{-\nu x}$$



- mean $EL = k/\nu$ CV=1/ $\sqrt{k} < 1$

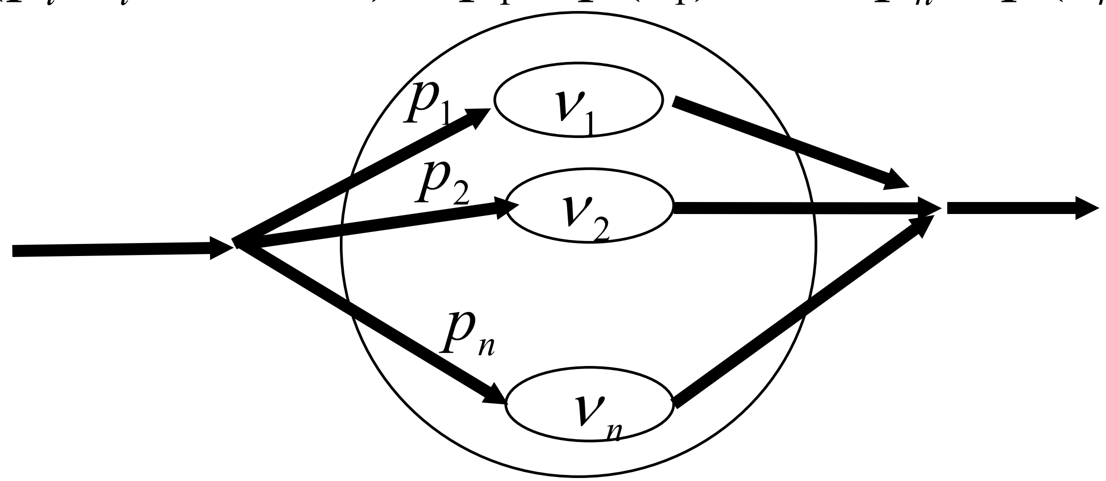
- Hyperexponential

$$Hyp(p_i, \nu_i, i = 1,..,n) = p_1 Exp(\nu_1) + ... + p_n Exp(\nu_n)$$

- mean

$$EL = \frac{p_1}{\nu_1} + ... + \frac{p_n}{\nu_n}$$

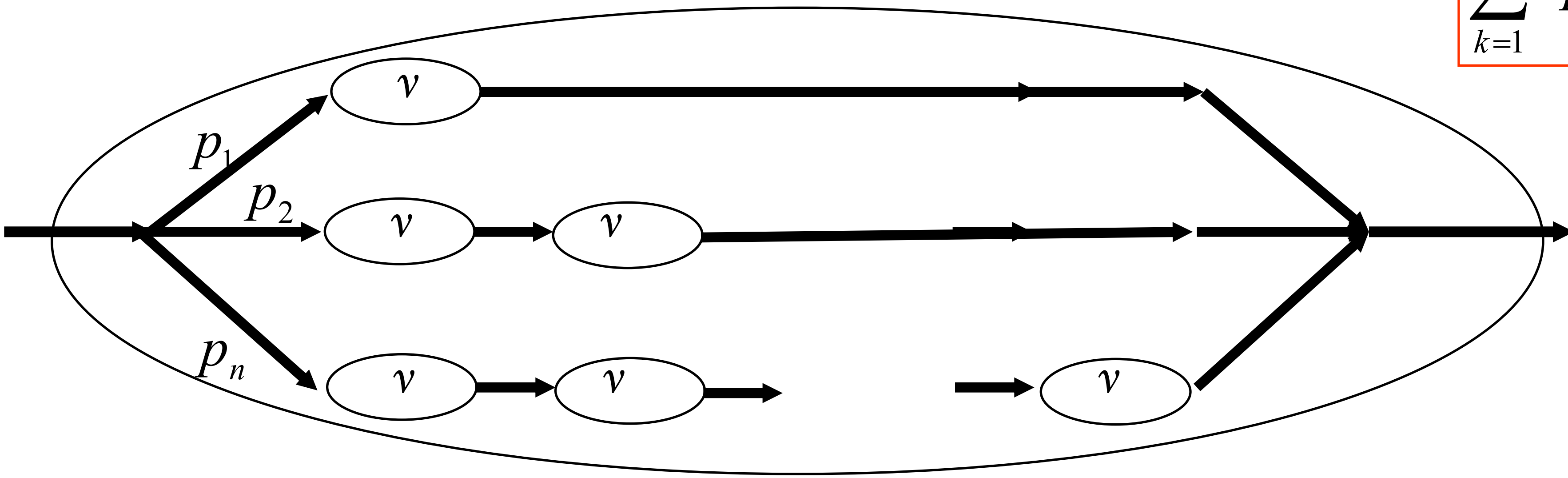
- CV > 1



General distribution: phase type distribution

$$\sum_{k=1}^{\infty} p_k = 1$$

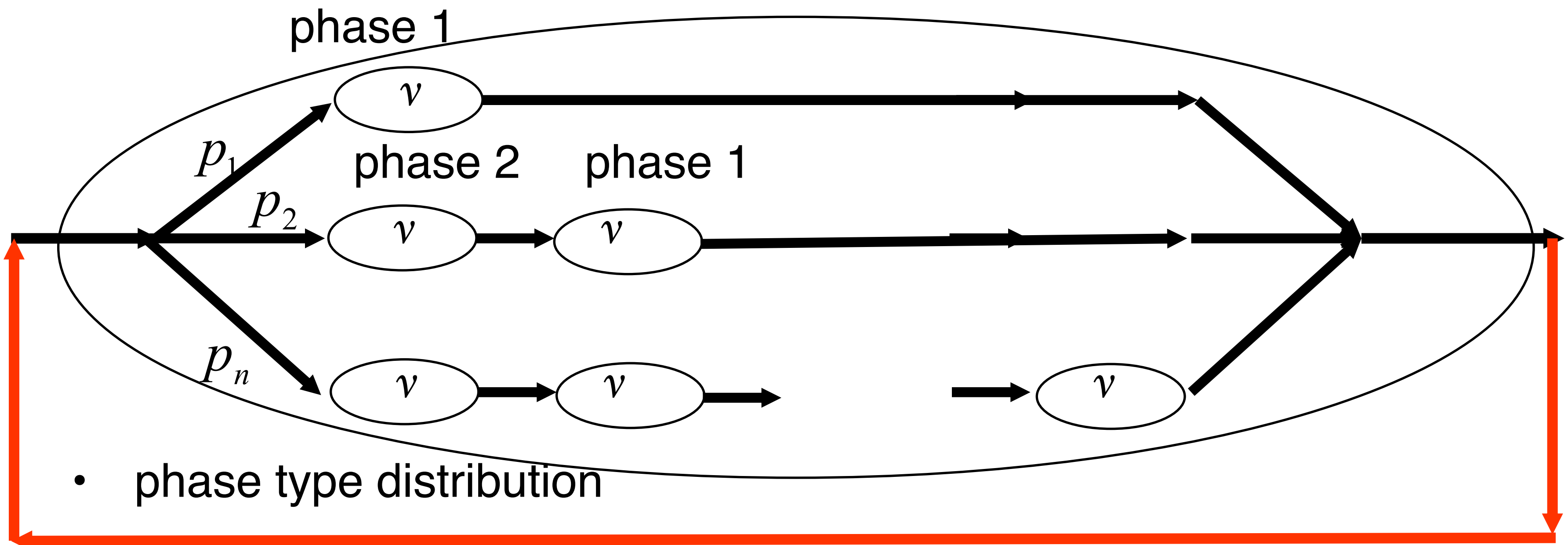
- With probability p_k Erlang(k, ν)



- phase type distribution $\quad F_L(x) = \sum_{k=1}^{\infty} p_k Erl(k,\nu)(x)$
- mean $\quad EL = \sum_{k=1}^{\infty} \frac{k\, p_k}{\nu}$

General distribution: phase type distribution

- With probability p_k Erlang(k, ν)



- phase type distribution
- dense in class of distributions with non-negative support

$$F_L(x) = \sum_{k=1}^{\infty} p_k Erl(k,\nu)(x)$$

General distribution: phase type distribution

- Markov chain that records the remaining number of phases and that restarts in phase k wp p_k each time phase 1 is completed
- state k records number of remaining phases of renewal process
- state space $S=\{1,2,\ldots\}$
- transition rates $q(k,k\text{-}1) = \nu$
  $q(1,k) = \nu \; p_k$

- Let $H(k)$ denote equilibrium distribution, then $H(k)$ satisfies global balance:

$$H(k)\,\nu = H(1)\,\nu \quad p_k + H(k+1)\,\nu, \qquad k=1,2,\ldots$$

- or discrete renewal equation (TK VII-6)

$$H(k) \; = H(1) \quad p_k \; + H(k+1), \qquad k=1,2,\ldots$$

- solution $\displaystyle H(k) = \frac{\mu}{\nu} \sum_{i=k}^{\infty} \; p_i$ where $\displaystyle \frac{1}{\mu} = \sum_{k=1}^{\infty} \; \frac{k\,p_k}{\nu}$

General distribution: phase type distribution

- $H(k) = \frac{\mu}{\nu} \sum_{i=k}^{\infty} p_i$ is distribution that satisfies
- discrete renewal equation

$$H(k) = H(1)\ p_k + H(k+1), \quad k=1,2,\ldots$$

- Proof
- insert $H(k)$ into equation: $\sum_{i=k}^{\infty} p_i = p_k \sum_{i=1}^{\infty} p_i + \sum_{i=k+1}^{\infty} p_i$
- show that $H(k)$ is distribution:

$$\sum_{k=1}^{\infty} H(k) = \frac{\mu}{\nu} \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} p_i = \frac{\mu}{\nu} \sum_{i=1}^{\infty} \sum_{k=1}^{i} p_i = \frac{\mu}{\nu} \sum_{i=1}^{\infty} i p_i = 1$$

Processor sharing queue

- Poisson arrivals rate λ
- Service request L mean $\tau=1/\mu$

- State n = # customers in queue
- State space $S = \{0,1,\dots\}$
- Markov chain $X = \{X(t),\ t\geq 0\}$
- birth rate $q(n,n+1)=\lambda$
- death rate $q(n,n-1)=\mu$

- Equilibrium distribution

$$\pi_n = (1-\lambda\tau)(\lambda\tau)^n \quad n = 0,1,2,\ldots$$

Proof: (exponential case)

equilibrium distribution $\pi_n = (1-\lambda\tau)(\lambda\tau)^n$

solution global balance

$$\pi_n[q(n,n+1)+q(n,n-1)] = \pi_{n-1}q(n-1,n)+\pi_{n+1}q(n+1,n)$$

rate out of state n = rate into state n

$$\pi_n[\lambda+\mu] = \pi_{n-1}\lambda 1(n>0)+\pi_{n+1}\mu \quad 0 \le n$$

detailed balance

$$\pi_n\lambda = \pi_{n+1}\mu \quad 0 \le n$$

Processor sharing queue: phase type service times

- Poisson arrivals rate λ
- service length L mean $\tau=1/\mu$
- State $(r_1,\ldots,r_n)$ customer i has r_i remaining phases;
- State space
- Markov chain $X = \{X(t), t\geq 0\}$
- Transition rates

$$F_L(x) = \sum_{k=1}^{\infty} p_k Erl(k,\nu)(x)$$

$$q((r_1,\ldots,r_n),(r_1,\ldots,r_i-1,\ldots,r_n)) = \frac{\nu}{n}\, 1(r_i>1)$$

$$q((r_1,\ldots,r_n),(r_1,\ldots,r_{i-1},r_{i+1},\ldots,r_n)) = \frac{\nu}{n}\, 1(r_i=1)$$

$$q((r_1,\ldots,r_n),(r_1,\ldots,r_i,r,r_{i+1},\ldots,r_n)) = \frac{\lambda}{n+1} p_r$$

- Equilibrium distribution

$$\pi(r_1,\ldots,r_n) = G^{-1}(\lambda\tau)^n \prod_{i=1}^{n} H(r_i) \qquad H(k) = \frac{\mu}{\nu}\sum_{i=k}^{\infty} p_i$$

- $H(k)$ is distribution of the remaining number of phases = remaining service time

Erlang loss queue: phase type service length

- Equilibrium distribution $\pi(r_1,\dots,r_n) = G^{-1}(\lambda\tau)^n \prod_{i=1}^{n} H(r_i)$ $\quad H(k) = \frac{\mu}{\nu}\sum_{i=k}^{\infty} p_i$

- Proof
- global balance $\pi_n[\lambda+\mu] = \pi_{n-1}\lambda 1(n>0) + \pi_{n+1}\mu \qquad 0 \le n$

$$\pi(r_1,\dots,r_n)[\lambda+\nu] = \sum_{i=1}^{n} \pi(r_1,\dots,r_{i-1},r_{i+1},\dots,r_n)\frac{\lambda}{n}\, p_{r_i} 1(n>0)$$

$$+\sum_{i=1}^{n} \pi(r_1,\dots,r_{i-1},r_i+1,r_{i+1},\dots,r_n)\frac{\nu}{n} + \sum_{i=0}^{n} \pi(r_1,\dots,r_i,1,r_{i+1},\dots,r_n)\frac{\nu}{n+1}$$

$$[\lambda+\nu] = \sum_{i=1}^{n} \{\lambda\tau H(r_i)\}^{-1}\frac{\lambda}{n}\, p_{r_i} 1(n>0)$$

$$+\sum_{i=1}^{n} \{H(r_i+1)/H(r_i)\}\frac{\nu}{n} + \sum_{i=0}^{n} \{\lambda\tau H(1)\}\frac{\nu}{n+1}$$

H(1)=μ/ν and use discrete renewal equation $H(r_i) = H(1)p_k + H(r_i+1), \qquad r_i = 1,2,\dots$

Processor sharing queue: phase type service

- Theorem 1
  Equilibrium distribution
- $$\pi(r_1,\ldots,r_n) = G^{-1}(\lambda\tau)^n \prod_{i=1}^{n} H(r_i)$$ where $$H(k) = \frac{\mu}{\nu}\sum_{i=k}^{\infty} p_i$$
- moreover, equilibrium distribution of number of customers depends on service time distribution only through its mean (insensitivity property):

$$\pi_n = (1-\lambda\tau)\ (\lambda\tau)^n$$

- Proof
- sum distribution over all possible configurations of phases

$$\pi_n = \sum_{\substack{r_i \geq 1 \\ i=1,\ldots,n}} \pi(r_1,\ldots,r_n) = \sum_{\substack{r_i \geq 1 \\ i=1,\ldots,n}} G^{-1}(\lambda\tau)^n \prod_{i=1}^{n} H(r_i) = G^{-1}(\lambda\tau)^n \prod_{i=1}^{n} \sum_{r_i \geq 1} H(r_i) = G^{-1}(\lambda\tau)^n$$

Networks of queues

Lecture 5:

- Insensitivity
- Arrival theorem
- Norton's theorem
- Optimal design of a Kelly Whittle network

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Arrival theorem

- PASTA:
  The distribution of the number of customers in the system seen by a a customer arriving to a system according to a Poisson process (i.e., at an arrival epoch) equals the distribution of the number of customers at an arbitrary epoch.

- Arrival theorem (open Jackson network):
  In an open network in equilibrium, a customer arriving to queue j observes the equilibrium distribution.

- Arrival theorem (closed Jackson network):
  In a closed networkin equilibrium, a customer arriving to queue j observes the equilibrium distribution of the network containing one customer less.

PASTA: Poisson Arrivals See Time Averages

- $P_{n',n}(t)$ fraction of time system in state n

- probability outside observer sees n customers at time t

- $P^0_{n',n}(t)$ probability that arriving customer sees n customers at time t

  (just before arrival at time t there are n customers in the system)

- in general $P_{n',n}(t) \neq P^0_{n',n}(t)$

PASTA: Poisson Arrivals See Time Averages

- For birth-death process:

- Let $C(t,t+h)$ event customer arrives in $(t,t+h)$

$$P^0_{n',n}(t) = \lim_{h\downarrow 0} \Pr\{X(t)=n \mid C(t,t+h), X(0)=n'\}$$

$$= \lim_{h\downarrow 0} \frac{\Pr\{C(t,t+h) \mid X(t)=n, X(0)=n'\}\Pr\{X(t)=n \mid X(0)=n'\}}{\sum_{k=0}^{\infty} \Pr\{C(t,t+h) \mid X(t)=k, X(0)=n'\}\Pr\{X(t)=k \mid X(0)=n'\}}$$

$$= \lim_{h\downarrow 0} \frac{[q(n,n+1)h+o(h)]P_{n',n}(t)}{\sum_{k=0}^{\infty} [q(k,k+1)h+o(h)]P_{n',k}(t)} = \frac{q(n,n+1)P_{n',n}(t)}{\sum_{k=0}^{\infty} q(k,k+1)P_{n',k}(t)}$$

- For Poisson arrivals $q(n,n+1)=\lambda$ so that $P_{n',n}(t) = P^0_{n',n}(t)$

- Alternative explanation; PASTA holds in general!

PASTA: Poisson Arrivals See Time Averages

- Transient

$$P^0_{n',n}(t) = \frac{q(n,n+1)P_{n',n}(t)}{\sum_{k=0}^{\infty} q(k,k+1)P_{n',k}(t)}$$

- In equilibrium

$$P^0_n = \frac{q(n,n+1)P_n}{\sum_{k=0}^{\infty} q(k,k+1)P_k}$$

- Ratio of flows

MUSTA: Moving Units See Time Averages

- Palm probabilities:

  Each type of transition $n \rightarrow n'$ for Markov chain associated with subset H of SxS \diag(SxS)

- Example:transition in which customer queue $i \rightarrow$ queue j

$$H_{ij} = \bigcup_{m} \left\{ (m+e_i, m+e_j), m+e_i, m+e_j \in S \right\}$$

- Transition in which customer leaves queue i

$$H_i^{out} = \bigcup_{j} H_{ij}$$

- Transition in which customer enters queue j

$$H_j^{in} = \bigcup_{i} H_{ij}$$

MUSTA: Moving Units See Time Averages

- N_H process counting the H-transitions
- Palm probability $P_H(C)$ of event C given that H occurs:

$$P_H(C) = \frac{\sum_{(n,n')\in C} \pi(n)q(n,n')}{\sum_{(n,n')\in H} \pi(n)q(n,n')}, \quad C \subseteq H$$

- Probability customer queue $i \rightarrow$ queue j sees state m

$$P_{ij}(m) = P_{H_{ij}}((m+e_i, m+e_j)) = \frac{\pi(m+e_i)q(m+e_i, m+e_j)}{\sum_{(n,n')\in H_{ij}} \pi(n)q(n,n')},$$

- Probability customer arriving to queue j sees state m

$$P_j(m) = P_{H_j^{in}}(\bigcup_i (m+e_i, m+e_j)) = \frac{\sum_i \pi(m+e_i)q(m+e_i, m+e_j)}{\sum_i \sum_{(n,n')\in H_{ij}} \pi(n)q(n,n')},$$

Kelly Whittle network

$$q(n, n-e_j+e_k) = \frac{\psi(n-e_j)}{\phi(n)} \mu_j p_{jk}$$

$$q(n, n-e_j) = \frac{\psi(n-e_j)}{\phi(n)} \mu_j p_{j0}$$

$$q(n, n+e_k) = \frac{\psi(n)}{\phi(n)} \lambda_k$$

Theorem: The equilibrium distribution for the Kelly Whittle network is

$$\pi(n) = B\phi(n)\prod_{j=1}^{J} \rho_j^{n_j} \quad \rho_j = \gamma_j / \mu_j \quad n \in S$$

where

$$\gamma_j = \lambda_j + \sum_k \gamma_k p_{kj}$$

and π satisfies partial balance

$$\sum_{k=0}^{J} \pi(n) q(n, n-e_j+e_k) = \sum_{k=0}^{J} \pi(n-e_j+e_k) q(n-e_j+e_k, n)$$

MUSTA : Kelly Whittle network

$$q(n, n-e_j+e_k) = \frac{\psi(n-e_j)}{\phi(n)}\mu_j p_{jk}$$

$$\pi(n) = B\phi(n)\prod_{j=1}^{J} \rho_j^{n_j} \quad \rho_j = \gamma_j / \mu_j \quad n \in S$$

$$q(n, n-e_j) = \frac{\psi(n-e_j)}{\phi(n)}\mu_j p_{j0}$$

$$q(n, n+e_k) = \frac{\psi(n)}{\phi(n)}\lambda_k$$

Theorem: The distribution seen by a customer moving from queue i to queue j is

$$P_{ij}(m) = B_{ij}\psi(m)p_{ij}\prod_{k=1}^{J} \rho_k^{m_k}, m \in S^d$$

Entering queue j is

$$P_j(m) = B_j\psi(m)\prod_{k=1}^{J} \rho_k^{m_k}, m \in S^d$$

where

$$S^d = \{m : \exists i, j : m+e_i, m+e_j \in S\}$$

$$P_j(m) = \frac{\sum_i \pi(m+e_i)q(m+e_i, m+e_j)}{\sum_i \sum_{(n,n')\in H_{ij}} \pi(n)q(n,n')},$$

$$= \frac{\sum_i \pi(m+e_i)q(m+e_i, m+e_j)}{\sum_i \sum_m \pi(m+e_i)q(m+e_i, m+e_j)}$$

$$= \frac{\sum_i \pi(m+e_j)q(m+e_j, m+e_i)}{\sum_i \sum_m \pi(m+e_j)q(m+e_j, m+e_i)}$$

$$= \frac{\psi(m)\prod_{k=1}^{J} \rho_k^{m_k} \sum_i \rho_j p_{ji}}{\sum_m \psi(m)\prod_{k=1}^{J} \rho_k^{m_k} \sum_i \rho_j p_{ji}}$$

$$= \frac{\psi(m)\prod_{k=1}^{J} \rho_k^{m_k}}{\sum_m \psi(m)\prod_{k=1}^{J} \rho_k^{m_k}}$$

Closed networks: MVA

Average queue length, average sojourn times?

$\lambda_m(i)$ arrival intensity queue i,
$F_m(i)$ expeceted sojourn time i,
$L_m(i)$ expected queue length queue i, when m cust in system

$$F_m(j) = \frac{1}{\mu_j} + L_{m-1}(j)\frac{1}{\mu_j}$$

Arrival theorem, FCFS

$$L_m(j) = \lambda_m(j) F_m(j)$$

Little's formula

Closed networks: MVA

$\lambda_m(i)$ arrival intensity queue i,
$F_m(i)$ expeceted sojourn time i,
$L_m(i)$ expected queue length queue i, when m cust in system

$$F_m(j) = \frac{1}{\mu_j} + L_{m-1}(j)\frac{1}{\mu_j}, \qquad L_m(j) = \lambda_m(j) F_m(j)$$

$$\lambda_m(i) = \sum_{j=1}^{N} \lambda_m(j) \cdot r_{ji} \qquad \pi_i = \sum_{j=1}^{N} \pi_j \cdot r_{ji} \quad , \quad \sum_{i=1}^{N} \pi_i = 1$$

$$\lambda_m(i) = \lambda_m \cdot \pi_i \quad , \quad \lambda_m = \sum_{i=1}^{N} \lambda_m(i)$$

- Little

$$m = \sum_{i=1}^{N} L_m(i) = \sum_{i=1}^{N} \lambda_m(i) \cdot F_m(i) = \lambda_m \cdot \sum_{i=1}^{N} \pi_i \cdot F_m(i)$$

- thus

$$\lambda_m = m \cdot \left\{ \sum_{i=1}^{N} \pi_i \cdot F_m(i) \right\}^{-1}$$

- Mean Value Analysis

  evaluates $\lambda_m(i)$, $F_m(i)$ en $L_m(i)$ for all m,i recursively

  - Find solution π of traffic equations
  - for m=1 : $F_1(i)=1/\mu_i$ for all i

  recursion

  - let $F_m(i)$ known for all i
  - Determine number of cust served per time unit at queue i :

$$\lambda_m(i) = \lambda_m \cdot \pi_i = m \cdot \left\{ \sum_{j=1}^{N} \pi_j \cdot F_m(j) \right\}^{-1} \cdot \pi_i$$

  - Determine average number of customers at queue i using Little

$$L_m(i) = \lambda_m(i) \cdot F_m(i)$$

  - Determine average sojourn time at queue i for system containing m+1 customers using arrival theorem

$$F_{m+1}(i) = \frac{1 + L_m(i)}{\mu_i}$$

Networks of queues

Lecture 5:

- Insensitivity
- Arrival theorem
- Norton's theorem
- Optimal design of a Kelly Whittle network

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Interpretation traffic equations

$$q(n, n-e_j+e_k) = \frac{\phi(n-e_j)}{\phi(n)}\mu_j p_{jk}$$

$$q(n, n-e_j) = \frac{\phi(n-e_j)}{\phi(n)}\mu_j p_{j0}$$

$$q(n, n+e_k) = \frac{\phi(n)}{\phi(n)}\mu_0 p_{0k}$$

Theorem: The equilibrium distribution for the Kelly Whittle network is

$$\pi(n) = B\phi(n)\prod_{j=1}^{J}\rho_j^{n_j} \quad \rho_j = \gamma_j / \mu_j \quad n \in S$$

where

$$\gamma_j = \lambda_j + \sum_k \gamma_k p_{kj}$$

and

$$\mathrm{E}q(n, n+e_k) = \lambda_k$$

$$\mathrm{E}q(n, n-e_j+e_k) = \gamma_j p_{jk} \qquad j = 1, \ldots, J$$

Intermezzo: mathematical programming

- Optimisation problem
$$\min f(x_1,\ldots,x_n)$$
$$\text{s.t.} \quad g_i(x_1,\ldots,x_n) = b_i \qquad i = 1,\ldots,m$$

- Lagrangian
$$L = f(x_1,\ldots,x_n) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x_1,\ldots,x_n))$$
$$\frac{\partial L}{\partial \lambda_i} = b_i - g_i(x_1,\ldots,x_n) = 0$$
$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}$$

- Lagrangian optimization problem $\min L(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_m)$

- Theorem : Under regularity conditions: any point $(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_m)$

  that satisfies Lagrangian optimization problem yields optimal solution $(x_1,\ldots,x_n)$

  of Optimisation problem

Intermezzo: mathematical programming (2)

- Optimisation problem

$$\min f(x_1,\ldots,x_n)$$

$$\text{s.t.} \quad g_i(x_1,\ldots,x_n) \le b_i \qquad i = 1,\ldots,m$$

- Introduce slack variables

- Kuhn-Tucker conditions:

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \bar{\lambda}_i \frac{\partial g_i}{\partial x_j} = 0, \quad j = 1,\ldots,n$$

$$\bar{\lambda}_i (b_i - g_i(x_1,\ldots,x_n)) = 0, \quad i = 1,\ldots,m$$

$$\bar{\lambda}_i \ge 0, \quad i = 1,\ldots,m$$

- Theorem : Under regularity conditions: any point $(x_1,\ldots,x_n)$ that satisfies

  Lagrangian optimization problem yields optimal solution of Optimisation problem

- Interpretation multipliers: shadow price for constraint.

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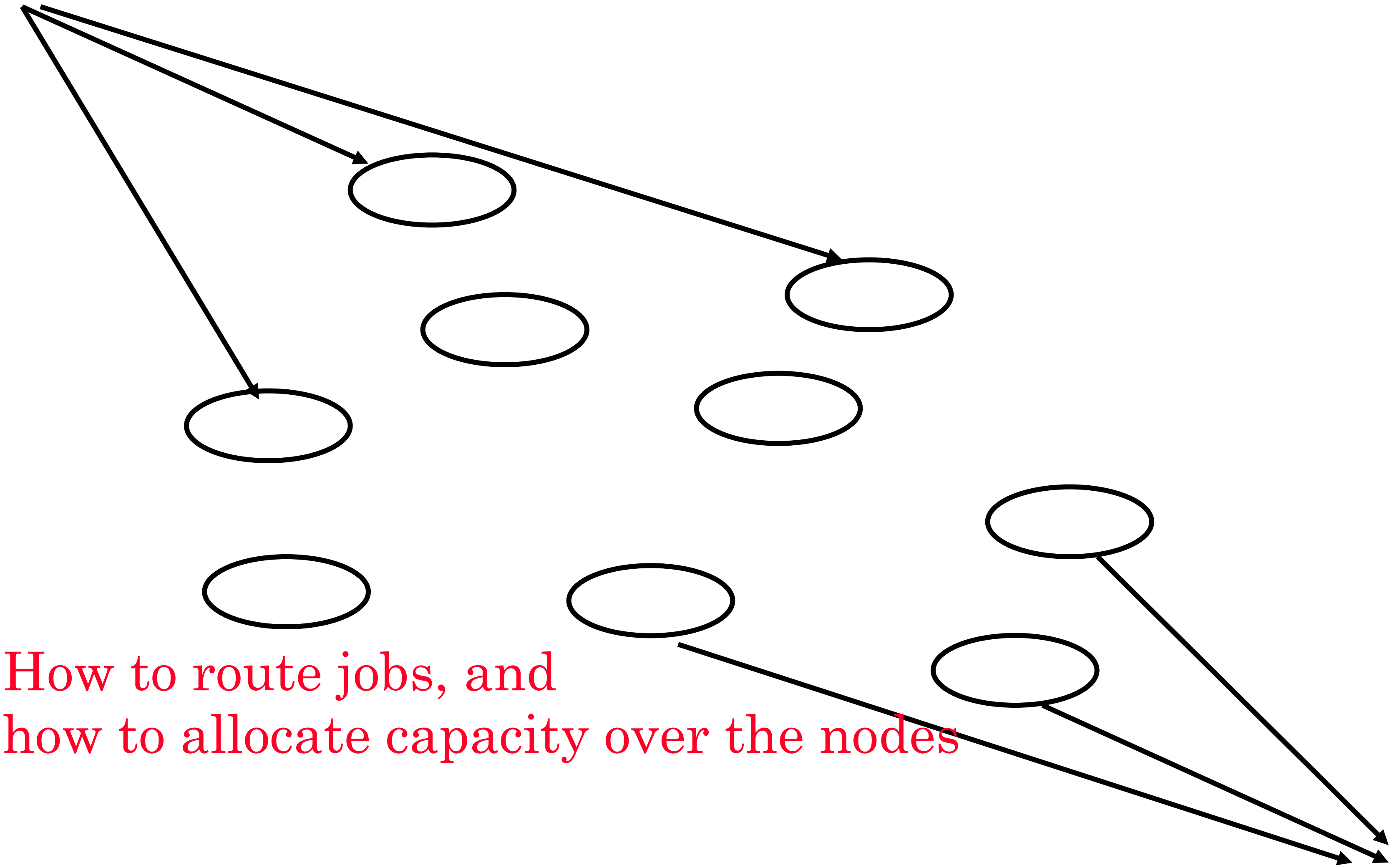
- Source



- How to route jobs, and
- how to allocate capacity over the nodes

- sink

Optimal design of Kelly / Whittle network (1)

- Transition rates

$$q(n, n-e_j+e_k) = \frac{\phi(n-e_j)}{\phi(n)}\lambda_{jk}$$

$$\lambda_{jk} = \mu_j p_{jk}$$

$$q(n, n-e_j) = \frac{\phi(n-e_j)}{\phi(n)}\lambda_{j0}$$

$$q(n, n+e_k) = \frac{\phi(n)}{\phi(n)}\lambda_{0k}$$

- Routing rules for open network to clear input traffic as efficiently as possible
- Cost per time unit in state n : *a(n)*
- Cost for routing j→k : b_{jk}
- Design : b_j0=+∞ : cannot leave from j; sequence of queues
- Expected cost rate

$$C = A(\alpha) + \sum_{j,k} b_{jk}\rho_j\lambda_{jk}$$

$$A(\rho) = \frac{\sum_{n\in S} a(n)\phi(n)\prod_{j=1}^{J}\rho_j^{n_j}}{\sum_{n\in S}\phi(n)\prod_{j=1}^{J}\rho_j^{n_j}}$$

Optimal design of Kelly / Whittle network (2)

- Transition rates $$q(n, n-e_j+e_k) = \frac{\phi(n-e_j)}{\phi(n)}\lambda_{jk}$$
- Given: input traffic $\mu_0 p_{0k}$
- Maximal service rate $\mu_j = \sum_k \lambda_{jk} \le \bar{\mu}_j$

$$A(\rho) = \frac{\sum_{n\in S} a(n)\phi(n)\prod_{j=1}^{J} \rho_j^{n_j}}{\sum_{n\in S} \phi(n)\prod_{j=1}^{J} \rho_j^{n_j}}$$

- Optimization problem : minimize costs $$C = A(\alpha) + \sum_{j,k} b_{jk}\rho_j\lambda_{jk}$$
- Under constraints

$$\sum_{k=0} \rho_j\lambda_{jk} = \sum_{k=0} \rho_k\lambda_{kj}, j = 1,\ldots,J$$

$$\sum_{k=0} \lambda_{jk} \le \bar{\mu}_j, j = 1,\ldots,J$$

$$\rho_j \ge 0, j = 1,\ldots,J$$

$$\rho_0 = 1$$

$$\lambda_{jk} \ge 0, j = 1,\ldots,J, k = 0,\ldots,J$$

$$\lambda_{0k} \text{ fixed}$$

Optimal design of Kelly / Whittle network (3)

- Optimisation problem

$$\min C(\{\rho_j, p_{jk}\}) = A(\rho) + \sum_{j,k} b_{jk}\rho_j\lambda_{jk}$$

- s.t.

$$\sum_{k=0} \rho_j\lambda_{jk} = \sum_{k=0} \rho_k\lambda_{kj}, j = 1,\ldots,J$$

$$\sum_{k=0} \lambda_{jk} \le \bar{\mu}_j, j = 1,\ldots,J$$

$$\rho_j \ge 0, j = 1,\ldots,J$$

$$\rho_0 = 1$$

$$\lambda_{jk} \ge 0, j = 1,\ldots,J, k = 0,\ldots,J$$

$$\lambda_{0k} \text{ fixed}$$

- Lagrangian form

$$L = C + \sum_{j=0} \sum_{k=0} \xi_j(\rho_k\lambda_{kj} - \rho_j\lambda_{jk})$$

$$+ \sum_{j=0} \eta_j(\sum_{k=0} \lambda_{jk} - \bar{\mu}_j) - \sum_{j=0} \kappa_j\rho_j - \sum_{j,k=0} \vartheta_{jk}\lambda_{jk}$$

$$\xi_0 = \eta_0 = \kappa_0 = \vartheta_{00} = 0$$

- KT-conditions

$$\frac{\partial L}{\partial \rho_j} = 0, \quad j = 1,\dots,J$$

$$\frac{\partial L}{\partial \lambda_{jk}} = 0, \quad j,k = 0,\dots,J$$

$$\sum_{k=0} \xi_j(\rho_k \lambda_{kj} - \rho_j \lambda_{jk}) = 0, \quad j = 1,\dots,J$$

$$\eta_j(\sum_{k=0} \lambda_{jk} - \overline{\mu}_j) = 0, \quad j = 1,\dots,J$$

$$\kappa_j \rho_j = 0, \quad j = 1,\dots,J$$

$$\vartheta_{jk} \lambda_{jk} = 0, \quad j = 1,\dots,J$$

$$\xi_j, \eta_j, \kappa_j, \vartheta_{jk} \geq 0$$

- Computing derivatives:

$$\frac{\partial L}{\partial \alpha_j} = c_j \lambda_j + \sum_k b_{jk} \lambda_{jk} - \lambda_j \xi_j + \sum_k \xi_k \lambda_{jk} - \kappa_j$$

$$\frac{\partial L}{\partial \lambda_{jk}} = b_{jk} \alpha_j - \xi_j \alpha_j + \xi_k \alpha_j + \eta_j - \vartheta_{jk}$$

$$c_j = \frac{1}{\lambda_j} \frac{\partial A(\alpha)}{\partial \alpha_j}$$

Optimal design of Kelly / Whittle network (5)

- Theorem : (i) the marginal costs of input satisfy

$$\xi_j \le c_j + \min_k (b_{jk} + \xi_k), j = 1, \ldots, J$$
$$\xi_0 = 0$$

- with equality for those nodes j which are used in the optimal design.
- (ii) If the routing j→k is used in the optimal design the equality holds in (i) and the minimum in the rhs is attained at given k.
- (iii) If node j is not used in the optimal design then $\alpha_j = 0$. If it is used but at less that full capacity then $c_j = 0$.
- Dynamic programming equations for nodes that are used

$$\xi_j = c_j + \min_k (b_{jk} + \xi_k)$$
$$\xi_0 = 0$$

Optimal design of Kelly / Whittle network (6)

- PROOF: Kuhn-Tucker conditions :

$$c_j \lambda_j + \sum_k b_{jk} \lambda_{jk} - \lambda_j \xi_j + \sum_k \xi_k \lambda_{jk} \geq 0 \qquad (*)$$

$$\text{and} = 0 \text{ if } \alpha_j > 0$$

$$b_{jk} \alpha_j - \xi_j \alpha_j + \xi_k \alpha_j + \eta_j \geq 0 \qquad (**)$$

$$\text{and} = 0 \text{ if } \lambda_{jk} > 0$$

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