UNIVERSITY OF TWENTE.

Markovian Queues and Stochastic Networks

Lecture 4

Richard J. Boucherie

Stochastic Operations Research

Time-reversed process and Kelly's Lemma – 1

- Stationary Markov chain $\{N(t)\}$
- Time-reversed process $\{N^r(t)\} = \{N(\tau - t)\}$.

$$\mathbb{P}(N(t) = \mathbf{n} | N(t{+}h) = \mathbf{n}') = \frac{\mathbb{P}(N(t) = \mathbf{n})}{\mathbb{P}(N(t+h) = \mathbf{n}')} \mathbb{P}(N(t{+}h) = \mathbf{n}' | N(t) = \mathbf{n}).$$

Theorem (4.1.2)

Let $\{N(t),\ t \in \mathbb{R}\}$ *be stationary Markov chain with transition rates* $q(\mathbf{n}, \mathbf{n}')$*,* $\mathbf{n}, \mathbf{n}' \in S$*, and equilibrium distribution* $\pi(\mathbf{n})$*,* $\mathbf{n} \in S$*. The time-reversed process* $\{N(\tau - t),\ t \in \mathbb{R}\}$ *is a conservative, stable, regular, irreducible continuous-time stationary Markov chain with transition rates* $q^r(\mathbf{n}, \mathbf{n}')$*,* $\mathbf{n}, \mathbf{n}' \in S$ *given by*

$$q^r(\mathbf{n}, \mathbf{n}') = \frac{\pi(\mathbf{n}')}{\pi(\mathbf{n})} q(\mathbf{n}', \mathbf{n}),$$

and the same equilibrium distribution $\pi(\mathbf{n})$*,* $\mathbf{n} \in S$*.*

Time-reversed process and Kelly's Lemma – 2

Theorem (4.1.3 Kelly's lemma)

Let $\{N(t),\ t \in \mathbb{R}\}$ *be a stationary Markov chain with transition rates* $q(\mathbf{n}, \mathbf{n}')$*,* $\mathbf{n}, \mathbf{n}' \in S$*. If we can find a collection of numbers* $q^r(\mathbf{n}, \mathbf{n}')$*,* $\mathbf{n}, \mathbf{n}' \in S$*, such that*

$$\sum_{\mathbf{n}' \neq \mathbf{n}} q(\mathbf{n}, \mathbf{n}') = \sum_{\mathbf{n}' \neq \mathbf{n}} q^r(\mathbf{n}, \mathbf{n}'), \quad \mathbf{n} \in S,$$

and a distribution $\pi = (\pi(\mathbf{n}),\ \mathbf{n} \in S)$ *such that*

$$\pi(\mathbf{n}) q^r(\mathbf{n}, \mathbf{n}') = \pi(\mathbf{n}') q(\mathbf{n}', \mathbf{n}), \quad \mathbf{n}, \mathbf{n}' \in S,$$

then $q^r(\mathbf{n}, \mathbf{n}')$*,* $\mathbf{n}, \mathbf{n}' \in S$*, are the transition rates of the time-reversed Markov chain* $\{N(\tau - t),\ t \in \mathbb{R}\}$ *and* $\pi(\mathbf{n})$*,* $\mathbf{n} \in S$*, is the equilibrium distribution of both Markov chains.*

Example: $M|M|1$ queue

- Poisson arrival rate λ and service rate μ, $\rho = \lambda/\mu < 1$.
- Departure rate λ.
- Guess for arrival rate time-reversed process: $q^r(\mathbf{n}, \mathbf{n}+1) = \lambda$.
- Further guess could then be $q^r(\mathbf{n}, \mathbf{n}-1) = \mu$
- Educated guess for the equilibrium distribution is $\pi(\mathbf{n}) = (1-\rho)\rho^{\mathbf{n}}$,
- Kelly's lemma 2 is satisfied.
- Time-reversed process is an $M|M|1$ queue with Poisson arrivals at rate λ and service rate μ.

Example: Kelly-Whittle network – 1

- Transition rates Kelly-Whittle network:

$$q(\mathbf{n},\mathbf{n}') = \begin{cases} \dfrac{\psi(\mathbf{n}-\mathbf{e}_i)}{\phi(\mathbf{n})}\mu_i p_{ij}, & \text{if } \mathbf{n}' = \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j,\ i,j = 0,\dots,J, \\ 0, & \text{otherwise.} \end{cases}$$

- Kelly's lemma: the time-reversed routing process is the Markov chain with transition probabilities

$$p^r_{ij} := \frac{\lambda_j}{\lambda_i} p_{ji}, \quad i,j = 0,\dots,J.$$

- Natural guess for the transition rates of the time-reversed process $\{N^r(t)\}$ is, for $\mathbf{n} \neq \mathbf{n}'$,

$$q^r(\mathbf{n},\mathbf{n}') = \begin{cases} \dfrac{\psi(\mathbf{n}-\mathbf{e}_i)}{\phi(\mathbf{n})}\mu_i p^r_{ij}, & \text{if } \mathbf{n}' = \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j,\ i,j = 0,\dots,J, \\ 0, & \text{otherwise.} \end{cases}$$

Example: Kelly-Whittle network – 2

- Observe that

$$
\begin{aligned}
\sum_{\mathbf{n}'\neq\mathbf{n}} q(\mathbf{n},\mathbf{n}') &= \sum_{i,j=0}^{J} \frac{\psi(\mathbf{n}-\mathbf{e}_i)}{\phi(\mathbf{n})}\mu_i p_{ij} = \sum_{i=0}^{J} \frac{\psi(\mathbf{n}-\mathbf{e}_i)}{\phi(\mathbf{n})}\mu_i, \\
\sum_{\mathbf{n}'\neq\mathbf{n}} q^r(\mathbf{n},\mathbf{n}') &= \sum_{i,j=0}^{J} \frac{\psi(\mathbf{n}-\mathbf{e}_i)}{\phi(\mathbf{n})}\mu_i p^r_{ij} = \sum_{i,j=0}^{J} \frac{\psi(\mathbf{n}-\mathbf{e}_i)}{\phi(\mathbf{n})}\mu_i \frac{\lambda_j}{\lambda_i} p_{ji} = \sum_{i=0}^{J} \frac{\psi(\mathbf{n}-\mathbf{e}_i)}{\phi(\mathbf{n})}\mu_i,
\end{aligned}
$$

- Educated guess: $\pi(\mathbf{n}) = G_{KW}\phi(\mathbf{n})\prod_{j=1}^{J}\rho_j^{n_j}$, $\mathbf{n}\in S$,
- Then

$$
\begin{aligned}
\pi(\mathbf{n})q^r(\mathbf{n},\mathbf{n}') &= G_{KW}\phi(\mathbf{n})\prod_{k=1}^{J}\rho_k^{n_k}\frac{\psi(\mathbf{n}-\mathbf{e}_i)}{\phi(\mathbf{n})}\mu_i\frac{\lambda_j}{\lambda_i}p_{ji} \\
&= G_{KW}\phi(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)\prod_{k=1}^{J}\rho_k^{n_k-\delta_{ki}+\delta_{kj}}\frac{\psi(\mathbf{n}-\mathbf{e}_i)}{\phi(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)}\mu_j p_{ji} \\
&= \pi(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j,\mathbf{n}).
\end{aligned}
$$

UNIVERSITY OF TWENTE.

Markovian Queues and Stochastic Networks

Lecture 4

Richard J. Boucherie

Stochastic Operations Research

Queue disciplines – 1

Definition (4.2.1 (κ, γ, δ)-protocol)

Customers ordered: if queue contains n customers then customers in positions $1, \ldots, n$, $n \in \mathbb{N}$. Queue operation:

- a customer of class c requires a negative-exponentially distributed amount of service with rate $\mu(c)$;
- if $n > 0$ customers present service at rate $\kappa(n) > 0$;
- fraction $\gamma(\ell, n)$ of service to customer in position ℓ; if customer in position ℓ completes service then customers in positions $\ell + 1, \ell + 2, \ldots, n$ move to $\ell, \ell + 1, \ldots, n - 1$;
- arriving customer into position ℓ with probability $\delta(\ell, n + 1)$; customers in positions $\ell, \ell + 1, \ldots, n$ move to positions $\ell + 1, \ell + 2, \ldots, n + 1$.

$$\sum_{\ell=1}^{n} \gamma(\ell, n) = 1, \quad \sum_{\ell=1}^{n} \delta(\ell, n) = 1.$$

Queue disciplines – 2

	$\kappa(n)$	$\gamma(\ell, n)$	$\delta(\ell, n)$
FIFO	$\mathbb{1}(n > 0)$	$\mathbb{1}(\ell = 1)$	$\mathbb{1}(\ell = n)$
LIFO-PR	$\mathbb{1}(n > 0)$	$\mathbb{1}(\ell = n)$	$\mathbb{1}(\ell = n)$
PS	$\mathbb{1}(n > 0)$	$1/n$	$1/n$
INF	n	$1/n$	$1/n$

Multi-class LIFO-PR queue – 1

- Customers of class c arrive according to a Poisson process with rate $\lambda(c)$, $\rho(c) = \lambda(c)/\mu(c) < 1$,
- Let $\mathbf{c} = (c(1), \ldots, c(n))$ record the class of the customers in position i, $i = 1, \ldots, n$,
- Let $\{N(t)\}$ record the state of the Markov chain at

$$S = \{\mathbf{c} : \mathbf{c} = (c(1), \ldots, c(n)),\ c(i) \in \{1, \ldots, C\}\}.$$

- Transition rates

$$q(\mathbf{c}, \mathbf{c}') = \begin{cases} \lambda(c), & \text{if } \mathbf{c}' = (c(1), \ldots, c(n), c), \\ \mu(c(n)), & \text{if } \mathbf{c}' = (c(1), \ldots, c(n-1)). \end{cases}$$

- Natural guess for time-reversed queue is the queue multi-class LIFO-PR queue with the same rates:

$$q^r(\mathbf{c}, \mathbf{c}') = \begin{cases} \lambda(c), & \text{if } \mathbf{c}' = (c(1), \ldots, c(n), c), \\ \mu(c(n)), & \text{if } \mathbf{c}' = ((c(1), \ldots, c(n-1)). \end{cases}$$

Multi-class LIFO-PR queue – 2

- Guess equilibrium distribution:

$$\pi(\mathbf{c}) = G\prod_{i=1}^{n}\rho(c(i)), \quad \mathbf{c} = (c(1),\ldots,c(n)) \in S.$$

- Check conditions Kelly's lemma:

$$\sum_{\mathbf{c}'\neq\mathbf{c}} q(\mathbf{c},\mathbf{c}') = \sum_{c=1}^{C}\lambda(c) + \mu(c(n)) = \sum_{\mathbf{c}'\neq\mathbf{c}} q^r(\mathbf{c},\mathbf{c}').$$

- For $\mathbf{c} = (c(1),\ldots,c(n))$, $\mathbf{c}' = (c(1),\ldots,c(n),c)$, $\mathbf{c}'' = (c(1),\ldots,c(n-1))$,

$$\begin{aligned} \pi(\mathbf{c})q^r(\mathbf{c},\mathbf{c}') = \pi(\mathbf{c}')q(\mathbf{c}',\mathbf{c}) \quad &\Leftrightarrow \quad \lambda(c) = \rho(c)\mu(c), \\ \pi(\mathbf{c})q^r(\mathbf{c},\mathbf{c}'') = \pi(\mathbf{c}'')q(\mathbf{c}'',\mathbf{c}) \quad &\Leftrightarrow \quad \rho(c(n))\mu(c(n)) = \lambda(c(n)), \end{aligned}$$

UNIVERSITY OF TWENTE.

Markovian Queues and Stochastic Networks

Lecture 4

Richard J. Boucherie
Stochastic Operations Research

Networks: customer types and fixed routes – 1

- Network of J queues.
- Customers of types $u = 1, \ldots, U$, arrive to a according to a Poisson process with rate $\mu_0(u)$, $u = 1, \ldots, U$.
- Customer type uniquely determines route through the network along the sequence of queues

$$r(u,1), r(u,2), \ldots, r(u, L(u)).$$

- Customer may visit the same queue at multiple stages.
- Queue j operates according to the $(\kappa_j, \gamma_j, \delta_j)$-protocol.
- Let $c_j(\ell) = (u_j(\ell),\ s_j(\ell))$, with $u_j(\ell)$ the type and $s_j(\ell)$ the stage of the customer in position ℓ in queue j.
- State of queue j is $\mathbf{c}_j = (c_j(1), \ldots, c_j(n_j))$.
- State of the network is $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_J)$.

Networks: customer types and fixed routes – 2

- Let $\{N(t)\}$ record state of Markov chain at state space $S = \{\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_J)\}$.
- For $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_J)$, let

  $C^{(u,s)}_{(\ell,j),(\ell',k)}\mathbf{c}$ denote state $\mathbf{c}'$ obtained from state $\mathbf{c}$ by removing customer of type u in stage s in position ℓ from queue j and adding that customer in position ℓ' to queue k.

- Transition rates (more precise in reader)

$$q(\mathbf{c}, \mathbf{c}') =$$

$$\begin{cases} \mu_0(u)\delta_k(\bar{\ell}', n_k + 1), & \text{if } \mathbf{c}' = C^{(u,0)}_{(0,0),(\ell',k)}\mathbf{c}, \\ \mu_j(u)\kappa_j(n_j)\gamma_j(\bar{\ell}, n_j)\delta_k(\bar{\ell}'_k, n_k + 1), & \text{if } \mathbf{c}' = C^{(u,s)}_{(\ell,j),(\ell',k)}\mathbf{c}, \\ \mu_j(u)\kappa_j(n_j)\gamma_j(\bar{\ell}, n_j), & \text{if } \mathbf{c}' = C^{(u,L(u))}_{(\ell,j),(0,0)}\mathbf{c}. \end{cases}$$

Networks: customer types and fixed routes – 3

- $\lambda_j(u,s)$: arrival rate of type u to queue $j = r(u,s)$. Then

$$\lambda_j(u,s) = \begin{cases} \mu_0(u), & \text{if } j = r(u,s), \\ 0, & \text{otherwise.} \end{cases}$$

- Mean amount of work arriving to queue j per unit time:

$$\rho_j = \sum_{u=1}^{U} \sum_{s=1}^{L(u)} \frac{\lambda_j(u,s)}{\mu_j(u)}, \quad j = 1, \dots, J.$$

- Let $\rho_j(c_j(\ell)) = \lambda_j(u_j(\ell), s_j(\ell))/\mu_j(u_j(\ell))$.

Networks: customer types and fixed routes – 4

Theorem (4.3.1 Network with fixed routes)

Let queue j operate according to the $(\kappa_j, \gamma_j, \delta_j)$-protocol. Negative-exponential(1) service requirements for all customers at all queues. Let

$$\pi_j(\mathbf{c}_j) = G_j \prod_{\ell=1}^{n} \frac{\rho_j(c_j(\ell))}{\kappa_j(\ell)}, \quad G_j = \left[\sum_{n=0}^{\infty} \prod_{\ell=1}^{n} \frac{\rho_j}{\kappa_j(\ell)} \right]^{-1} < \infty,$$

Then

$$\pi(\mathbf{c}) = \prod_{j=1}^{J} \pi_j(\mathbf{c}_j), \quad \mathbf{c} \in S.$$

Networks: customer types and fixed routes – 5

Proof. Natural guess for the reversed process:

- customers of type u arrive according to a Poisson process with rate $\mu_0(u)$ to queue $L(u)$
- and follow the reversed route $r(u, L(u)), \ldots, r(u, 1)$,
- and that the transition rates have the role of γ and δ reversed:

$$q^r(\mathbf{c}', \mathbf{c}) =$$

$$\begin{cases} \kappa_k(n_k + 1)\delta_k(\bar{\ell}', n_k + 1), & \text{if } \mathbf{c}' = C^{(u,0)}_{(0,0),(\ell',k)}\mathbf{c}, \\ \kappa_k(n_k + 1)\delta_k(\bar{\ell}'_k, n_k + 1)\gamma_j(\bar{\ell}, n_j), & \text{if } \mathbf{c}' = C^{(u,s)}_{(\ell,j),(\ell',k)}\mathbf{c}, \\ \mu_0(u)\gamma_j(\bar{\ell}, n_j), & \text{if } \mathbf{c}' = C^{(u,L(u))}_{(\ell,j),(0,0)}\mathbf{c}. \end{cases}$$

Networks: customer types and fixed routes – 6

$$\begin{aligned}
\sum_{\mathbf{c}'} q(\mathbf{c}, \mathbf{c}') &= \sum_{u=1}^{U} \mu_0(u) + \sum_{j=1}^{J} \sum_{\ell_j=1}^{n_j} \kappa_j(n_j)\gamma_j(\ell_j, n_j), \\
\sum_{\mathbf{c}'} q^r(\mathbf{c}, \mathbf{c}') &= \sum_{u=1}^{U} \mu_0(u) + \sum_{k=1}^{J} \sum_{\ell_k=1}^{n_k} \kappa_k(n_k)\delta_k(\ell_k, n_k),
\end{aligned}$$

For $\mathbf{c}' = C^{(u,s)}_{(\ell,j),(\ell',k)}\mathbf{c}$, with $j, k \neq 0$, we have

$$\pi(\mathbf{c})q(\mathbf{c}, \mathbf{c}') = \pi(\mathbf{c})\kappa_j(n_j)\gamma_j(\bar{\ell}, n_j)\delta_k(\bar{\ell}'_k, n_k + 1),$$

$$\begin{aligned}
&\pi(\mathbf{c}')q^r(\mathbf{c}', \mathbf{c}) \\
&= \pi(\mathbf{c}) \frac{\rho_k(c_k(\ell'_k))}{\rho_j(c_j(\ell_j))} \frac{\kappa_j(n_j)}{\kappa_k(n_k + 1)} \kappa_k(n_k + 1)\delta_k(\bar{\ell}'_k, n_k + 1)\gamma_j(\bar{\ell}, n_j).
\end{aligned}$$

UNIVERSITY OF TWENTE.

Markovian Queues and Stochastic Networks

Lecture 4

Richard J. Boucherie

Stochastic Operations Research

Quasi-reversibility – 1

- Burke's theorem: output process from a reversible queue before t, the input process after t and the state at t independent.
- Quasi-reversibility formalises this independence property.
- $\{N(t),\ t \in \mathbb{R}\}$ Markov process, state space S, states $\mathbf{n} \in S$, transition rates $q(\mathbf{n}, \mathbf{n}')$, equilibrium distribution $\pi(\mathbf{n})$.
- Let $S(c, \mathbf{n}) \subset S$ denote the set of states that may be obtained from state $\mathbf{n}$ when a customer of class c arrives to the queue.
- Let $\{A_c(t),\ t \in \mathbb{R}\}$ and $\{D_c(t),\ t \in \mathbb{R}\}$ record the arrival and departure processes of customers of class c.

Quasi-reversibility – 2

Definition (4.4.1 Quasi-reversibility)

The stationary Markov chain $\{N(t)\}$ is quasi-reversible if for all $t \in \mathbb{R}$ the state at time t, $N(t)$, is independent of $\{A_c(s),\ s > t\}$, the arrival process of class c customers after time t, and independent of $\{D_c(s),\ s < t\}$, the departure process of class c customers prior to time t, $c = 1, \ldots, C$.

Theorem (4.4.2)

If $\{N(t)\}$ is a quasi-reversible Markov chain, then

(i) *the arrival processes $\{A_c(t),\ t \in \mathbb{R}\}$, $c = 1, \ldots, C$, form independent Poisson processes;*

(ii) *the departure processes $\{D_c(t),\ t \in \mathbb{R}\}$, $c = 1, \ldots, C$, form independent Poisson processes.*

Quasi-reversibility – 3

Algebraic characterisation of quasi-reversibility:

$$\begin{aligned}\lambda(c) &= \sum_{\mathbf{n}' \in S(c,\mathbf{n})} q(\mathbf{n}, \mathbf{n}'), \\ \lambda(c) &= \sum_{\mathbf{n}' \in S(c,\mathbf{n})} q^r(\mathbf{n}, \mathbf{n}'),\end{aligned}$$

so that

$$\sum_{\mathbf{n}' \in S(c,\mathbf{n})} \pi(\mathbf{n}) q(\mathbf{n}, \mathbf{n}') = \sum_{\mathbf{n}' \in S(c,\mathbf{n})} \pi(\mathbf{n}') q(\mathbf{n}', \mathbf{n}).$$

- In equilibrium the flow out of state **n** due to a customer of type *c* arriving to the queue balances with the probability flow into state **n** due to a customer of type *c* departing from the queue.

Symmetric queue – 1

Definition (4.2.6 Symmetric queue)

A queue that operates under the (κ, γ, δ)-protocol is called symmetric if

$$\gamma(\ell, n) = \delta(\ell, n), \quad \ell = 1, \ldots, n,\ n \in \mathbb{N}.$$

Theorem (4.4.6)

Let $\{N(t)\}$ record the state of a symmetric queue to which customers of class c arrive according to independent Poisson processes with rate $\lambda(c)$, $c = 1, \ldots, C$. Then $\{N(t)\}$ is quasi-reversible.

Symmetric queue – 2

Proof.

- Transition rates, for $\mathbf{c} = (c(1), \ldots, c(n))$, $\mathbf{c}' \neq \mathbf{c}$,

$$q(\mathbf{c}, \mathbf{c}') = \begin{cases} \lambda(c)\gamma(\ell, n+1), & \text{if } \mathbf{c}' = (c(1), \ldots, c(\ell), c, c(\ell+1), \ldots, c(n)), \\ \mu_{c(\ell)}\kappa(n)\gamma(\ell, n), & \text{if } \mathbf{c}' = c(1), \ldots, c(\ell-1), c(\ell+1), \ldots, c(n)). \end{cases}$$

- Arrivals of class c customers independent Poisson processes $\Rightarrow N(t)$ independent of $\{A_c(s),\ s > t\}$.
- Transition rates of time-reversed queue: $q^r = q$.
- Arrival process to the time-reversed queue is Poisson process.
- Arrivals in the time-reversed process coincide with departures of $\{N(t)\} \Rightarrow N(t)$ is independent of $\{D_c(s),\ s < t\}$.

UNIVERSITY OF TWENTE.

Markovian Queues and Stochastic Networks

Lecture 4

Richard J. Boucherie

Stochastic Operations Research

Quasi-reversible queues and fixed routes – 1

- Network of J quasi-reversible queues.
- Customers of types $u = 1, \ldots, U$, arrive to a according to a Poisson process with rate $\mu_0(u)$, $u = 1, \ldots, U$.
- Customer type uniquely determines route along the sequence of queues $r(u,1), r(u,2), \ldots, r(u, L(u))$.
- State of queue j: $\{N_j(t)\}$, state space S_j, transition rates $q_j(\mathbf{c}_j, \mathbf{c}_j')$, customers of class (u,s) arrive according to Poisson process with rate

$$\lambda_j(u,s) \sum_{\mathbf{c}_j' \in S_j((u,s),\mathbf{c}_j)} q_j(\mathbf{c}_j, \mathbf{c}_j'),$$

- Equilibrium distribution $\pi_j = (\pi_j(\mathbf{c}_j),\ \mathbf{c}_j \in S_j)$ satisfies

$$\sum_{\mathbf{c}_j' \in S_j(c,\mathbf{c}_j)} \pi_j(\mathbf{c}_j) q_j(\mathbf{c}_j, \mathbf{c}_j') = \sum_{\mathbf{c}_j' \in S_j(c,\mathbf{c}_j)} \pi_j(\mathbf{c}_j') q_j(\mathbf{c}_j', \mathbf{c}_j).$$

Quasi-reversible queues and fixed routes – 2

- For $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_J)$, and $j, k = 0, \ldots, J$, let

  $C^{(u,s)}_{j,k}\mathbf{c}$ denote the set of states $\mathbf{c}'$ obtained from state $\mathbf{c}$ by removing the customer of type u in stage s from queue j and adding that customer in stage $s+1$ to queue k:

$$(C^{(u,s)}_{j,k}\mathbf{c})_i = \begin{cases} \{\mathbf{c}_i\}, & \text{if } i \neq j, k, \\ S_k((u, s+1), \mathbf{c}_k), & \text{if } i = k, \\ \{\mathbf{c}'_j \text{ s.t. } \mathbf{c}_j \in S_j((u,s), \mathbf{c}'_j)\}, & \text{if } i = j, \end{cases}$$

Quasi-reversible queues and fixed routes – 3

- Transition rates, for $u = 1, \ldots, U$, $\mathbf{c} \neq \mathbf{c}'$, $\mathbf{c}, \mathbf{c}' \in S$,

$$q(\mathbf{c}, \mathbf{c}') =$$

$$\begin{cases} q_k(\mathbf{c}_k, \mathbf{c}'_k), & \text{if } \mathbf{c}' \in C_{0,k}^{(u,1)}\mathbf{c}, & \text{(arrival)} \\ q_j(\mathbf{c}_j, \mathbf{c}'_j) \dfrac{q_k(\mathbf{c}_k, \mathbf{c}'_k)}{\sum_{\mathbf{c}'_k \in S_k((u,s+1),\mathbf{c}_k)} q_k(\mathbf{c}_k, \mathbf{c}'_k)}, & \text{if } \mathbf{c}' \in C_{j,k}^{(u,s)}\mathbf{c}, & \text{(routing)} \\ q_j(\mathbf{c}_j, \mathbf{c}'_j), & \text{if } \mathbf{c}' \in C_{j,0}^{(u,L(u))}\mathbf{c}, & \text{(departure)} \\ q_j(\mathbf{c}_j, \mathbf{c}'_j), & \text{if } \mathbf{c}_j, \mathbf{c}'_j \in S_j,\ \mathbf{c}'_i = \mathbf{c}_i,\ i \neq j, & \text{(internal)} \end{cases}$$

- Quasi-reversibility implies that

$$\frac{q_k(\mathbf{c}_k, \mathbf{c}'_k)}{\sum_{\mathbf{c}'_k \in S_k((u,s+1),\mathbf{c}_k)} q_k(\mathbf{c}_k, \mathbf{c}'_k)} = \frac{q_k(\mathbf{c}_k, \mathbf{c}'_k)}{\lambda_k(u, s+1)}.$$

Quasi-reversible queues and fixed routes – 3

Theorem (4.5.1)

Let $\{N(t)\} = \{(N_1(t), \ldots, N_J(t))\}$ *record the state of a network of J quasi-reversible queues to which customers of types* $u = 1, \ldots, U$ *arrive according to independent Poisson processes with rates* $\mu_0(u)$ *to follow a fixed route* $r(u,1), r(u,2), \ldots, r(u, L(u))$*,* $u = 1, \ldots, U$*. Let* S_j*,* q_j*, and* π_j *denote the state space, transition rates and equilibrium distribution of queue j,* $j = 1, \ldots, J$*. Then* $\{N(t)\}$ *has equilibrium distribution*

$$\pi(\mathbf{c}_1, \ldots, \mathbf{c}_J) = \prod_{j=1}^{J} \pi_j(\mathbf{c}_j), \quad (\mathbf{c}_1, \ldots, \mathbf{c}_J) \in S = S_1 \times \cdots \times S_J.$$

Quasi-reversible queues and fixed routes – 4

Proof. Natural guess for time-reversed process:

- customers of types $u = 1, \ldots, U$ arrive according to a Poisson process with rate $\mu_0(u)$,
- route through the network along the sequence of queues in reversed order $r(u, L(u)), \ldots, r(u, 1)$
- each queue operates according to its time-reversed transition rates: for $u = 1, \ldots, U$, $\mathbf{c} \neq \mathbf{c}'$, $\mathbf{c}, \mathbf{c}' \in S$,

$$q^r(\mathbf{c}', \mathbf{c}) =$$

$$\begin{cases} q_k^r(\mathbf{c}_k', \mathbf{c}_k), & \text{if } \mathbf{c}' \in C_{0,k}^{(u,1)}\mathbf{c}, & \text{(departure)} \\ q_k^r(\mathbf{c}_k', \mathbf{c}_k)\dfrac{q_j^r(\mathbf{c}_j', \mathbf{c}_j)}{\lambda_j(u,s)}, & \text{if } \mathbf{c}' \in C_{j,k}^{(u,s)}\mathbf{c}, & \text{(routing)} \\ q_j^r(\mathbf{c}_j', \mathbf{c}_j), & \text{if } \mathbf{c}' \in C_{j,0}^{(u,L(u))}\mathbf{c}, & \text{(arrival)} \\ q_j^r(\mathbf{c}_j', \mathbf{c}_j), & \text{if } \mathbf{c}_j, \mathbf{c}_j' \in S_j,\ \mathbf{c}_i' = \mathbf{c}_i,\ i \neq j. & \text{(internal)} \end{cases}$$

Quasi-reversible queues and fixed routes – 5

- For a routing transition from queue $j = r(u, s)$ to queue $k = r(u, s+1)$ it must be that $\lambda_j(u, s) = \lambda_k(u, s+1)$, which implies that

$$\pi_j(\mathbf{c}_j)\pi_k(\mathbf{c}_k)q_j(\mathbf{c}_j, \mathbf{c}_j')\frac{q_k(\mathbf{c}_k, \mathbf{c}_k')}{\lambda_k(u, s+1)} = \pi^r(\mathbf{c}_j')\pi_k^r(\mathbf{c}_k')q_k^r(\mathbf{c}_k', \mathbf{c}_k)\frac{q_j^r(\mathbf{c}_j', \mathbf{c}_j)}{\lambda_j(u, s)}.$$

UNIVERSITY OF TWENTE.

Markovian Queues and Stochastic Networks

Lecture 4

Richard J. Boucherie
Stochastic Operations Research