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Markovian Queues and Stochastic Networks

Lecture 3

Richard J. Boucherie

Stochastic Operations Research

Open network of $M|M|1$ queues

- Evolution number of customers in the queues recorded by Markov chain $\{N(t) = (N_1(t), \dots, N_J(t)),\ t \in \mathbb{R}\}$
- State space $S \subseteq \mathbb{N}_0^J$, states $\mathbf{n} = (n_1, \dots, n_J)$.
- If $\{N(t)\}$ is in state $\mathbf{n}$ and a customer routes from queue i to queue j then the next state is $\mathbf{n} - e_i + e_j$, $i, j = 0, \dots, J$.
- Queue 0 is introduced to represent the outside.
- If a customer routes from queue i to queue 0 then this customer leaves the network
- and if a customer routes from queue 0 to queue j then this customers enters the network at queue j, $j = 1, \dots, J$.
- State space $S = \mathbb{N}_0^J$.
- The transition rates of $\{N(t)\}$ for an open network are, for $\mathbf{n} \neq \mathbf{n}'$, $\mathbf{n}, \mathbf{n}' \in S$,

$$q(\mathbf{n}, \mathbf{n}') = \begin{cases} \mu_i p_{ij}, & \text{if } \mathbf{n}' = \mathbf{n} - e_i + e_j,\ i, j = 0, \dots, J, \\ 0, & \text{otherwise.} \end{cases}$$

Open network of $M|M|1$ queues

Theorem (3.1.4 Equilibrium distribution)

Consider the Markov chain $\{N(t)\}$ *at state space* $S = \mathbb{N}_0^J$ *for the open network of* $M|M|1$ *queues. Assume the routing matrix* $P = (p_{ij})$ *is irreducible and let* $\{\lambda_j\}$ *be the unique solution of the traffic equations. If* $\rho_j := \lambda_j/\mu_j < 1$, $j = 1, \ldots, J$, *then* $\{N(t)\}$ *has unique product-form equilibrium distribution*

$$\pi(\mathbf{n}) = \prod_{j=1}^{J}(1-\rho_j)\rho_j^{n_j} = \prod_{j=1}^{J}\pi_j(n_j), \quad \mathbf{n} \in S.$$

Moreover, the equilibrium distribution satisfies partial balance, for all $\mathbf{n} \in S$, $i = 0, \ldots, J$,

$$\sum_{j=0}^{J}\{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) - \pi(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j, \mathbf{n})\} = 0.$$

Proof of Theorem 3.1.4

$$\sum_{j=0}^{J} \{m(\mathbf{n})q(\mathbf{n},\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) - m(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j,\mathbf{n})\}$$

$$= \sum_{j=0}^{J} \left\{ \prod_{k=1}^{J} \rho_k^{n_k} \mu_i p_{ij} \mathbb{1}(\mathbf{n}-\mathbf{e}_i \in \mathbb{N}_0^J) - \prod_{k=1}^{J} \rho_k^{n_k-\delta_{ki}+\delta_{kj}} \mu_j p_{ji} \mathbb{1}(\mathbf{n}-\mathbf{e}_i \in \mathbb{N}_0^J) \right\}$$

$$\sum_{j=0}^{J} \{m(\mathbf{n})q(\mathbf{n},\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) - m(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j,\mathbf{n})\} \mathbb{1}(i=0)$$

$$= \left\{ \mu_0 - \sum_{j=1}^{J} \lambda_j p_{j0} \right\} \prod_{k=1}^{J} \rho_k^{n_k} \mathbb{1}(\mathbf{n} \in \mathbb{N}_0^J) = 0,$$

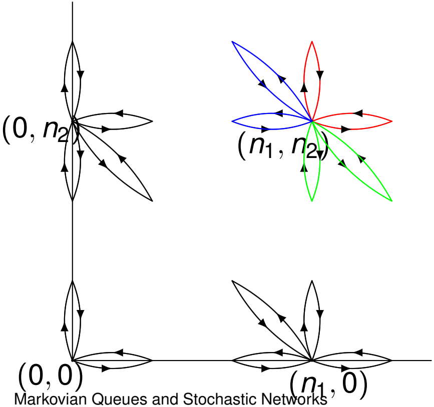
$$\sum_{j=0}^{J} \{m(\mathbf{n})q(\mathbf{n},\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) - m(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j,\mathbf{n})\} \mathbb{1}(i \neq 0)$$

$$= \left\{ \lambda_i - \mu_0 p_{0i} - \sum_{j=1}^{J} \lambda_j p_{ji} \right\} \prod_{k=1}^{J} \rho_k^{n_k-\delta_{ki}} \mathbb{1}(\mathbf{n}-\mathbf{e}_i \in \mathbb{N}_0^J) \mathbb{1}(i \neq 0) = 0.$$

Partial balance

Moreover, the equilibrium distribution satisfies partial balance, for all $\mathbf{n} \in S$, $i = 0, \ldots, J$,

$$\sum_{j=0}^{J} \{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\} = 0.$$



Closed network of $M|M|1$ queues

Theorem (3.1.5 Equilibrium distribution)

Consider Markov chain $\{N(t)\}$ *at state space* $S = S_M = \{\mathbf{n} : \sum_{j=1}^{J} n_j = M\}$ *for the closed network of* $M|M|1$ *queues containing* M *customers. Assume* $P = (p_{ij})$ *is irreducible and let* $\{\lambda_j\}$ *be the unique solution of the traffic equations such that* $\sum_{j=1}^{J} \lambda_j = 1$*. Let* $\rho_j := \lambda_j/\mu_j$*. Then* $\{N(t)\}$ *has unique product-form equilibrium distribution*

$$\pi(\mathbf{n}) = G_M \prod_{j=1}^{J} \rho_j^{n_j}, \quad \mathbf{n} \in S, \quad G_M = \left[\sum_{\mathbf{n}\in S} \prod_{j=1}^{J} \rho_i^{n_j}\right]^{-1}.$$

Moreover, the equilibrium distribution satisfies partial balance, for all $\mathbf{n} \in S$, $i = 1, \ldots, J$,

$$\sum_{j=1}^{J} \{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\} = 0.$$

Closed network of $M|M|1$ queues

Algorithm (3.1.8 Buzen's Algorithm)

Define $G(m,j)$, $m = 0,\dots,M$, $j = 1,\dots,J$. Set

$$\begin{aligned} G(0,j) &= 1, \quad j = 1,\dots,J, \\ G(m,1) &= \rho_1^m, \quad m = 0,\dots,M. \end{aligned}$$

For $j = 2,\dots,J$, $m = 1,\dots,M$, do

$$G(m,j) = G(m,j-1) + \rho_j G(m-1,j).$$

Then $G_M = G(M,J)^{-1}$.

- Buzen's algorithm yields G_m, $m = 1,\dots,M$, and marginals and means:

$$\begin{aligned} \pi_j(n_j) &= G_M \rho_j^{n_j} [G_{M-n_j}^{-1} - \rho_j G_{M-n_j-1}^{-1}], \quad n_j = 0,\dots,M-1, \\ \pi_j(M) &= G_M \rho_j^{n_j}, \\ \mathbb{E}[N_j] &= \textstyle\sum_{m=1}^{M} \rho_j^m \frac{G_M}{G_{M-m}}. \end{aligned}$$

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Stochastic Operations Research

Kelly-Whittle networks – 1

$$\sum_{j=0}^{J} \{m(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - m(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\}$$

$$= \sum_{j=0}^{J} \left\{ \prod_{k=1}^{J} \rho_k^{n_k} \mu_i p_{ij} \mathbb{1}(\mathbf{n} - \mathbf{e}_i \in \mathbb{N}_0^J) - \prod_{k=1}^{J} \rho_k^{n_k - \delta_{ki} + \delta_{kj}} \mu_j p_{ji} \mathbb{1}(\mathbf{n} - \mathbf{e}_i \in \mathbb{N}_0^J) \right\}$$

$$\sum_{j=0}^{J} \{m(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - m(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\} \mathbb{1}(i \neq 0)$$

$$= \left\{ \sum_{j=0}^{J} \lambda_i p_{ij} - \mu_0 p_{0i} - \sum_{j=1}^{J} \lambda_j p_{ji} \right\} \prod_{k=1}^{J} \rho_k^{n_k - \delta_{ki}} \mathbb{1}(\mathbf{n} - \mathbf{e}_i \in \mathbb{N}_0^J) \mathbb{1}(i \neq 0) = 0.$$

Kelly-Whittle networks – 2

$$\sum_{j=0}^{J} \{m(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - m(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\}$$

state dependent sojourn time in state **n**: $q(\mathbf{n}) \to \dfrac{q(\mathbf{n})}{\phi(\mathbf{n})}$

$$\neq \quad \sum_{j=0}^{J} \left\{ m(\mathbf{n}) \frac{q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)}{\phi(\mathbf{n})} - m(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) \frac{q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})}{\phi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)} \right\}$$

Kelly-Whittle networks – 3

$$\sum_{j=0}^{J} \{m(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - m(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\}$$

state dependent sojourn time in state **n**: $q(\mathbf{n}) \to \dfrac{q(\mathbf{n})}{\phi(\mathbf{n})}$

also scale $m(\mathbf{n})$: fraction of time spent in state **n**

$$= \sum_{j=0}^{J} \left\{ \phi(\mathbf{n})m(\mathbf{n})\frac{q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)}{\phi(\mathbf{n})} - \phi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)m(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)\frac{q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})}{\phi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)} \right\}$$

Kelly-Whittle networks – 4

$$\sum_{j=0}^{J} \{m(\mathbf{n})q(\mathbf{n},\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) - m(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j,\mathbf{n})\}$$

state dependent sojourn time in state **n**: $q(\mathbf{n}) \to \frac{q(\mathbf{n})}{\phi(\mathbf{n})}$

also scale $m(\mathbf{n})$: fraction of time spent in state **n**

and add a function $\psi(\mathbf{n}-\mathbf{e}_i)$ to the rates

$$= \sum_{j=0}^{J} \left\{ \phi(\mathbf{n})m(\mathbf{n})\psi(\mathbf{n}-\mathbf{e}_i)\frac{q(\mathbf{n},\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)}{\phi(\mathbf{n})} \right.$$

$$\left. -\phi(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)m(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)\psi(\mathbf{n}-\mathbf{e}_i)\frac{q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j,\mathbf{n})}{\phi(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)} \right\}$$

Kelly-Whittle networks – 5

Markov chain $\{N(t)\}$ at state space $S \subseteq \mathbb{N}_0^J$ with transition rates, for $\mathbf{n}' \neq \mathbf{n}$,

$$q(\mathbf{n}, \mathbf{n}') = \begin{cases} \dfrac{\psi(\mathbf{n} - \mathbf{e}_i)}{\phi(\mathbf{n})} \mu_i p_{ij}, & \text{if } \mathbf{n}' = \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j,\ i, j = 0, \ldots, J, \\ 0, & \text{otherwise,} \end{cases}$$

where $\psi : \mathbb{N}_0^J \to [0, \infty)$ and $\phi : \mathbb{N}_0^J \to (0, \infty)$.

We will consider closed networks as special case of open networks with $\mu_0 = 0$ and $p_{i0} = 0$, $i = 1, \ldots, J$.

Kelly-Whittle networks – 6

Theorem (Equilibrium distribution: Kelly-Whittle network)

Consider the Kelly-Whittle network $\{N(t)\}$ *at state space* $S \subseteq \mathbb{N}_0^J$. *Assume the routing matrix* $P = (p_{ij})$ *is irreducible and let* $\{\lambda_j\}$ *be solution of traffic equations. Let* $\rho_j = \lambda_j/\mu_j$. *Assume*

$$G_{KW}^{-1} = \sum_{\mathbf{n}\in S} \phi(\mathbf{n}) \prod_{j=1}^{J} \rho_j^{n_j} < \infty,$$

and that $\{N(t)\}$ *is irreducible. Then*

$$\pi(\mathbf{n}) = G_{KW}\phi(\mathbf{n}) \prod_{j=1}^{J} \rho_j^{n_j}, \quad \mathbf{n} \in S.$$

Moreover, π *satisfies* partial balance, *for all* $\mathbf{n} \in S$, $i = 0, \ldots, J$,

$$\sum_{j=0}^{J} \left\{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\right\} = 0.$$

Kelly-Whittle networks – 7

- Poisson arrivals
- Independent queues:

$$q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \begin{cases} \kappa_i(n_i)\mu_i p_{ij}, & i, j = 1, \ldots, J, \\ \kappa_i(n_i)\mu_i p_{i0}, & i = 1, \ldots, J, \\ \mu_0 p_{0j}, & j = 1, \ldots, J, \end{cases}$$

for $\kappa_i : \mathbb{N}_0 :\rightarrow (0, \infty)$, $i = 1, \ldots, J$.
Typical examples are, for $n \in \mathbb{N}$, $i = 1, \ldots, J$,

$$\begin{array}{rcll} \kappa_i(n) & = & 1, & \text{single server queue,} \\ \kappa_i(n) & = & \min(n, s), & s \text{ server queue,} \\ \kappa_i(n) & = & n, & \text{infinite server queue.} \end{array}$$

Kelly-Whittle networks – 8

- Poisson arrivals
- Independent queues:

$$q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \begin{cases} \kappa_i(n_i)\mu_i p_{ij}, & i, j = 1, \ldots, J, \\ \kappa_i(n_i)\mu_i p_{i0}, & i = 1, \ldots, J, \\ \mu_0 p_{0j}, & j = 1, \ldots, J, \end{cases}$$

for $\kappa_i : \mathbb{N}_0 :\to (0, \infty)$, $i = 1, \ldots, J$.
Let $\eta_i : \mathbb{N}_0 \to (0, \infty)$, $i = 1, \ldots, J$,

$$\eta_i(n)^{-1} = \prod_{r=1}^{n} \kappa_i(r), \quad n \in \mathbb{N}_0,\ i = 1, \ldots, J.$$

Then

$$\kappa_i(n) = \frac{\eta_i(n-1)}{\eta_i(n)}, \quad n \in \mathbb{N},\ i = 1, \ldots, J,$$

Partial balance – 2

Transition balance:

$$q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n}), \quad i, j = 0, \ldots, J,$$

Detailed balance:

$$\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n}), \quad i, j = 0, \ldots, J,$$

Partial balance:

$$\sum_{j=0}^{J} \pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \sum_{j=0}^{J} \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n}), \ i = 0, \ldots, J,$$

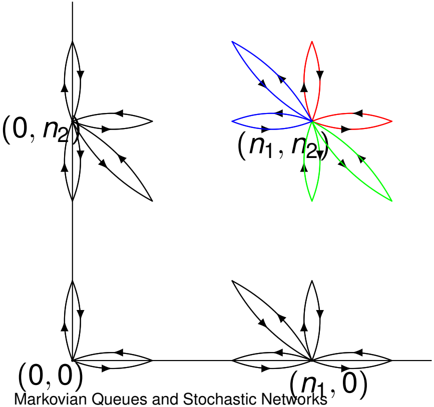
Global balance:

$$\sum_{i,j=0}^{J} \pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \sum_{i,j=0}^{J} \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n}).$$

Partial balance -1

Moreover, the equilibrium distribution satisfies partial balance, for all $\mathbf{n} \in S$, $i = 0, \dots, J$,

$$\sum_{j=0}^{J} \{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\} = 0.$$



Interpretation of the traffic equations

- Average number of customers moving from queue i to queue j is (see reader for proper definition)

$$\lambda_{ij} = \mathbb{E}q(N, N - \mathbf{e}_i + \mathbf{e}_j) = \sum_{\mathbf{n} \in S} \pi(\mathbf{n}) q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j).$$

- For network with Poisson arrivals $\psi(\mathbf{n}) = \phi(\mathbf{n})$, $\mathbf{n} \in \mathbb{N}_0^J$
- Then, with $\lambda_0 = \mu_0$,

$$\begin{aligned}\lambda_{ij} &= \sum_{\mathbf{n} \in S} G_{KW} \phi(\mathbf{n}) \prod_{j=1}^{J} \rho_j^{n_j} \frac{\phi(\mathbf{n} - \mathbf{e}_i)}{\phi(\mathbf{n})} \mu_i p_{ij} \\ &= \lambda_i p_{ij} \sum_{\mathbf{n} \in S, n_i > 0} G_{KW} \phi(\mathbf{n} - \mathbf{e}_i) \prod_{j=1}^{J} \rho_j^{n_j - \delta_{ij}} = \lambda_i p_{ij}.\end{aligned}$$

- We may interpret the solution λ_j, $j = 1, \ldots, J$, of traffic equations as the arrival rate of customers.

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State-dependent routing; blocking protocols – 1

A Kelly-Whittle network with state-dependent routing is a Markov chain $\{N(t)\}$ at state space $S \subseteq \mathbb{N}_0^J$ with transition rates, for $\mathbf{n}' \neq \mathbf{n}$,

$$q(\mathbf{n}, \mathbf{n}') = \begin{cases} \dfrac{\psi(\mathbf{n} - \mathbf{e}_i)\theta_i(\mathbf{n} - \mathbf{e}_i)}{\phi(\mathbf{n})} \mu_i b_{ij}(\mathbf{n} - \mathbf{e}_i), & \mathbf{n}' = \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \\ 0, & \text{otherwise,} \end{cases}$$

where $\phi : S \to (0, \infty)$ and $\psi, \theta_i, b_{ij} : S^b \to [0, \infty)$, and S^b is the set of base states:

$$S^b = \{\mathbf{m} \in \mathbb{N}_0^J : \exists i, j \in \{0, \dots, J\},\ i \neq j \text{ s.t. } \mathbf{m}+\mathbf{e}_i \text{ and } \mathbf{m}+\mathbf{e}_j \in S\}.$$

Without loss of generality $\sum_{j=0}^J b_{ij}(\mathbf{m}) = 1$.

State-dependent routing; blocking protocols – 2

Theorem (3.4.1 Equilibrium distribution)

Consider the Kelly-Whittle network with state-dependent routing $\{N(t)\}$ *at state space* $S \subseteq \mathbb{N}_0^J$*. Assume a solution* $H : S \to [0, \infty)$ *exists of the state-dependent traffic equations, for* $\mathbf{n} \in S$*,* $i = 0, \ldots, J$*:*

$$\sum_{j=0}^{J} H(\mathbf{n})\theta_i(\mathbf{n} - \mathbf{e}_i)b_{ij}(\mathbf{n} - \mathbf{e}_i) = H(\mathbf{n} - \mathbf{e}_i)\theta_0(\mathbf{n} - \mathbf{e}_i)\mu_0 b_{0i}(\mathbf{n} - \mathbf{e}_i)$$

$$+ \sum_{j=1}^{J} H(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)\theta_j(\mathbf{n} - \mathbf{e}_i)b_{ji}(\mathbf{n} - \mathbf{e}_i).$$

Assume that $G^{-1} = \sum_{\mathbf{n} \in S} \phi(\mathbf{n}) \prod_{j=1}^{J} \left(\frac{1}{\mu_j}\right)^{n_j} H(\mathbf{n}) < \infty$*, and that* $\{N(t)\}$ *is irreducible. Then*

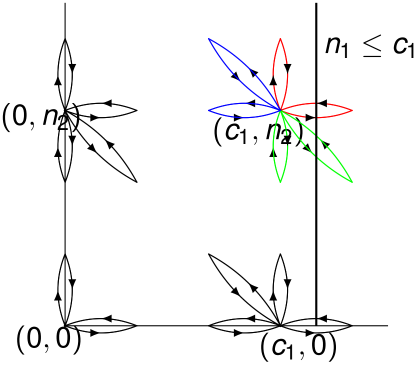
$$\pi(\mathbf{n}) = G\phi(\mathbf{n}) \prod_{j=1}^{J} \left(\frac{1}{\mu_j}\right)^{n_j} H(\mathbf{n}), \quad \mathbf{n} \in S.$$

State-dependent routing; blocking protocols – 3

- Product-form
$$\pi(\mathbf{n}) = G\phi(\mathbf{n}) \prod_{j=1}^{J} \left(\frac{1}{\mu_j}\right)^{n_j} H(\mathbf{n}), \quad \mathbf{n} \in S.$$
- State-dependent traffic equations just as difficult to solve as the partial balance equations.
- In applications, often Markov routing probabilities p_{ij}, $i, j = 0, \ldots, J$, and a function of the base state:
$$b_{ij}(\mathbf{m}) = p_{ij} f(\mathbf{m}), \quad \mathbf{m} \in S^b,$$
for some $f : S^b \to [0, \infty)$.
- With λ_j, $j = 1, \ldots, J$, solution of the traffic equations, solution of the state-dependent traffic equations:
$$H(\mathbf{n}) = \prod_{j=1}^{J} \lambda_j^{n_j}, \quad \mathbf{n} \in S.$$

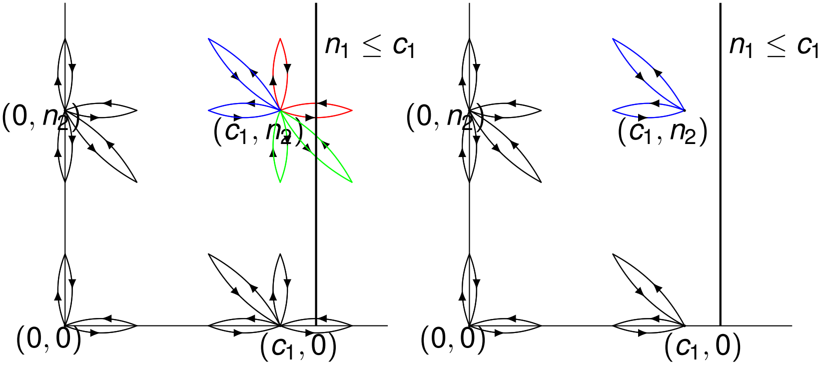
Product-form?

- Why product-form useful?
- Capacity constraints: no product-form
  Tandem of 2 queues. Queue 1 has capacity restriction c_1.
  If $n_1 = c_1$ customers arriving customer discarded.



Blocking protocols: Stop-protocol – 1

- If queue i in a Kelly-Whittle network with finite capacity constraints becomes saturated ($n_i = c_i$) then stop service at all other queues $j = 1, \ldots, J$, $j \neq i$, and stop the arrival process to the network.



Blocking protocols: Stop-protocol – 2

If queue i in a Kelly-Whittle network with finite capacity constraints becomes saturated ($n_i = c_i$) then stop service at all other queues, and stop the arrival process to the network.
For the open network the state space is

$$S_{\mathbf{c},o} = \{\mathbf{n} \in \mathbb{N}_0^J : n_j \leq c_j,\ n_i + n_j < c_i + c_j,\ i \neq j,\ i,j = 1,\ldots,J\}.$$

Transition rates

$$q(\mathbf{n},\mathbf{n}') = \begin{cases} \dfrac{\psi(\mathbf{n}-\mathbf{e}_i)\theta_i(\mathbf{n}-\mathbf{e}_i)}{\phi(\mathbf{n})}\mu_i b_{ij}(\mathbf{n}-\mathbf{e}_i), & \mathbf{n}' = \mathbf{n}-\mathbf{e}_i+\mathbf{e}_j, \\ 0, & \text{otherwise,} \end{cases}$$

with

$$\begin{aligned} \theta_i(\mathbf{m}) &= 1, \quad i = 0,\ldots,J,\ \mathbf{m} \in S^b_{\mathbf{c},o}, \\ f(\mathbf{m}) &= \mathbb{1}(m_j < c_j,\ j = 1,\ldots,J), \quad \mathbf{m} \in S^b_{\mathbf{c},o}, \\ b_{ij}(\mathbf{m}) &= p_{ij} f(\mathbf{m}), \quad i,j = 0,\ldots,J,\ \mathbf{m} \in S^b_{\mathbf{c},o}, \end{aligned}$$

and

$$S^b_{\mathbf{c},o} = \{\mathbf{m} \in \mathbb{N}_0^J : 0 \leq m_j \leq c_j - 1\}.$$

Blocking protocols: Stop-protocol – 3

The state-dependent traffic equations now reduce to the traffic equations, and

$$H(\mathbf{n}) = \prod_{j=1}^{J} \lambda_j^{n_j}, \quad \mathbf{n} \in S_{\mathbf{c},o},$$

satisfies the state-dependent traffic equations. Assume that

$$G_{\mathbf{c},o}^{-1} = \sum_{\mathbf{n} \in S_{\mathbf{c},o}} \phi(\mathbf{n}) \prod_{j=1}^{J} \rho_j^{n_j} < \infty,$$

and that $\{N(t)\}$ is irreducible. Then $\{N(t)\}$ has unique equilibrium distribution

$$\pi(\mathbf{n}) = G_{\mathbf{c},o} \phi(\mathbf{n}) \prod_{j=1}^{J} \rho_j^{n_j}, \quad \mathbf{n} \in S_{\mathbf{c},o}.$$

Blocking protocols: Jump-over-protocol – 1

- **Jump-over-blocking** If queue i in a Kelly-Whittle network with finite capacity constraints becomes saturated ($n_i = c_i$) then a customer arriving to queue i will immediately select a new station j with probability p_{ij}, $j = 0, \ldots, J$, $i = 1, \ldots, J$.
- **Generalised jump-over-blocking** A customer arriving at station i when n_i customers are present will be accepted with probability $a_i(n_i)$, and will jump over the station with probability $1 - a_i(n_i)$. A rejected customer selects a new station j with probability p_{ij}, $j = 0, \ldots, J$, $i = 1, \ldots, J$.

Blocking protocols: Jump-over-protocol – 2

- **Generalised jump-over-blocking** A customer arriving at station i when n_i customers are present will be accepted with probability $a_i(n_i)$, and will jump over the station with probability $1 - a_i(n_i)$. A rejected customer selects a new station j with probability p_{ij}, $j = 0, \ldots, J$, $i = 1, \ldots, J$.
- Let $c_j = \inf\{k : a_j(k) = 0,\ k = 0, 1, 2, \ldots\}$, $j = 1, \ldots, J$.
- For the open network the state space is

$$S_{jo,\mathbf{c}} = \{\mathbf{n} \in \mathbb{N}_0^J : 0 \leq n_j \leq c_j,\ i = 1, \ldots, J\}$$

- Let $P(\mathbf{m}) = (p_{ij} a_j(m_j),\ i, j = 0, \ldots, J)$,
  and $P_*(\mathbf{m}) = (p_{ij}(1 - a_j(m_j)),\ i, j = 0, \ldots, J)$.
- Transition rates, $\mathbf{m} \in S_{\mathbf{c}}^b$,

$$\begin{aligned} b_{ij}(\mathbf{m}) &= p_{ij} a_j(m_j) + (P_*(\mathbf{m})P(\mathbf{m}))_{ij} + (P_*^2(\mathbf{m})P(\mathbf{m}))_{ij} + \cdots \\ &= \sum_{k=0}^{\infty} (P_*^k(\mathbf{m})P(\mathbf{m}))_{ij}. \end{aligned}$$

Blocking protocols: Jump-over-protocol – 3

For $a_j(m_j) = \mathbb{1}(m_j \leq c_j)$

$$H(\mathbf{n}) = \prod_{j=1}^{J} \lambda_j^{n_j}, \quad \mathbf{n} \in S_{jo,\mathbf{c}},$$

satisfies the state-dependent traffic equations. Assume that

$$G_{jo,\mathbf{c}}^{-1} = \sum_{\mathbf{n} \in S_{jo,\mathbf{c}}} \phi(\mathbf{n}) \prod_{j=1}^{J} \rho_j^{n_j} < \infty,$$

and that $\{N(t)\}$ is irreducible. Then $\{N(t)\}$ has unique equilibrium distribution

$$\pi(\mathbf{n}) = G_{jo,\mathbf{c}} \phi(\mathbf{n}) \prod_{j=1}^{J} \rho_j^{n_j}, \quad \mathbf{n} \in S_{jo,\mathbf{c}}.$$

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Markovian Queues and Stochastic Networks

Lecture 3

Richard J. Boucherie

Stochastic Operations Research