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Markovian Queues and Stochastic Networks

Lecture 2

Richard J. Boucherie
Stochastic Operations Research

Detailed balance – 1

Definition (2.2.1 Detailed balance)

A Markov chain $\{N(t)\}$ at state space S with transition rates $q(\mathbf{n}, \mathbf{n}')$, $\mathbf{n}, \mathbf{n}' \in S$, satisfies detailed balance if a distribution $\pi = (\pi(\mathbf{n}),\ \mathbf{n} \in S)$ exists that satisfies for all $\mathbf{n}, \mathbf{n}' \in S$ the detailed balance equations:

$$\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n}') - \pi(\mathbf{n}')q(\mathbf{n}', \mathbf{n}) = 0.$$

Theorem (2.2.2)

If the distribution π satisfies the detailed balance equations then π is the equilibrium distribution.

- The detailed balance equations state that the probability flow between each pair of states is balanced.

Detailed balance – 2

Lemma (2.2.3, 2.2.4 Kolmogorov's criterion)

$\{N(t)\}$ satisfies detailed balance if and only if for all $r \in \mathbb{N}$ and any finite sequence of states $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_r \in S$, $\mathbf{n}_r = \mathbf{n}_1$,

$$\prod_{i=1}^{r-1} q(\mathbf{n}_i, \mathbf{n}_{i+1}) = \prod_{i=1}^{r-1} q(\mathbf{n}_{r-i+1}, \mathbf{n}_{r-i}).$$

If $\{N(t)\}$ satisfies detailed balance, then

$$\pi(\mathbf{n}) = \pi(\mathbf{n}') \frac{q(\mathbf{n}_1, \mathbf{n}_2) q(\mathbf{n}_2, \mathbf{n}_3)}{q(\mathbf{n}_2, \mathbf{n}_1) q(\mathbf{n}_3, \mathbf{n}_2)} \cdots \frac{q(\mathbf{n}_{r-1}, \mathbf{n}_r)}{q(\mathbf{n}_r, \mathbf{n}_{r-1})},$$

for arbitrary $\mathbf{n}' \in S$ for all $r \in \mathbb{N}$ and any path $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_r \in S$ such that $\mathbf{n}_1 = \mathbf{n}'$, $\mathbf{n}_r = \mathbf{n}$ for which the denominator is positive.

- Direct generalisation of the result for birth-death process.

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Reversibility – 1

Definition (Stationary process)

A stochastic process $\{N(t),\ t \in \mathbb{R}\}$ is stationary if $(N(t_1), N(t_2), \ldots, N(t_k))$ has the same distribution as $(N(t_1+\tau), N(t_2+\tau), \ldots, N(t_k+\tau))$ for all $k \in \mathbb{N}$, $t_1, t_2, \ldots, t_k \in T$, $\tau \in T$

Definition (2.4.1 Reversibility)

A stochastic process $\{N(t),\ t \in \mathbb{R}\}$ is reversible if $(N(t_1), N(t_2), \ldots, N(t_k))$ has the same distribution as $(N(\tau - t_1), N(\tau - t_2), \ldots, N(\tau - t_k))$ for all $k \in \mathbb{N}$, $t_1, t_2, \ldots, t_k \in \mathbb{R}$, $\tau \in \mathbb{R}$.

Theorem (2.4.2)

If $\{N(t)\}$ is reversible then $\{N(t)\}$ is stationary.

Reversibility – 2

Theorem (2.4.3 Reversibility and detailed balance)

Let $\{N(t),\ t \in \mathbb{R}\}$ be a stationary Markov chain with transition rates $q(\mathbf{n}, \mathbf{n}')$, $\mathbf{n}, \mathbf{n}' \in S$. $\{N(t)\}$ is reversible if and only if there exists a distribution $\pi = (\pi(\mathbf{n}),\ \mathbf{n} \in S)$ that satisfies the detailed balance equations. When there exists such a distribution π, then π is the equilibrium distribution of $\{N(t)\}$.

Example: Departures from the $M|M|1$ queue

- Arrival process to the $M|M|1$ queue is a Poisson process with rate λ.
- If $\lambda < \mu$ departure process from $M|M|1$ queue has rate λ.

- $\{N(t)\}$ recording the number of customers in $M|M|1$ with arrival rate λ and service rate μ satisfies detailed balance.
- Markov chain $\{N^r(t)\}$ in reversed time has Poisson arrivals at rate λ and service rate μ.
- Therefore $\{N^r(t)\}$ is the Markov chain of an $M|M|1$ queue with Poisson arrivals at rate λ and negative-exponential service at rate μ.
- Epochs of the arrival process for the reversed queue coincide with the epochs of the arrival process for the original queue, it must be that the departure process from the $M|M|1$ queue is a Poisson *process* with rate λ.

Reversibility – 3

Theorem (2.4.3 Reversibility and detailed balance)

Let $\{N(t),\ t \in \mathbb{R}\}$ be a stationary Markov chain with transition rates $q(\mathbf{n}, \mathbf{n}')$, $\mathbf{n}, \mathbf{n}' \in S$. $\{N(t)\}$ is reversible if and only if there exists a distribution $\pi = (\pi(\mathbf{n}),\ \mathbf{n} \in S)$ that satisfies the detailed balance equations. When there exists such a distribution π, then π is the equilibrium distribution of $\{N(t)\}$.

Proof. If $\{N(t)\}$ is reversible, then for all $t, h \in \mathbb{R}$, $\mathbf{n}, \mathbf{n}' \in S$:

$$\mathbb{P}(N(t+h) = \mathbf{n}',\ N(t) = \mathbf{n}) = \mathbb{P}(N(t) = \mathbf{n}',\ N(t+h) = \mathbf{n}).$$

$\{N(t),\ t \in \mathbb{R}\}$ is stationary. Let $\pi(\mathbf{n}) = \mathbb{P}(N(t) = \mathbf{n})$, $t \in \mathbb{R}$.

$$\frac{\mathbb{P}(N(t+h) = \mathbf{n}' | N(t) = \mathbf{n})}{h} \pi(\mathbf{n}) = \frac{\mathbb{P}(N(t+h) = \mathbf{n} | N(t) = \mathbf{n}')}{h} \pi(\mathbf{n}').$$

Letting $h \to 0$ yields the detailed balance equations.

Proof continued

Assume $\pi = (\pi(\mathbf{n}),\ \mathbf{n} \in S)$ satisfies detailed balance.
Consider $\{N(t)\}$ for $t \in [-H, H]$. Suppose $\{N(t)\}$ moves along the sequence of states $\mathbf{n}_1, \dots, \mathbf{n}_k$ and has sojourn time h_i in $\mathbf{n}_i$, $i = 1, \dots, k-1$, and remains in $\mathbf{n}_k$ for at least h_k until time H.
With probability $\pi(\mathbf{n}_1) = \mathbb{P}(N(-H) = \mathbf{n}_1)$ $\{N(t)\}$ starts in $\mathbf{n}_1$.
Probability density with respect to $h_1, \dots, h_k$ for this sequence

$$\pi(\mathbf{n}_1)q(\mathbf{n}_1)e^{-q(\mathbf{n}_1)h_1}p(\mathbf{n}_1, \mathbf{n}_2)\cdots q(\mathbf{n}_{k-1})e^{-q(\mathbf{n}_{k-1})h_{k-1}}p(\mathbf{n}_{k-1}, \mathbf{n}_k)e^{-q(\mathbf{n}_k)h_k},$$

Kolmogorov's criterion implies that

$$\pi(\mathbf{n}_1)q(\mathbf{n}_1, \mathbf{n}_2)\cdots q(\mathbf{n}_{k-1}, \mathbf{n}_k) = \pi(\mathbf{n}_k)q(\mathbf{n}_k, \mathbf{n}_{k-1})\cdots q(\mathbf{n}_2, \mathbf{n}_1),$$

probability density equals the probability density for the reversed path that starts in $\mathbf{n}_k$ at time H.
Thus $(N(t_1), N(t_2), \dots, N(t_k)) \sim (N(-t_1), N(t_2), \dots, N(-t_k))$
Stationarity completes the proof,.

Burke's theorem and feedforward networks – 1

Theorem (2.5.1 Burke's theorem)

Let $\{N(t)\}$ record the number of customers in the $M|M|1$ queue with arrival rate λ and service rate μ, $\lambda < \mu$. Let $\{D(t)\}$ record the customers' departure process from the queue. In equilibrium the departure process $\{D(t)\}$ is a Poisson process with rate λ, and $N(t)$ is independent of $\{D(s),\ s < t\}$.

Proof. $M|M|1$ reversible: epochs at which $\{N(-t)\}$ jumps up form Poisson process with rate λ.
If $\{N(-t)\}$ jumps up at time t^* then $\{N(t)\}$ jumps down at t^*.
Departure process forms a Poisson process with rate λ.
$\{N(t)\}$ reversible: departure process up to t^* and $N(t^*)$ have same distribution as arrival process after $-t^*$ and $N(-t^*)$.
Arrival process is Poisson process: arrival process after $-t^*$ independent of $N(-t^*)$.
Hence, the departure process up to t^* independent of $N(t^*)$.

Burke's theorem and feedforward networks – 2

- Tandem network of two $M|M|1$ queues
- Poisson λ arrival process to queue 1, service rates μ_i.
- Provided $\rho_i = \lambda/\mu_i < 1$, marginal distributions $\pi_i(n_i) = (1-\rho_i)\rho_i^{n_i}$, $n_i \in \mathbb{N}_0$.
- Burke's theorem: departure process from queue 1 before t^* and $N_1(t^*)$, are independent.
- Hence, in equilibrium, the at time t^* the random variables $N_1(t^*)$ and $N_2(t^*)$ are independent:

$$\pi(\mathbf{n}) = \prod_{i=1}^{2} \pi_i(n_i), \quad \mathbf{n} \in S = \mathbb{N}_0^2.$$

Burke's theorem and feedforward networks – 3

- Customer leaving queue j can route to any of the queues $j+1,\ldots,J$, or may leave the network.
- p_{ij} fraction of customers from queue i to queue $j > i$, p_{i0} fraction leaving the network.
- Arrival process is Poisson process with rate μ_0.
- Fraction p_{0j} of these customers is routed to queue j.
- The service rate at queue j is μ_j.
- Burke's theorem implies that all flows of customers among the queues are Poisson flows.
- Arrival rate λ_j of customers to queue j is obtained from superposition and random splitting of Poisson processes:

$$\lambda_j = \mu_0 p_{0j} + \sum_{i=1}^{j-1} \lambda_i p_{ij}, \quad j = 1,\ldots,J,$$

- traffic equations: the mean flow of customers.

Burke's theorem and feedforward networks – 4

Theorem (2.5.4 Equilibrium distribution)

Let $\{N(t) = (N_1(t), \ldots, N_J(t))\}$ *at state space* $S = \mathbb{N}_0^J$, *where* $\mathbf{n} = (n_1, \ldots, n_J)$ *and* n_j *the number of customers in queue* j, $j = 1, \ldots, J$, *record the number of customers in the feedforward network of* J $M|M|1$ *queues described above. If* $\rho_j = \lambda_j/\mu_j < 1$, *with* λ_j *the solution of the traffic equations,* $j = 1, \ldots, J$, *then the equilibrium distribution is the product of the marginal distributions of the queues:*

$$\pi(\mathbf{n}) = \prod_{j=1}^{J}(1-\rho_j)\rho_j^{n_j}, \quad n_j \in \mathbb{N}_0,\ j = 1, \ldots, J. \tag{1}$$

- Next: networks of $M|M|1$ queues.

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Open network of $M|M|1$ queues – 1

- Customer leaving queue j can route to any of the queues $1, \ldots, J$, or may leave the network.
- p_{ij} fraction of customers from queue i to queue j, p_{i0} fraction leaving the network.
- Arrival process is Poisson process with rate μ_0.
- Fraction p_{0j} of these customers is routed to queue j.
- The service rate at queue j is μ_j.
- Arrival rate λ_j of customers to queue j is obtained from the traffic equations

$$\lambda_j = \mu_0 p_{0j} + \sum_{i=1}^{J} \lambda_i p_{ij}, \quad j = 1, \ldots, J,$$

Open network of $M|M|1$ queues – 2

- Evolution number of customers in the queues recorded by Markov chain $\{N(t) = (N_1(t), \ldots, N_J(t)),\ t \in \mathbb{R}\}$
- State space $S \subseteq \mathbb{N}_0^J$, states $\mathbf{n} = (n_1, \ldots, n_J)$.
- If $\{N(t)\}$ is in state $\mathbf{n}$ and a customer routes from queue i to queue j then the next state is $\mathbf{n} - e_i + e_j$, $i, j = 0, \ldots, J$.
- Queue 0 is introduced to represent the outside.
- If a customer routes from queue i to queue 0 then this customer leaves the network
- and if a customer routes from queue 0 to queue j then this customers enters the network at queue j, $j = 1, \ldots, J$.
- State space $S = \mathbb{N}_0^J$.
- The transition rates of $\{N(t)\}$ for an open network are, for $\mathbf{n} \neq \mathbf{n}'$, $\mathbf{n}, \mathbf{n}' \in S$,

$$q(\mathbf{n}, \mathbf{n}') = \begin{cases} \mu_i p_{ij}, & \text{if } \mathbf{n}' = \mathbf{n} - e_i + e_j,\ i, j = 0, \ldots, J, \\ 0, & \text{otherwise.} \end{cases}$$

Closed network of $M|M|1$ queues – 1

- Queueing network is closed if arrivals to the network and departures from the network are not possible.
- Closed network by setting $\mu_0 = 0$ and $p_{j0} = 0$, $j = 1, \dots, J$.
- Number of customers in a closed network is constant: $S = S_M = \{\mathbf{n} : \sum_{j=1}^{J} n_j = M\}$ for some M, the number of customers in the network.
- The transition rates of $\{N(t)\}$ for a closed network are, for $\mathbf{n} \neq \mathbf{n}'$, $\mathbf{n}, \mathbf{n}' \in S$,

$$q(\mathbf{n}, \mathbf{n}') = \begin{cases} \mu_i p_{ij}, & \text{if } \mathbf{n}' = \mathbf{n} - e_i + e_j,\ i, j = 1, \dots, J, \\ 0, & \text{otherwise.} \end{cases}$$

Open network of $M|M|1$ queues – 3

Lemma (3.1.1 Traffic equations: open network)

Consider an open network. Assume that the routing matrix $P = (p_{ij},\ i, j = 0, \ldots, J)$ is irreducible. The traffic equations

$$\lambda_j = \mu_0 p_{0j} + \sum_{i=1}^{J} \lambda_i p_{ij}, \quad j = 1, \ldots, J,$$

have a unique non-negative solution $\{\lambda_j,\ j = 1, \ldots, J\}$.

Proof. Let $\lambda_0 = \mu_0$. Observe that the traffic equations also imply a traffic equation for queue 0: $\mu_0 = \sum_{j=1}^{J} \lambda_j p_{j0}$. Then the traffic equations for the open network read

$$\sum_{i=0}^{J} \lambda_j p_{ji} = \sum_{i=0}^{J} \lambda_i p_{ij}, \quad j = 0, \ldots, J.$$

Open network of $M|M|1$ queues – 4

Theorem (3.1.4 Equilibrium distribution)

Consider the Markov chain $\{N(t)\}$ at state space $S = \mathbb{N}_0^J$ for the open network of $M|M|1$ queues. Assume the routing matrix $P = (p_{ij})$ is irreducible and let $\{\lambda_j\}$ be the unique solution of the traffic equations. If $\rho_j := \lambda_j/\mu_j < 1$, $j = 1, \ldots, J$, then $\{N(t)\}$ has unique product-form equilibrium distribution

$$\pi(\mathbf{n}) = \prod_{j=1}^{J}(1-\rho_j)\rho_j^{n_j} = \prod_{j=1}^{J}\pi_j(n_j), \quad \mathbf{n} \in S.$$

Moreover, the equilibrium distribution satisfies partial balance, for all $\mathbf{n} \in S$, $i = 0, \ldots, J$,

$$\sum_{j=0}^{J}\{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) - \pi(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j, \mathbf{n})\} = 0.$$

Proof of Theorem 3.1.4

$$\sum_{j=0}^{J} \{m(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - m(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\}$$

$$= \sum_{j=0}^{J} \left\{ \prod_{k=1}^{J} \rho_k^{n_k} \mu_i p_{ij} \mathbb{1}(\mathbf{n} - \mathbf{e}_i \in \mathbb{N}_0^J) - \prod_{k=1}^{J} \rho_k^{n_k - \delta_{ki} + \delta_{kj}} \mu_j p_{ji} \mathbb{1}(\mathbf{n} - \mathbf{e}_i \in \mathbb{N}_0^J) \right\}$$

$$\sum_{j=0}^{J} \{m(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - m(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\} \mathbb{1}(i = 0)$$

$$= \left\{ \mu_0 - \sum_{j=1}^{J} \lambda_j p_{j0} \right\} \prod_{k=1}^{J} \rho_k^{n_k} \mathbb{1}(\mathbf{n} \in \mathbb{N}_0^J) = 0,$$

$$\sum_{j=0}^{J} \{m(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - m(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\} \mathbb{1}(i \neq 0)$$

$$= \left\{ \sum_{j=0}^{J} \lambda_i p_{ij} - \mu_0 p_{0i} - \sum_{j=1}^{J} \lambda_j p_{ji} \right\} \prod_{k=1}^{J} \rho_k^{n_k - \delta_{ki}} \mathbb{1}(\mathbf{n} - \mathbf{e}_i \in \mathbb{N}_0^J) \mathbb{1}(i \neq 0) = 0.$$

Traffic equations

$$\sum_{j=0}^{J}\{m(\mathbf{n})q(\mathbf{n},\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)-m(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j,\mathbf{n})\}$$

$$= \sum_{j=0}^{J}\left\{\prod_{k=1}^{J}\rho_k^{n_k}\mu_i p_{ij}\mathbb{1}(\mathbf{n}-\mathbf{e}_i\in\mathbb{N}_0^J)-\prod_{k=1}^{J}\rho_k^{n_k-\delta_{ki}+\delta_{kj}}\mu_j p_{ji}\mathbb{1}(\mathbf{n}-\mathbf{e}_i\in\mathbb{N}_0^J)\right\}$$

$$\sum_{j=0}^{J}\{m(\mathbf{n})q(\mathbf{n},\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)-m(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j,\mathbf{n})\}\,\mathbb{1}(i=0)$$

$$= \left\{\mu_0-\sum_{j=1}^{J}\lambda_j p_{j0}\right\}\prod_{k=1}^{J}\rho_k^{n_k}\mathbb{1}(\mathbf{n}\in\mathbb{N}_0^J)=0,$$

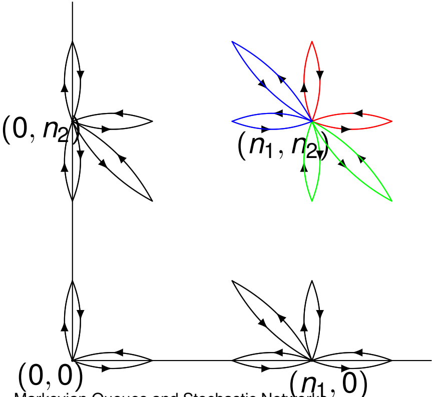
$$\sum_{j=0}^{J}\{m(\mathbf{n})q(\mathbf{n},\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)-m(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j)q(\mathbf{n}-\mathbf{e}_i+\mathbf{e}_j,\mathbf{n})\}\,\mathbb{1}(i\neq 0)$$

$$= \left\{\lambda_i-\mu_0 p_{0i}-\sum_{j=1}^{J}\lambda_j p_{ji}\right\}\prod_{k=1}^{J}\rho_k^{n_k-\delta_{ki}}\mathbb{1}(\mathbf{n}-\mathbf{e}_i\in\mathbb{N}_0^J)\mathbb{1}(i\neq 0)=0.$$

Partial balance -1

Moreover, the equilibrium distribution satisfies partial balance, for all $\mathbf{n} \in S$, $i = 0, \ldots, J$,

$$\sum_{j=0}^{J} \{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\} = 0.$$



Closed network of $M|M|1$ queues – 2

Theorem (3.1.5 Equilibrium distribution)

Consider Markov chain $\{N(t)\}$ at state space $S = S_M = \{\mathbf{n} : \sum_{j=1}^{J} n_j = M\}$ for the closed network of $M|M|1$ queues containing M customers. Assume $P = (p_{ij})$ is irreducible and let $\{\lambda_j\}$ be the unique solution of the traffic equations such that $\sum_{j=1}^{J} \lambda_j = 1$. Let $\rho_j := \lambda_j/\mu_j$. Then $\{N(t)\}$ has unique product-form equilibrium distribution

$$\pi(\mathbf{n}) = G_M \prod_{j=1}^{J} \rho_j^{n_j}, \quad \mathbf{n} \in S, \quad G_M = \left[\sum_{\mathbf{n}\in S} \prod_{j=1}^{J} \rho_i^{n_j}\right]^{-1}.$$

Moreover, the equilibrium distribution satisfies partial balance, for all $\mathbf{n} \in S$, $i = 1, \ldots, J$,

$$\sum_{j=1}^{J} \{\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - \pi(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j)q(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j, \mathbf{n})\} = 0.$$

Closed network of $M|M|1$ queues – 3

Algorithm (3.1.8 Buzen's Algorithm)

Define $G(m,j)$, $m = 0, \dots, M$, $j = 1, \dots, J$. Set

$$\begin{aligned} G(0,j) &= 1, \quad j = 1, \dots, J, \\ G(m,1) &= \rho_1^m, \quad m = 0, \dots, M. \end{aligned}$$

For $j = 2, \dots, J$, $m = 1, \dots, M$, do

$$G(m,j) = G(m,j-1) + \rho_j G(m-1,j).$$

Then $G_M = G(M,J)^{-1}$.

- Buzen's algorithm yields G_m, $m = 1, \dots, M$, and marginals and means:

$$\begin{aligned} \pi_j(n_j) &= G_M \rho_j^{n_j} [G_{M-n_j}^{-1} - \rho_j G_{M-n_j-1}^{-1}], \quad n_j = 0, \dots, M-1, \\ \pi_j(M) &= G_M \rho_j^{n_j}, \\ \mathbb{E}[N_j] &= \textstyle\sum_{m=1}^{M} \rho_j^m \frac{G_M}{G_{M-m}}. \end{aligned}$$

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