UNIVERSITY OF TWENTE.

Markovian Queues and Stochastic Networks

Lecture 1

Richard J. Boucherie

Stochastic Operations Research

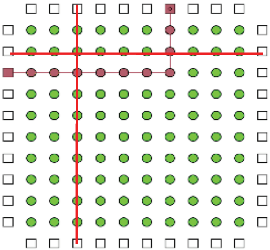
Overview MQSN

- Background on Markov chains
- Reversibility, output theorem, tandem networks, feedforward networks
- Partial balance, Markovian routing, Kelly-Whittle networks
- Kelly's lemma, time-reversed process, networks with fixed routes
- Advanced topics

Literature

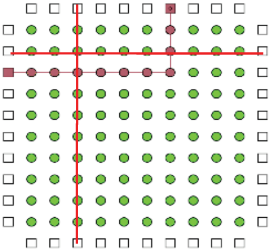
- R.D. Nelson, Probability, Stochastic Processes, and Queueing Theory, 1995, chapter 10
- F.P. Kelly, Reversibility and stochastic networks, 1979, chapters 1–4
  www.statslab.cam.ac.uk/∼frank/BOOKS/kelly_book.html
- R.W. Wolff, Stochastic Modeling and the Theory of Queues, Prentice Hall, 1989
- R.J. Boucherie, N.M. van Dijk (editors), Queueing Networks - A Fundamental Approach, International Series in Operations Research and Management Science Vol 154, Springer, 2011
- Reader: R.J. Boucherie, Markovian queueing networks, 2018 (work in progress)

Internet of Things: optimal route in Jackson network



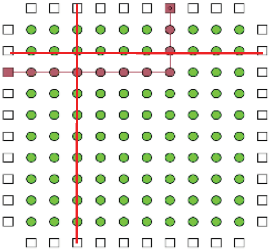
- Jobs arrive at outside nodes with given destination
- Each node single server queue minimize sojourn time
- Optimal route selection
- Inform jobs in neighbouring node
- alternative route

Internet of Things: optimal route in Jackson network



- Tandem of $M|M|1$ queues
- Sojourn time
- Average sojourn time queue i: $\mathbb{E}S_i = (\mu_i - \lambda_i)^{-1}$
- On route $\mathbb{E}S = \sum_i (\mu_i - \lambda_i)^{-1}$

Internet of Things: optimal route in Jackson network



- For fixed routes via set of queues
- On route r
  $\mathbb{E}S_r = \sum_i (\mu_i - \lambda_i)^{-1} \mathbb{1}(i \text{ on } r)$

Challenge

- Grid $N \times N$
- On each side k flows arrive from sources at randomly selected (but fixed) nodes with destination a randomly selected (but fixed) node on one of the 4 sides
- At each gridpoint a single server queue handles and forwards packets
- Packets select their route from source to destination to minimize their travelling time (no travelling time on link)
- Packets may communicate with neighbours to avoid congestions and change their route accordingly
- Poisson arrivals of packets; general processing time at nodes; one destination on each side
- Develop decentralized routing algorithm to minimize mean travelling times and demonstrate that it outperforms shortest (and fixed) route selection

Continuous-time Markov chain

- Stochastic process $\{N(t),\ t \in T\}$ records evolution of random variable, $T = \mathbb{R}$
- State space $S \subseteq \mathbb{N}_0^J$, state $\mathbf{s} = (n_1, \ldots, n_J)$
- Stationary process if $(N(t_1), N(t_2), \ldots, N(t_k))$ has the same distribution as $(N(t_1 + \tau), N(t_2 + \tau), \ldots, N(t_k + \tau))$ for all $k \in \mathbb{N}$, $t_1, t_2, \ldots, t_k \in T$, $\tau \in T$
- Markov proces satisfies the Markov property: for every $k \geq 1$, $0 \leq t_1 < \cdots < t_k < t_{k+1}$, and any $\mathbf{s}_1, \ldots, \mathbf{s}_{k+1}$ in S, the joint distribution of $(N(t_1), \ldots, N(t_{k+1}))$ is such that

$$\mathbb{P}\{N(t_{k+1}) = \mathbf{s}_{k+1} | N(t_1) = \mathbf{s}_1, \ldots, N(t_k) = \mathbf{s}_k\} = \mathbb{P}\{N(t_{k+1}) = \mathbf{s}_{k+1} | N(t_k) = \mathbf{s}_k\},$$

whenever the conditioning event $(N(t_1) = \mathbf{s}_1, \ldots, N(t_k) = \mathbf{s}_k)$ has positive probability.

Continuous-time Markov chain – 2

- A Markov process is time-homogeneous if the conditional probability $\mathbb{P}\{N(s+t) = \mathbf{s}' | N(s) = \mathbf{s}\}$ is independent of t for all $s, t > 0$, $\mathbf{s}, \mathbf{s}' \in S$.
- For a time-homogeneous Markov process the transition probability from state $\mathbf{s}$ to state $\mathbf{s}'$ in time t is defined as

$$P(\mathbf{s}, \mathbf{s}'; t) = \mathbb{P}\left\{N(s+t) = \mathbf{s}' | N(s) = \mathbf{s}\right\}, \quad t > 0.$$

Continuous-time Markov chain – 3

- The transition matrix $P(t) = (P(\mathbf{s}, \mathbf{s}'; t),\ \mathbf{s}, \mathbf{s}' \in S)$ has non-negative entries (1) and row sums equal to one (2).
- The *Markov property* implies that the transition probabilities satisfy the Chapman-Kolmogorov equations (3). Assume that the transition matrix is *standard* (4). For all $\mathbf{s}, \mathbf{s}' \in S$, $s, t \in T$, a standard transition matrix satisfies:

$$P(\mathbf{s}, \mathbf{s}'; t) \geq 0; \tag{1}$$

$$\sum_{\mathbf{s}' \in S} P(\mathbf{s}, \mathbf{s}'; t) = 1; \tag{2}$$

$$P(\mathbf{s}, \mathbf{s}''; t + s) = \sum_{\mathbf{s}' \in S} P(\mathbf{s}, \mathbf{s}'; t) P(\mathbf{s}', \mathbf{s}''; s); \tag{3}$$

$$\lim_{t \downarrow 0} P(\mathbf{s}, \mathbf{s}'; t) = \delta_{\mathbf{s}, \mathbf{s}'}. \tag{4}$$

Continuous-time Markov chain – 4

- For a standard transition matrix the transition rate from state $\mathbf{s}$ to state $\mathbf{s}'$ can be defined as
$$q(\mathbf{s},\mathbf{s}') = \lim_{h \downarrow 0} \frac{P(\mathbf{s},\mathbf{s}';h) - \delta_{\mathbf{s},\mathbf{s}'}}{h}.$$
- For all $\mathbf{s},\mathbf{s}' \in S$ this limit exists.
- Markov process is called continuous-time Markov chain if for all $\mathbf{s},\mathbf{s}' \in S$ the limit exists and is finite (5).
- Assume that the rate matrix $Q = (q(\mathbf{s},\mathbf{s}'),\ \mathbf{s},\mathbf{s}' \in S)$ is stable (6), and conservative (7)

$$0 \le q(\mathbf{s},\mathbf{s}') < \infty, \quad \mathbf{s}' \neq \mathbf{s}; \tag{5}$$

$$0 \le q(\mathbf{s}) := -q(\mathbf{s},\mathbf{s}) < \infty; \tag{6}$$

$$\sum_{\mathbf{s}' \in S} q(\mathbf{s},\mathbf{s}') = 0. \tag{7}$$

Continuous-time Markov chain – 5

- If the rate matrix is stable the transition probabilities can be expressed in the transition rates: for $\mathbf{s}, \mathbf{s}' \in S$,

$$P(\mathbf{s}, \mathbf{s}'; h) = \delta_{\mathbf{s},\mathbf{s}'} + q(\mathbf{s}, \mathbf{s}')h + \text{o}(h) \quad \text{for } h \downarrow 0, \qquad (8)$$

  where $\text{o}(h)$ denotes a function $g(h)$ with the property that $g(h)/h \to 0$ as $h \to 0$.
- For small positive values of h, for $\mathbf{s}' \neq \mathbf{s}$, $q(\mathbf{s}, \mathbf{s}')h$ may be interpreted as the conditional probability that the Markov chain $\{N(t)\}$ makes a transition to state $\mathbf{s}'$ during $(t, t+h)$ given that the process is in state $\mathbf{s}$ at time t.

Continuous-time Markov chain – 6

- For every initial state $N(0) = \mathbf{s}$, $\{N(t),\ t \in T\}$ is a pure-jump process: the process jumps from state to state and remains in each state a *strictly positive* sojourn-time with probability 1.
- Markov chain remains in state $\mathbf{s}$ for an exponential sojourn-time with mean $q(\mathbf{s})^{-1}$.
- Conditional on the process departing from state $\mathbf{s}$ it jumps to state $\mathbf{s}'$ with probability $p(\mathbf{s}, \mathbf{s}') = q(\mathbf{s}, \mathbf{s}')/q(\mathbf{s})$.
- The Markov chain represented via the holding times $q(\mathbf{s})$ and transition probabilities $p(\mathbf{s}, \mathbf{s}')$, $\mathbf{s}, \mathbf{s}' \in S$, is referred to as the Markov jump chain.
- The Markov chain with transition rates $q(\mathbf{s}, \mathbf{s}')$ is obtained from the Markov jump chain with holding times with mean $q(\mathbf{s})^{-1}$ and transition probabilities $p(\mathbf{s}, \mathbf{s}')$ as $q(\mathbf{s}, \mathbf{s}') = q(\mathbf{s})p(\mathbf{s}, \mathbf{s}')$, $\mathbf{s}, \mathbf{s}' \in S$.

Continuous-time Markov chain – 7

- From the Chapman-Kolmogorov equations

$$P(\mathbf{s}, \mathbf{s}''; t+s) = \sum_{\mathbf{s}' \in S} P(\mathbf{s}, \mathbf{s}'; t) P(\mathbf{s}', \mathbf{s}''; s)$$

  two systems of differential equations for the transition probabilities can be obtained:
- Conditioning on the first jump of the Markov chain in $(0, t]$ yields the so-called Kolmogorov backward equations (9), whereas conditioning on the last jump in $(0, t]$ gives the Kolmogorov forward equations (10), for $\mathbf{s}, \mathbf{s}' \in S$, $t \geq 0$,

$$\frac{dP(\mathbf{s}, \mathbf{s}'; t)}{dt} = \sum_{\mathbf{s}'' \in S} q(\mathbf{s}, \mathbf{s}'') P(\mathbf{s}'', \mathbf{s}'; t), \quad (9)$$

$$\frac{dP(\mathbf{s}, \mathbf{s}'; t)}{dt} = \sum_{\mathbf{s}'' \in S} P(\mathbf{s}, \mathbf{s}''; t) q(\mathbf{s}'', \mathbf{s}'). \quad (10)$$

Continuous-time Markov chain – 8

- Derivation Kolmogorov forward equations (regular)

$$
\begin{aligned}
P(\mathbf{s},\mathbf{s}';t+h) &= \sum_{\mathbf{s}''} P(\mathbf{s},\mathbf{s}'';t)P(\mathbf{s}'',\mathbf{s}';h) \\
P(\mathbf{s},\mathbf{s}';t+h) - P(\mathbf{s},\mathbf{s}';t) &= \sum_{\mathbf{s}''\neq\mathbf{s}'} P(\mathbf{s},\mathbf{s}'';t)P(\mathbf{s}'',\mathbf{s}';h) + P(\mathbf{s},\mathbf{s}';t)[P(\mathbf{s}',\mathbf{s}';h) - 1] \\
&= \sum_{\mathbf{s}''\neq\mathbf{s}'} \{P(\mathbf{s},\mathbf{s}'';t)P(\mathbf{s}'',\mathbf{s}';h) - P(\mathbf{s},\mathbf{s}';t)P(\mathbf{s}',\mathbf{s}'';h)\} \\
\frac{dP(\mathbf{s},\mathbf{s}';t)}{dt} &= \sum_{\mathbf{s}''\neq\mathbf{s}'} \{P(\mathbf{s},\mathbf{s}'';t)q(\mathbf{s}'',\mathbf{s}') - P(\mathbf{s},\mathbf{s}';t)q(\mathbf{s}',\mathbf{s}'')\}
\end{aligned}
$$

Explosion in a pure birth process

- Consider the Markov chain at state space $S = \mathbb{N}_0$ with transition rates
$$q(\mathbf{s}, \mathbf{s}') = \begin{cases} q(\mathbf{s}), & \text{if } \mathbf{s}' = \mathbf{s} + 1, \\ -q(\mathbf{s}), & \text{if } \mathbf{s}' = \mathbf{s}, \\ 0, & \text{otherwise,} \end{cases}$$
with initial distribution $\mathbb{P}(N(0) = \mathbf{s}) = \delta(\mathbf{s}, 0)$.
- Let $\xi(\mathbf{s})$ denote the time spent in state $\mathbf{s}$; $\xi = \sum_{\mathbf{s}=0}^{\infty} \xi(\mathbf{s})$
- Let $q(\mathbf{s}) = 2^{\mathbf{s}}$, then
$$\mathbb{E}\{\xi\} = \sum_{\mathbf{s}=0}^{\infty} \mathbb{E}\{\xi(\mathbf{s})\} = \sum_{\mathbf{s}=0}^{\infty} 2^{-\mathbf{s}} = 2$$
As $\mathbb{E}\{\xi\} < \infty$ it must be that $\mathbb{P}(\xi < \infty) = 1$ and therefore $\{N(t)\}$ is explosive (diverges to infinity in finite time).

Continuous-time Markov chain – 9

Theorem (1.1.2)

For a conservative, stable, regular, continuous-time Markov chain the forward equations (10) and the backward equations (9) have the same unique solution $\{P(\mathbf{s}, \mathbf{s}'; t),\ \mathbf{s}, \mathbf{s}' \in S,\ t \geq 0\}$. *Moreover, this unique solution is the transition matrix of the Markov chain.*

- The transient distribution $p(\mathbf{s}, t) = \mathbb{P}\{N(t) = \mathbf{s}\}$ can be obtained from the Kolmogorov forward equations for $\mathbf{s} \in S$, $t \geq 0$,

$$\begin{cases} \dfrac{dp(\mathbf{s}, t)}{dt} = \displaystyle\sum_{\mathbf{s}' \neq \mathbf{s}} \{p(\mathbf{s}', t)q(\mathbf{s}', \mathbf{s}) - p(\mathbf{s}, t)q(\mathbf{s}, \mathbf{s}')\}, \\ p(\mathbf{s}, 0) = p_{(0)}(\mathbf{s}). \end{cases}$$

Continuous-time Markov chain – 10

- A measure $m = (m(\mathbf{s}),\ \mathbf{s} \in S)$ such that $0 \leq m(\mathbf{s}) < \infty$ for all $\mathbf{s} \in S$ and $m(\mathbf{s}) > 0$ for some $\mathbf{s} \in S$ is called a stationary measure if for all $\mathbf{s} \in S$, $t \geq 0$,

$$m(\mathbf{s}) = \sum_{\mathbf{s}' \in S} m(\mathbf{s}')P(\mathbf{s}', \mathbf{s}; t),$$

  and is called an invariant measure if for all $\mathbf{s} \in S$,

$$\sum_{\mathbf{s}' \neq \mathbf{s}} \left\{ m(\mathbf{s})q(\mathbf{s}, \mathbf{s}') - m(\mathbf{s}')q(\mathbf{s}', \mathbf{s}) \right\} = 0.$$

- $\{N(t)\}$ is ergodic if it is positive-recurrent with stationary measure having finite mass
- Global balance; interpretation

Continuous-time Markov chain – 11

Theorem (1.1.4 Equilibrium distribution)

Let $\{N(t),\ t \geq 0\}$ *be a conservative, stable, regular, irreducible continuous-time Markov chain.*

(i) *If a positive finite mass invariant measure m exists then the Markov chain is positive-recurrent (ergodic). In this case* $\pi(\mathbf{s}) = m(\mathbf{s}) \left[\sum_{\mathbf{s}\in S} m(\mathbf{s})\right]^{-1}$, $\mathbf{s} \in S$, *is the unique stationary distribution and* π *is the equilibrium distribution, i.e., for all* $\mathbf{s}, \mathbf{s}' \in S$,

$$\lim_{t\to\infty} P(\mathbf{s}, \mathbf{s}'; t) = \pi(\mathbf{s}').$$

(ii) *If a positive finite mass invariant measure does not exist then for all* $\mathbf{s}, \mathbf{s}' \in S$,

$$\lim_{t\to\infty} P(\mathbf{s}, \mathbf{s}'; t) = 0.$$

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The birth-death process – 1

- A birth-death process is a Markov chain $\{N(t),\ t \in T\}$, $T = [0, \infty)$, or $T = \mathbb{R}$, with state space $S \subseteq \mathbb{N}_0$ and transition rates for $\lambda, \mu : S \to [0, \infty)$

$$q(\mathbf{s}, \mathbf{s}') = \begin{cases} \lambda(\mathbf{s}) & \text{if } \mathbf{s}' = \mathbf{s} + 1, \quad \text{(birth rate)} \\ \mu(\mathbf{s})\mathbb{1}(\mathbf{s} > 0), & \text{if } \mathbf{s}' = \mathbf{s} - 1, \quad \text{(death rate)} \\ -\lambda(\mathbf{s}) - \mu(\mathbf{s}), & \text{if } \mathbf{s}' = \mathbf{s},\ \mathbf{s} > 0, \\ -\lambda(\mathbf{s}), & \text{if } \mathbf{s} = 0. \end{cases}$$

- Kolmogorov forward equations

$$\frac{dP(\mathbf{s}, t)}{dt} = P(\mathbf{s} - 1, t)\lambda(\mathbf{s} - 1) - P(\mathbf{s}, t)[\lambda(\mathbf{s}) + \mu(\mathbf{s})] + P(\mathbf{s} + 1, t)\mu(\mathbf{s} + 1), \quad \mathbf{s} > 0,$$

$$\frac{dP(\mathbf{s}, t)}{dt} = -P(\mathbf{s}, t)\lambda(\mathbf{s}) + P(\mathbf{s} + 1, t)\mu(\mathbf{s} + 1), \quad \mathbf{s} = 0.$$

The birth-death process – 2

- Global balance equations

$$0 = \pi(\mathbf{s}-1)\lambda(\mathbf{s}-1) - \pi(\mathbf{s})[\lambda(\mathbf{s})+\mu(\mathbf{s})] + \pi(\mathbf{s}+1)\mu(\mathbf{s}+1), \quad \mathbf{s}>0,$$
$$0 = -\pi(0)\lambda(0) + \pi(1)\mu(1).$$

- π satisfies the detailed balance equations

$$\pi(\mathbf{s})\lambda(\mathbf{s}) = \pi(\mathbf{s}+1)\mu(\mathbf{s}+1), \quad \mathbf{s} \in S.$$

Theorem (2.1.1)

Let $\{N(t)\}$ be a birth-death process with state space $S = \mathbb{N}_0$, birth rates $\lambda(\mathbf{s})$ and death rates $\mu(\mathbf{s})$. If
$\pi(0) := \left[\sum_{\mathbf{s}=0}^{\infty} \prod_{\mathbf{r}=0}^{\mathbf{s}-1} \frac{\lambda(\mathbf{r})}{\mu(\mathbf{r}+1)}\right]^{-1} > 0,$
then the equilibrium distribution is

$$\pi(\mathbf{s}) = \pi(0)\prod_{\mathbf{r}=0}^{\mathbf{s}-1}\frac{q(\mathbf{r},\mathbf{r}+1)}{q(\mathbf{r}+1,\mathbf{r})} = \pi(0)\prod_{\mathbf{r}=0}^{\mathbf{s}-1}\frac{\lambda(\mathbf{r})}{\mu(\mathbf{r}+1)}, \quad \mathbf{s} \in S.$$

Example: The $M|M|1$ queue

- Customers arrive to a queue according to a Poisson process (the arrival process) with rate λ.
- A single server serves the customers in order of arrival.
- Customers' service times have an exponential distribution with mean μ^{-1} and are independent of each other and of the arrival process.
- $\{N(t),\ t \in T\}$, $T = [0, \infty)$ recording number of customers in the queue is a birth-death process at $S = \mathbb{N}_0$ with

$$q(\mathbf{s}, \mathbf{s}') = \begin{cases} \lambda(\mathbf{s}) = \lambda & \text{if } \mathbf{s}' = \mathbf{s} + 1, \quad \text{(birth rate)} \\ \mu(\mathbf{s}) = \mu \mathbb{1}(\mathbf{s} > 0), & \text{if } \mathbf{s}' = \mathbf{s} - 1, \quad \text{(death rate)} \end{cases}$$

and equilibrium distribution

$$\pi(\mathbf{s}) = (1 - \rho)\rho^{\mathbf{s}}, \quad \mathbf{s} \in S,$$

provided that the queue is *stable*: $\rho := \dfrac{\lambda}{\mu} < 1$.

Example: The $M|M|1|c$ queue

- $M|M|1$ queue, but now with finite waiting room that may contain at most $c-1$ customers.
- $\{N(t),\ t \in T\}$, $T = [0, \infty)$ recording the number of customers in the queue is a birth-death process at $S = \{0, 1, 2, \ldots, c\}$ with

$$q(\mathbf{s}, \mathbf{s}') = \begin{cases} \lambda(\mathbf{s}) = \lambda \mathbb{1}(\mathbf{s} < c) & \text{if } \mathbf{s}' = \mathbf{s} + 1, \quad \text{(birth rate)} \\ \mu(\mathbf{s}) = \mu \mathbb{1}(\mathbf{s} > 0), & \text{if } \mathbf{s}' = \mathbf{s} - 1, \quad \text{(death rate).} \end{cases}$$

- Detailed balance equations are truncated at state c
- The equilibrium distribution is truncated to S:

$$\pi(\mathbf{s}) = \pi(0)\rho^{\mathbf{s}}, \quad \mathbf{s} \in \{0, 1, \ldots, c\},$$

with

$$\pi(0) = \left[\sum_{\mathbf{s}=0}^{c} \rho^{\mathbf{s}}\right]^{-1} = \frac{1-\rho}{1-\rho^{c+1}}.$$

Detailed balance – 1

Definition (2.2.1 Detailed balance)

A Markov chain $\{N(t)\}$ at state space S with transition rates $q(\mathbf{s}, \mathbf{s}')$, $\mathbf{s}, \mathbf{s}' \in S$, satisfies detailed balance if a distribution $\pi = (\pi(\mathbf{s}),\ \mathbf{s} \in S)$ exists that satisfies for all $\mathbf{s}, \mathbf{s}' \in S$ the detailed balance equations:

$$\pi(\mathbf{s})q(\mathbf{s}, \mathbf{s}') - \pi(\mathbf{s}')q(\mathbf{s}', \mathbf{s}) = 0.$$

Theorem (2.2.2)

If the distribution π satisfies the detailed balance equations then π is the equilibrium distribution.

- The detailed balance equations state that the probability flow between each pair of states is balanced.

Detailed balance – 2

Theorem (2.2.5 Truncation)

Consider $\{N(t)\}$ *at state space* S *with transition rates* $q(\mathbf{s},\mathbf{s}')$, $\mathbf{s},\mathbf{s}' \in S$, *and equilibrium distribution* π. *Let* $V \subset S$.
Let $r > 0$. *If the transition rates are altered from* $q(\mathbf{s},\mathbf{s}')$ *to* $rq(\mathbf{s},\mathbf{s}')$ *for* $\mathbf{s} \in V$, $\mathbf{s}' \in S \setminus V$, *then the resulting Markov chain* $\{N_r(t)\}$ *satisfies detailed balance and has equilibrium distribution (G is the normalizing constant)*

$$\pi_r(\mathbf{s}) = \begin{cases} G\pi(\mathbf{s}), & \mathbf{s} \in V, \\ Gr\pi(\mathbf{s}), & \mathbf{s} \in S \setminus V, \end{cases}$$

If $r = 0$ *then the Markov chain is* *truncated* *to* V *and*

$$\pi_0(\mathbf{s}) = \pi(\mathbf{s}) \left[\sum_{\mathbf{s} \in V} \pi(\mathbf{s}) \right]^{-1}, \quad \mathbf{s} \in V.$$

Example: Network of parallel $M|M|1$ queues – 1

- Network of two $M|M|1$ queues in parallel.
- Queue j has arrival rate λ_j and service rate μ_j, $j = 1, 2$.
- $\{N_j(t)\}$, $j = 1, 2$, are assumed independent.
- $\{N(t) = (N_1(t), N_2(t))\}$, state space $S = \mathbb{N}_0^2$, $\mathbf{s} = (n_1, n_2)$,
- Transition rates, for $\mathbf{s}, \mathbf{s}' \in S$, $\mathbf{s}' \neq \mathbf{s}$,

$$q(\mathbf{s}, \mathbf{s}') = \begin{cases} \lambda_j & \text{if } \mathbf{s}' = \mathbf{s} + \mathbf{e}_j, \quad j = 1, 2, \\ \mu_j, & \text{if } \mathbf{s}' = \mathbf{s} - \mathbf{e}_j, \quad j = 1, 2. \end{cases}$$

- Random variables $N_j := N_j(\infty)$ recording the equilibrium number of customers in queue j are independent.

$$\pi(\mathbf{s}) = \prod_{j=1}^{2} \pi_j(n_j), \quad \mathbf{s} \in S,$$

$$\pi_j(n_j) = (1 - \rho_j)\rho_j^{n_j},\ n_j \in \mathbb{N}_0, \text{ provided } \rho_j := \frac{\lambda_j}{\mu_j} < 1.$$

Example: Network of parallel $M|M|1$ queues – 2

- Common capacity restriction $n_1 + n_2 \leq c$.
- Customers arriving to the network with c customers present are discarded.
- The Markov chain $\{N(t) = (N_1(t), N_2(t))\}$ has state space $S_c = \{(n_1, n_2) : n_j \geq 0,\ j = 1, 2,\ n_1 + n_2 \leq c\}$ and transition rates truncated to S_c.
- Invoking Truncation Theorem:

$$\pi(\mathbf{s}) = G \prod_{j=1}^{2} \rho_j^{n_j}, \quad \mathbf{s} \in S_c,$$

with normalising constant

$$G = \left[\sum_{n_1=0}^{c} \sum_{n_2=0}^{c-n_1} \prod_{i=1}^{2} \rho_i^{n_i} \right]^{-1}.$$

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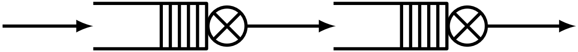
Tandem network



- Tandem network of two $M|M|1$ queues
- Poisson λ arrival process to queue 1, service rates μ_i.
- Provided $\rho_i = \lambda/\mu_i < 1$, marginal distributions $\pi_i(n_i) = (1-\rho_i)\rho_i^{n_i}$, $n_i \in \mathbb{N}_0$.
- In equilibrium:

$$\pi(\mathbf{s}) = \prod_{i=1}^{2} \pi_i(n_i), \quad \mathbf{s} \in S = \mathbb{N}_0^2.$$

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