Lecure 8 : Loss networks

1 Motivation: multiple services, single cell / link -- HSCSD

2 Generalised stochastic knapsack

3 Admission control

4 GSM network / PSTN network

5 Model, Equilibrium distribution

6 Blocking probabilities

7 Reduced load approximation

8 Monte-Carlo summation

9 Summary and exercises

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Intensive Care Unit, Ward, …

Patients with different medical condition require different nurse to patient ratio

Ranging from 2:1 (2 nurses for 1 patient – ICU) to 1:6 (medical ward)

Mixing of patient classes may be possible in some wards, and (in future) we may use flexible assignment of nurses to wards

Patient requires b nurses $1/6 \le b \le 2$

Patient accepted iff minimum requirement b is met: loss system

GSM /HSCSD: High Speed Circuit Switched Data

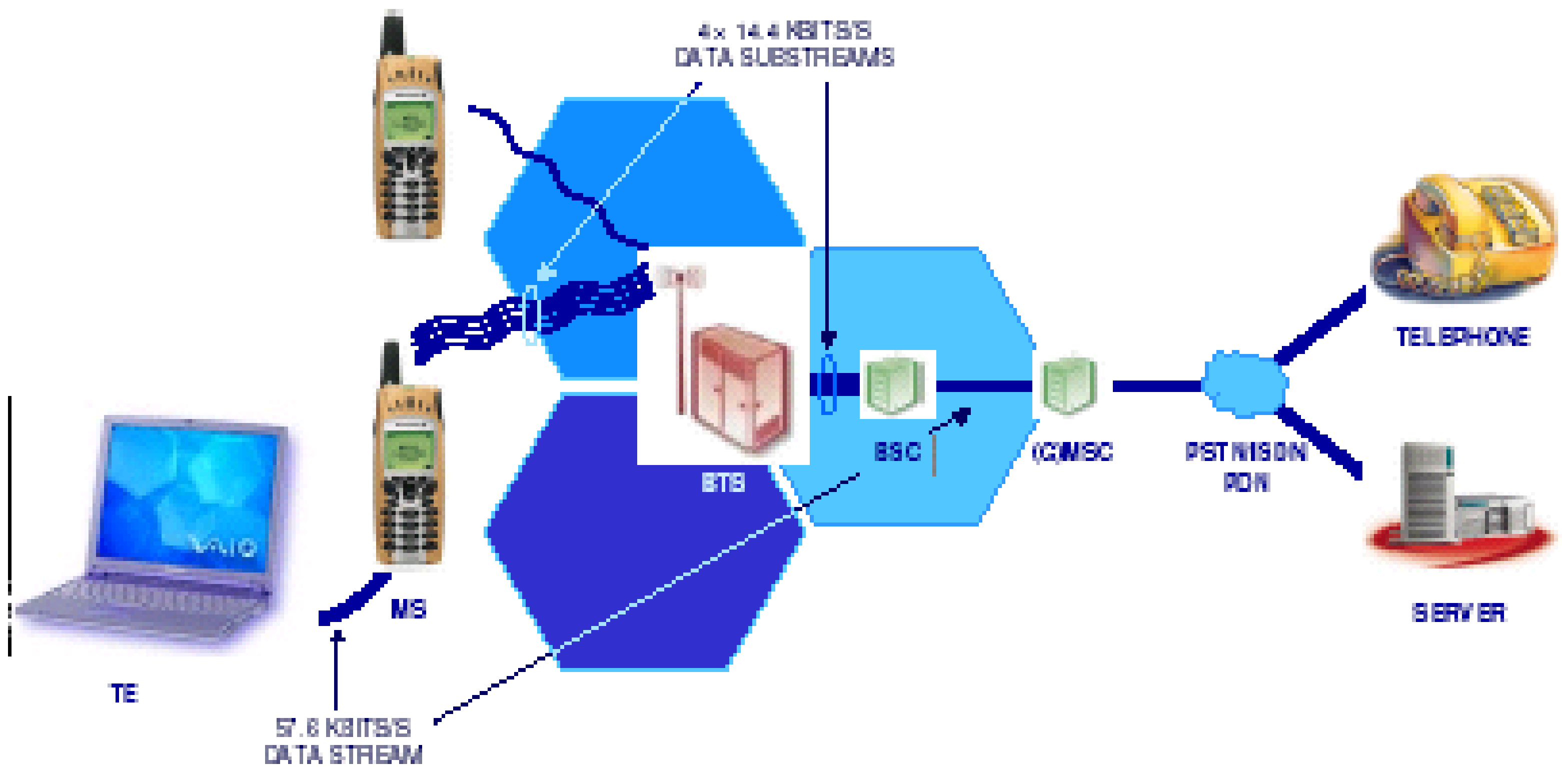

Figure 2.1 GSM/HSCSD network architecture: the illustration shows an example data call at an information bit rate of 4×14.4 kbits/s, maintained between an HSCSD terminal and a remote server.

HSCSD characteristics

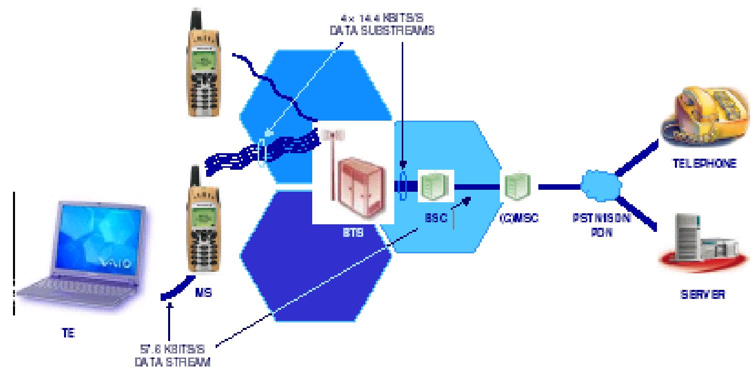

Figure 2.1 GSM/HSCSD network architecture: the illustration shows an example data call at an information bit rate of 4×14.4 kbits/s, maintained between an HSCSD terminal and a remote server.

Multiple types (speech, video, data)
circuit switched: each call gets number of channels

GSM speech: 1 channel
data: 1 channel (CS, data rate 9.6 kbps)

GSM/HSCSD speech: 1 channel
data: $1 \leq b,\ldots,B \leq 8$ channels (technical requirements, data rate 14.4 kbps)

Call accepted iff minimum channel requirement b is met: loss system

Up / downgrading:
data calls may use more channels (up to B) when other services are not using these channels

video: better picture quality, but same video length
data: faster transmission rate, thus smaller transmission time

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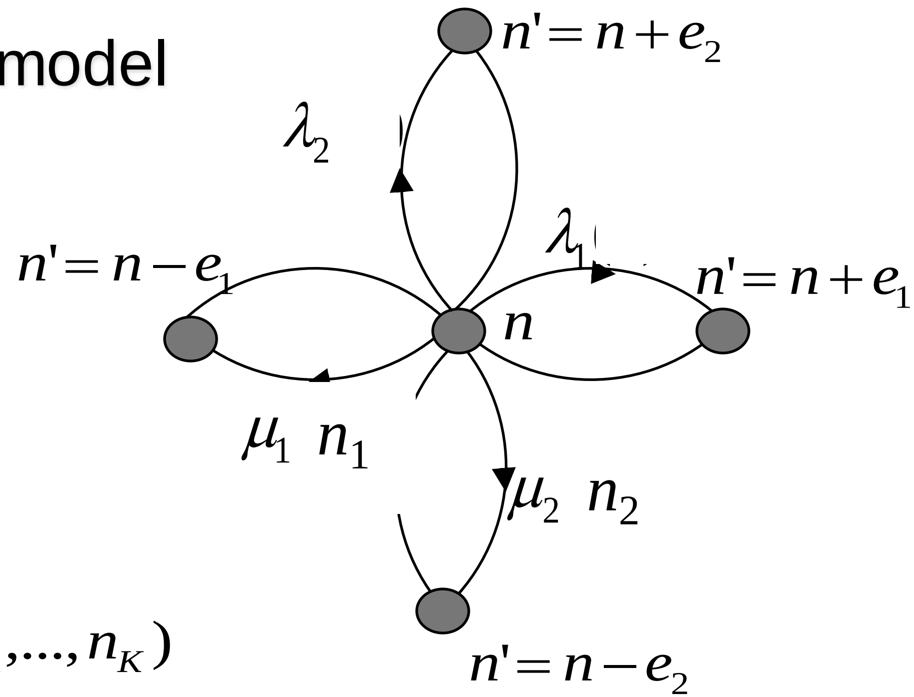
Generalised stochastic knapsack: model



number of resource units C

number of object classes K

class k arrival rate λ_k

class k mean holding time (exp) $1/\mu_k$

class k size b_k

state (number of objects) $n = (n_1, \ldots, n_K)$

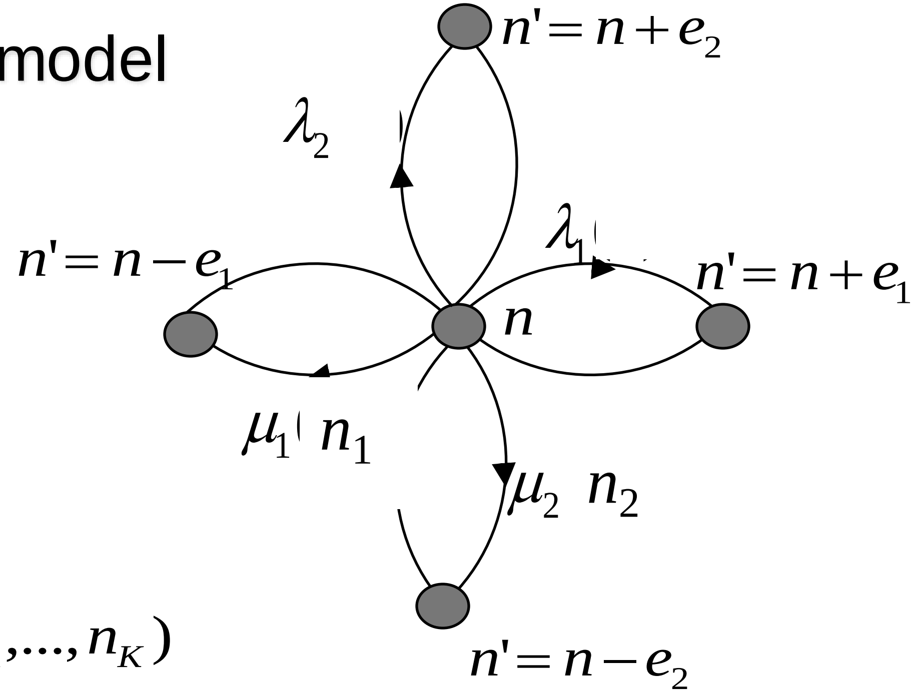
state of process at time t $X(t) = (X_1(t), \ldots, X_K(t))$

stationary Markov process $\{X(t), t \geq 0\}$

transition rates

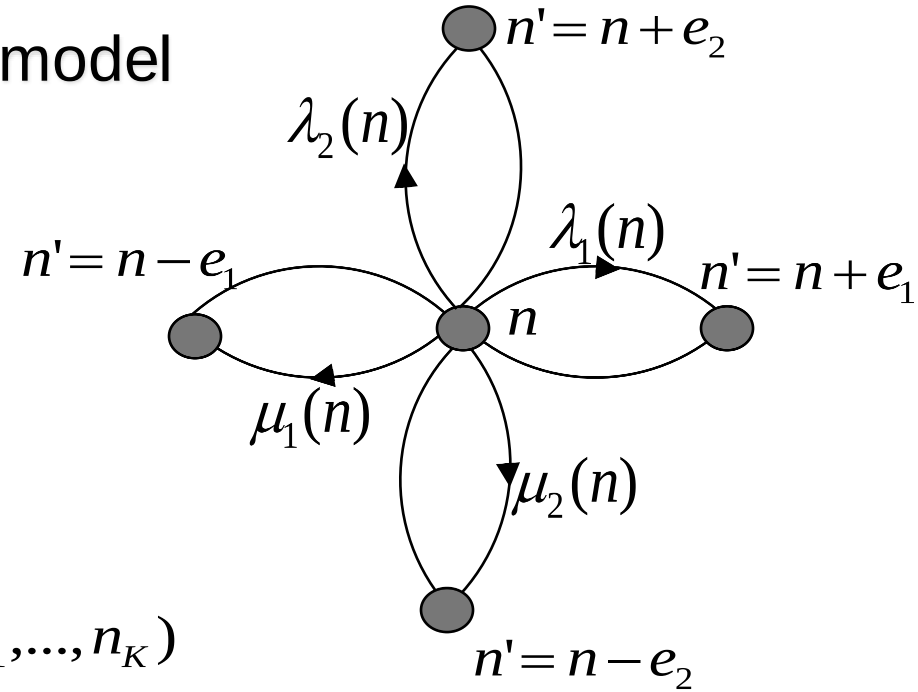
$$q(n, n') = \begin{cases} \lambda_k & n' = n + e_k \\ \mu_k \; n_k & n' = n - e_k \end{cases}$$

Generalised stochastic knapsack: model



number of resource units	C
number of object classes	K
class k arrival rate	λ_k
class k mean holding time (exp)	$1/\mu_k$
class k size	b_k
state (number of objects)	$n = (n_1, \ldots, n_K)$
state space	$S = \{n \in \mathbf{N}_0^K : b \cdot n \le C\}$
object of class k accepted only if	$b_k \le C - b \cdot n$
state of process at time t	$X(t) = (X_1(t), \ldots, X_K(t))$
stationary Markov process	$\{X(t), t \ge 0\}$
transition rates	$q(n, n') = \begin{cases} \lambda_k(n) 1(b_k \le C - b \cdot n) & n' = n + e_k \\ \mu_k \, n_k & n' = n - e_k \end{cases}$

Generalised stochastic knapsack: model



number of resource units	C
number of object classes	K
class k arrival rate	$\lambda_k(n)$
class k mean holding time (exp)	$1/\mu_k(n)$
class k size	b_k
state (number of objects)	$n=(n_1,\ldots,n_K)$
state space	$S=\{n\in\mathbf{N}_0^K : b\cdot n\le C\}$
object of class k accepted only if	$b_k\le C-b\cdot n$
state of process at time t	$X(t)=(X_1(t),\ldots,X_K(t))$
stationary Markov process	$\{X(t),t\ge 0\}$
transition rates	$q(n,n')=\begin{cases}\lambda_k(n)1(b_k\le C-b\cdot n) & n'=n+e_k\\ \mu_k(n) & n'=n-e_k\end{cases}$

Generalised stochastic knapsack: equilibrium distribution

Theorem 1:

For the generalised stochastic knapsack, a necessary and sufficient condition for reversibility of $X(t)=(X_1(t),\dots,X_K(t))$ is that

$$\frac{\lambda_k(n)}{\mu_k(n+e_k)}=\frac{\Psi(n+e_k)}{\Psi(n)} \qquad \text{for all } n\in S\setminus T_k,\ k=1,\dots,K$$

for some function $\Psi: S\to[0,\infty)$. Moreover, when such a function Ψ exists, the equilibrium distribution for the generalised stochastic knapsack is given by

$$\pi(n)=\frac{\Psi(n)}{\sum_{n\in S}\Psi(n)}, \qquad n\in S$$

Generalised stochastic knapsack: equilibrium distribution

Proof:

We have to verify detailed balance:

$$\pi(n)q(n,n+e_k)=\pi(n+e_k)q(n+e_k,n)$$

$$\Leftrightarrow$$

$$\pi(n)\lambda_k(n)=\pi(n+e_k)\mu_k(n+e_k)$$

$$\Leftrightarrow$$

$$\frac{\lambda_k(n)}{\mu_k(n+e_k)}=\frac{\pi(n+e_k)}{\pi(n)}$$

If π exists that satisfies the last expression, then π satisfies detailed balance. As the right hand side of this expression is independent of the index k it must be that the condition of the theorem involving $\Psi:S\to[0,\infty)$ is satisfied. Conversely, assume that the condition involving $\Psi:S\to[0,\infty)$ is satisfied. Then

$$\pi(n)=\frac{\Psi(n)}{\sum_{n\in S}\Psi(n)},\qquad n\in S$$ is the equilibrium distribution.

Generalised stochastic knapsack: examples

Stochastic knapsack

$$\lambda_k(n) = \lambda_k 1(b_k \leq C - b \cdot n) \quad n' = n + e_k$$

$$\mu_k(n) = n_k \mu_k \quad n' = n - e_k$$

$$\Psi(n) = \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!} 1(b \cdot n \leq C)$$

Finite source input

$$\lambda_k(n) = (M_k - n_k)\lambda_k 1(n_k \leq M_k) \quad n' = n + e_k$$

$$\mu_k(n) = n_k \mu_k \quad n' = n - e_k$$

$$\Psi(n) = \prod_{k=1}^{K} \binom{M_k}{n_k} \rho_k^{n_k}$$

State space constraints

$$\lambda_k(n) = \lambda_k 1(b_k \leq C - b \cdot n;\ n_k \leq C_k) \quad n' = n + e_k$$

$$\mu_k(n) = n_k \mu_k \quad n' = n - e_k$$

$$\Psi(n) = \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!} 1(b \cdot n \leq C;\ n_k \leq C_k)$$

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Admission control

Admission class k whenever sufficient room $b_k \leq C - b \cdot n$ Complete sharing

Simple, but

may be unfair (some classes monopolize the knapsack resources)

may lead to poor long-run average revenue (admitted objects may not contribute to revenue)

admission policies: restrict access even when sufficient room available

calculate performance under policy

determine optimal policy

Coordinate convex policies

In general: Markov decision theory

Stochastic knapsack under admission control

Admission policy

$$f = (f_1, \ldots, f_K) \quad f_k : S \to \{0,1\}$$

$$f_k(n) = \begin{cases} 1 & \text{class } k \text{ accepted in state } n \\ 0 & \text{class } k \text{ rejected in state } n \end{cases}$$

Transition rates

$$q(n, n') = \begin{cases} \lambda_k f_k(n) 1(b_k \leq C - b \cdot n) & n' = n + e_k \\ n_k \mu_k & n' = n - e_k \end{cases}$$

Recurrent states

$$S(f) \subseteq S = \{n : b \cdot n \leq C\}$$

Examples: complete sharing

$$f_k(n) = \begin{cases} 1 & b \cdot n \leq C - b_k \\ 0 & \text{otherwise} \end{cases}$$

trunk reservation

$$f_k(n) = \begin{cases} 1 & b \cdot n \leq C - b_k - t_k \\ 0 & \text{otherwise} \end{cases}$$

Stochastic knapsack under trunk reservation

trunk reservation admits class k object iff after admittance at least t_k resource units remain available

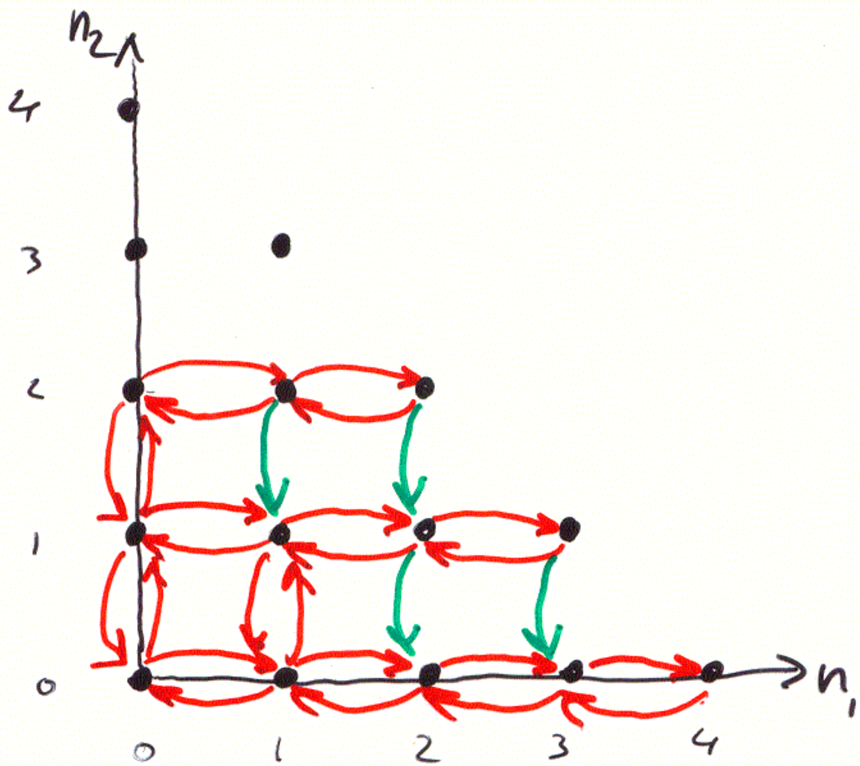
$C = 4$

$K = 2$

$b_1 = b_2 = 1$

$t_1 = 0$

$t_2 = 2$



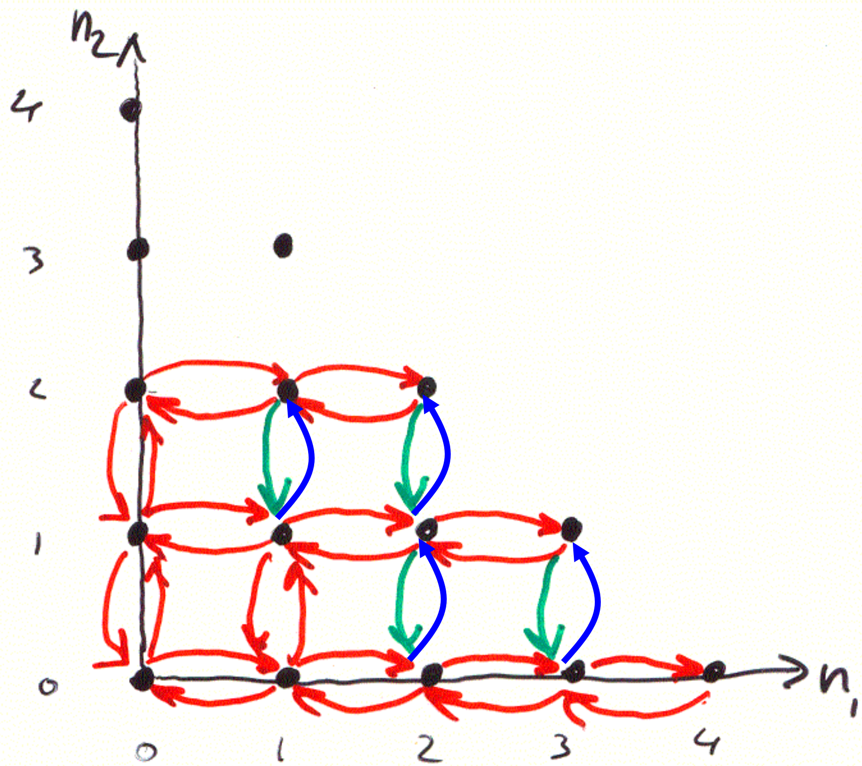
$$f_k(n) = \begin{cases} 1 & b \cdot n \le C - b_k - t_k \\ 0 & \text{otherwise} \end{cases}$$

Not reversible: so that equilibrium distribution usually not available in closed form

Stochastic knapsack coordinate convex policies

Coordinate convex set $\Omega \subseteq S \;:\; n \in \Omega,\;\; n_k > 0 \Rightarrow n - e_k \in \Omega$

Coordinate convex policy: admit object iff state process remains in Ω



$$f_k(n) = 1 \quad \text{iff} \quad n + e_k \in \Omega$$

Theorem:

Under the coordinate convex policy f the state process

$$X_f(t) = (X_1(t), \ldots, X_K(t))$$

is reversible, and

$$\pi_f(n) = \frac{1}{G_f} \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!}, \qquad n \in \underline{\Omega}$$

Coordinate convex policies: examples

Note: Not all policies are coordinate convex, e.g. trunk reservation

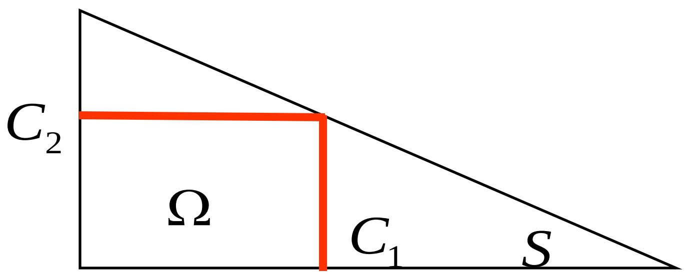
Complete sharing: always admit if room available $\Omega = S$

Complete partitioning: accept class k iff

$$b_k(n_k + 1) \le C_k$$

$$\Omega = \{0,\dots,\left\lfloor \frac{C_1}{b_1} \right\rfloor\} \times \dots \times \{0,\dots,\left\lfloor \frac{C_K}{b_K} \right\rfloor\}$$

$$C_1 + \dots + C_K \le C$$

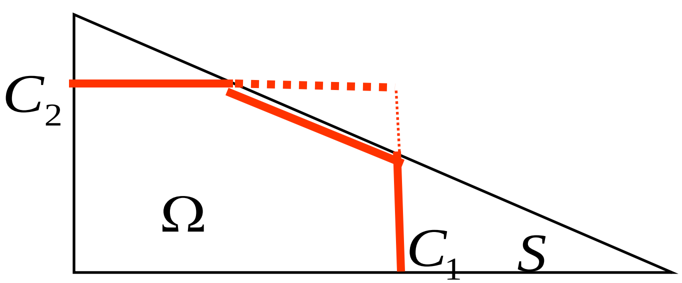


Threshold policies: accept class k iff

$$b_k(n_k + 1) \le C_k$$

$$b \cdot n + b_k \le C$$

$$\Omega = \{n : b \cdot n \le C, n_k \le \left\lfloor \frac{C_k}{b_k} \right\rfloor, k = 1,\dots,K\}$$



Coordinate convex policies: revenue optimization

revenue in state n $$r(n) = \sum_{k=1}^{K} r_k n_k$$

long run average revenue $$W(f) = \sum_{n \in \Omega} r(n) \pi_f(n)$$

example: long run average utilization $r_k = b_k$

long run average throughput $r_k = \mu_k$

Intuition: optimal policy in special cases

$\lambda_k \downarrow 0 \ \forall k$	blocking obsolete $\Rightarrow$ complete sharing	

$\lambda_k \to \infty \ \forall k$ complete partitioning with $C_k = b_k s_k^*$

where $(s_1^*, \ldots, s_K^*)$ is the optimal solution of the knapsack problem

$$\max \sum_{k=1}^{K} r_k s_k \quad \text{subject to} \quad \sum_{k=1}^{K} b_k s_k \le C \quad s_k \in \mathrm{N}_0$$

If C / b_{k^*} integer, where k^* maximizes per unit revenue r_k / b_k then $s_k^* = \begin{cases} r_k / b_k & \text{for } k = k^* \\ 0 & \text{otherwise} \end{cases}$

Coordinate convex policies: optimal policies

number of coordinate convex policies is finite

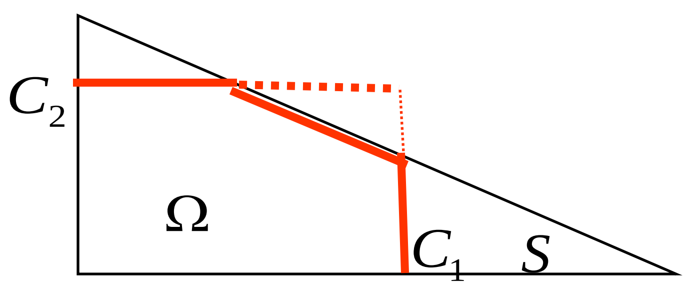
thus for each coordinate convex policy *f* compute *W(f)*

and select *f* with highest *W(f)*

infeasible as number of policies grows as $O(C_1 ... C_K)$

show that optimal policy is in certain class

often threshold policies



$$b \cdot n + b_k \leq C$$

$$\Omega = \{n : b \cdot n \leq C, n_k \leq \left\lfloor \frac{C_k}{b_k} \right\rfloor, k = 1, \ldots, K\}$$

then problem reduces to finding optimal thresholds

in full generality: Markov decision theory

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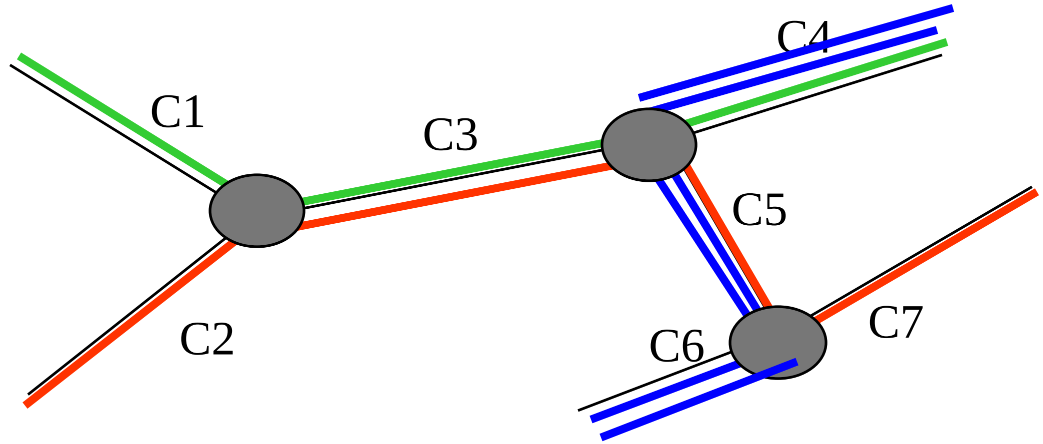
Loss networks

So far: stochastic knapsack
- equilibrium distribution
- blocking probabilities
- throughput
- admission control
  - coordinate convex policy -- complex state space

model for multi service single link

multi service multiple links / networks

PSTN / ISDN



Link: connection between switches
Route: number of links
Capacity Ci of link i
Call class: route, bandwidth requirement per link

or in medical setting: patient simultaneously requires various resources

Stochastic knapsack: special case = model for single link
But: is special case of generalised stochastic knapsack

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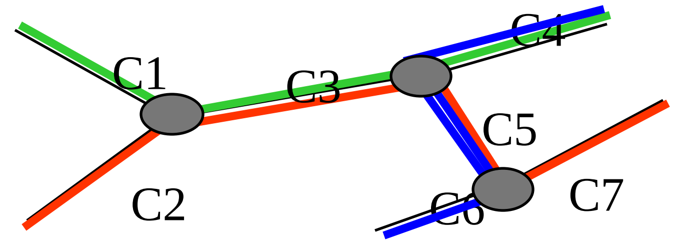
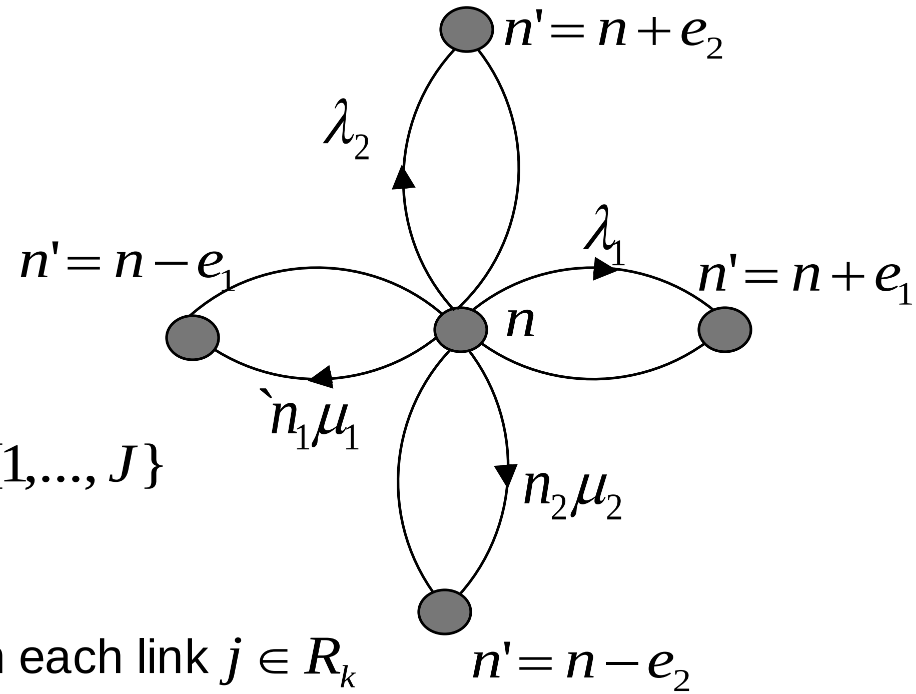


Loss network : notation

number of links	J
capacity (bandwidth units) link j	C_j
number of object classes	K
class k arrival rate	λ_k
class k mean holding time (exp)	$1/\mu_k$
bandwidth req. class k on link j	b_{jk}
Route	$R_k \subseteq \{1, \ldots, J\}$



Class k admitted iff b_{jk} bandwidth units free in each link $j \in R_k$

Otherwise call is blocked and cleared

Admitted call occupies b_{jk} bandwidth units in each link $j \in R_k$ for duration of its holding time

Loss network : notation

Set of classes $\mathrm{K} = \{1,\ldots,K\}$

set of classes that uses link *j* $\mathrm{K}_{\mathrm{j}} = \{k \in \mathrm{K} : j \in R_k\}$

state $n = (n_1,\ldots,n_K)$

state space $$S = \{n \in \mathbf{N}_0^K : \sum_{k \in \mathrm{K}_{\mathrm{j}}} b_{jk} n_k \le C_j, \quad j = 1,\ldots,J\}$$

$$S = \{n \in \mathbf{N}_0^K : An \le C\} \quad A = (b_{jk})$$

class *k* blocked $$T_k = \{n \in S : \sum_{\ell \in \mathrm{K}_{\mathrm{j}}} b_{j\ell} n_\ell + b_{jk} > C_j, \ \text{ some } \ j\}$$

$$T_k = \{n \in \mathbf{N}_0^K : An \le C, \ A(n + e_k) \not\le C\}$$

Loss network : notation

state of process at time t $X(t) = (X_1(t),\ldots,X_K(t))$

stationary Markov process $\{X(t), t \geq 0\}$

aperiodic,irreducible

equilibrium state $X = (X_1,\ldots,X_K)$

utilization of link j $U_j := \sum_{k \in K_j} b_{jk} X_k$

long run fraction of blocked calls $B_k = 1 - \Pr\{U_j \leq C_j - b_{jk}, j \in R_k\}$ PASTA

long run throughput $TH_k = \lambda_k (1 - B_k) = \mu_k E[X_k]$ Little

unconstrained cousin (∞ capacity) $X^\infty = (X^\infty{}_1,\ldots,X^\infty{}_K)$

Poisson r.v. with mean $\rho_k = \lambda_k / \mu_k$

unconstrained cousin of utilization $U^\infty{}_j := \sum_{k \in K_j} b_{jk} X^\infty{}_k$

Equilibrium distribution

Theorem: Product form equilibrium distribution

$$\Pr\{X=n\}=\frac{1}{G}\prod_{k=1}^{K}\frac{\rho_k^{\,n_k}}{n_k!}=\frac{\Pr\{X^{\infty}=n\}}{\Pr\{U^{\infty}\le C\}},\qquad n\in S \qquad\qquad G=\sum_{n\in S}\prod_{k=1}^{K}\frac{\rho_k^{\,n_k}}{n_k!},\qquad n\in S$$

Blocking probability of class k call

$$B_k=1-\frac{\sum_{n\in S\setminus T_k}\Pr\{X=n\}}{\sum_{n\in S}\Pr\{X=n\}}=1-\frac{\sum_{n\in S\setminus T_k}\prod_{\ell=1}^{K}\frac{\rho_\ell^{\,n_\ell}}{n_\ell!}}{\sum_{n\in S}\prod_{\ell=1}^{K}\frac{\rho_\ell^{\,n_\ell}}{n_\ell!}}$$

V, C vectors

The Markov chain is reversible, PASTA holds, and the equilibrium distribution (and the blocking probabilities) are insensitive.

PROOF: special case of generalised stochastic knapsack!

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Computing blocking probabilities

Direct summation is possible, but complexity $O(KC_1 ... C_J)$

Recursion is possible (KR), but complexity $O(KC_1 ... C_J)$

Bounds for single service loss networks ($b_{jk} \in \{0,1\}$)

For link j in loss network: probability that call on link j is blocked

$$L_j \leq Er(\bar{\rho}_j, C_j) = \frac{\bar{\rho}_j^{C_j} / C_j!}{\sum_{k=0}^{C_j} \bar{\rho}_j^{\,k} / k!}, \qquad \bar{\rho}_j = \sum_{k \in K_j} \rho_k = \sum_{k \in K} b_{jk} \rho_k$$

load offered to link j

facility bound

Call of class k blocked if not accepted at all links

$$B_k \leq 1 - \prod_{j \in R_k} (1 - Er(\bar{\rho}_j, C_j))$$

product bound

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Computing blocking probabilities: single service networks
Reduced load approximation

Facility bound $L_j \le Er(\sum_{k \in K_j} \rho_k, C_j)$

Part of offered load $\sum_{k \in K_j} \rho_k$ blocked on other links : $L_j = Er(\sum_{k \in K_j} t_k(j)\rho_k, C_j)$

$t_k(j)$ probability at least one unit of bandwidth available in each link in $R_k \setminus \{j\}$

$\rho_k t_k(j)$ reduced load

approximation: blocking independent from link to link $t_k(j) = \prod_{i \in R_k \setminus \{j\}} (1 - L_i)$

reduced load approximation

$$L_j = Er(\sum_{k \in K_j} \rho_k \prod_{i \in R_k \setminus \{j\}} (1 - L_i), C_j), \quad j = 1, \ldots, J$$

$$B_k = 1 - \prod_{j \in R_k} (1 - L_j), \quad k = 1, \ldots, K$$

existence, uniqueness fixed point
repeated substitution
accuracy

Reduced load approximation: existence and uniqueness

Notation: $L := (L_1, \ldots, L_J)$

$$T_j(L) := Er\Big(\sum_{k \in K_j} \rho_k \prod_{i \in R_k \setminus \{j\}} (1 - L_i), C_j\Big)$$

$$T(L) := (T_1(L), \ldots, T_J(L))$$

Theorem There exists a unique solution L^* to the fixed point equation $L = T(L)$

Proof The mapping $T : [0,1]^J \to [0,1]^J$ is continuous, so existence from Brouwer's theorem

T is not a contraction!

Reduced load approximation: uniqueness

Notation:

$Er^{-1}{}_j(B)$ inverse of Er for capacity C_j : value of ρ such that $B = Er(\rho, C_j)$

is strictly increasing function of B

Therefore $\int_0^B Er^{-1}{}_j(z)dz$ is strictly convex function of B for $B \in [0,1]$

Proof of uniqueness:

consider fixed point $L = T(L)$ and apply $Er^{-1}{}_j$: $Er^{-1}{}_j(L_j) := \sum_{k \in K_j} r_k \prod_{i \in R_k \setminus \{j\}} (1 - L_i)$ (*)

Define $\forall L \in [0,1]^J$

$$\psi(L) := \sum_{k \in K_j} r_k \prod_{i \in R_k} (1 - L_i) + \sum_{j=1}^{J} \int_0^{L_j} Er^{-1}{}_j(z)dz$$

which is strictly convex.

Thus, if $L^* \in [0,1]^J$ is solution of $\frac{\partial \psi(L)}{\partial L_i} = 0, \quad i = 1,\dots,J$

Then L^* is unique minimum of ψ over $[0,1]^J$

writing out the partial derivatives yields (*)

Reduced load approximation: repeated substitution

$$T_j(L) := Er\Big(\sum_{k \in K_j} r_k \prod_{i \in R_k \setminus \{j\}} (1-L_i), C_j\Big)$$

Start $L \in [0,1]^J$

Repeat $L^0 := L$

$$L^m := T(L^{m-1}), \quad m = 1,2,\dots$$

Theorem Let $L^0 = (1,\dots,1)$

Then $(0,\dots,0) = L^1 < L^{2n+1} < L^{2n+3} < L^* < L^{2n+2} < L^{2n} < L^0 = (1,\dots,1)$

Thus $L^{2n} \to L^+, \quad L^{2n+1} \to L^-, \quad L^- \le L^* \le L^+$

Proof

T is *decreasing* operator: $T(L) < T(L')$ if $L' < L$ componentwise

Thus $T^{2n}(L) < T^{2n}(L')$ and

$T^{2n+1}(L) > T^{2n+1}(L')$ if $L' < L$ componentwise

Reduced load approximation: accuracy

$$T_j(L) := Er\Big(\sum_{k\in \mathcal{K}_j} r_k \prod_{i\in R_k\setminus\{j\}} (1-L_i), C_j\Big)$$

Corollary $$L_j^* \le Er\Big(\sum_{k\in \mathcal{K}_j} r_k, C_j\Big)$$

so that $$B_k = 1-\prod_{j\in R_k}(1-L_j^*) \le 1-\prod_{j\in R_k} Er\Big(\sum_{k\in \mathcal{K}_j} r_k, C_j\Big), \qquad k=1,\ldots,K$$

Proof $$T_j^2(1\ldots1) = T(0\ldots0) = Er\Big(\sum_{k\in \mathcal{K}_j} r_k, C_j\Big)$$

$$(0,\ldots,0) = L^1 < L^{2n+1} < L^* < L^{2n} < L^0 = (1,\ldots,1)$$

$$L^2 = T(0,\ldots,0)$$

Indeed B_k from reduced load approximation does not violate the upper bound

Plan for today

1 multiple services, single cell / link -- HSCSD
2 Generalised stochastic knapsack
3 Admission control
4 GSM network / PSTN network
5 Model, Equilibrium distribution
6 Blocking probabilities
7 Reduced load approximation
8 Monte-Carlo summation
9 Summary and exercises

Monte Carlo summation

For performance measures: compute equilibrium distribution

$$\Pr\{X = n\} = \frac{\prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!}}{\sum_{n \in S} \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!}}, \qquad n \in S$$

for blocking probability of class k call

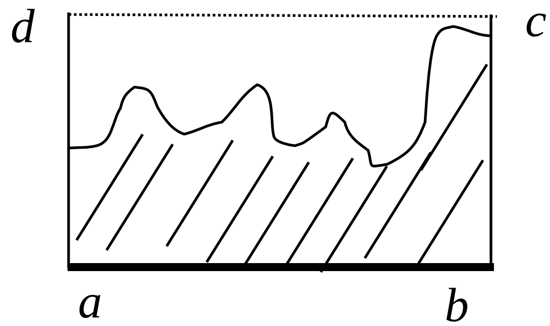
$$B_k = \frac{\sum_{n \in T_k} \prod_{\ell=1}^{K} \frac{\rho_\ell^{n_\ell}}{n_\ell!}}{\sum_{n \in S} \prod_{\ell=1}^{K} \frac{\rho_\ell^{n_\ell}}{n_\ell!}}$$

In general: evaluate $\sum_{n \in U} \prod_{\ell=1}^{K} \frac{\rho_\ell^{n_\ell}}{n_\ell!}$

difficult due to size of the state space : use Monte Carlo summation

Monte Carlo summation

Example: evaluate integral



$$\int_a^b f(x)dx$$

Monte Carlo summation:

draw points at random in box abcd

points under curve / # points is measure for surface //// = value integral

Method

Let $X \stackrel{d}{=} U(a,b)$ $Y \stackrel{d}{=} U(0,c)$ indep, and let $Z = 1(Y \leq f(X))$

draw n indep. Samples $Z_1,\ldots,Z_n$

Then $\bar{Z} = \dfrac{Z_1 + \ldots + Z_n}{n}$ unbiased estimator of $\dfrac{1}{c(b-a)}\int_a^b f(x)dx$

with 95% confidence interval $\left[\bar{Z} \pm 1.96\sqrt{S^2(n)/n}\right]$

Powerful method: as accurate as desired

Monte Carlo summation

For blocking probabilities

$$B_k = \frac{\sum_{n \in T_k} \prod_{\ell=1}^{K} \frac{\rho_\ell^{n_\ell}}{n_\ell!} \, e^{-\rho_\ell}}{\sum_{n \in S} \prod_{\ell=1}^{K} \frac{\rho_\ell^{n_\ell}}{n_\ell!} \, e^{-\rho_\ell}}$$

ratio of multidimensional Poisson(ρ) distributed r.v.

Let $X \overset{d}{=} \text{Poisson}(\rho)$ then $B_k = \frac{\Pr\{X(\rho) \in T_k\}}{\Pr\{X(\rho) \in S\}}$

Estimate enumerator and denominator via Monte Carlo summation:

draw V_i from $\text{Poisson}(\rho)$, $i = 1,\ldots,n$ iid

Let $g(V_i) = 1(V_i \in U)$

$$Eg(V) = \Pr\{X(\rho) \in U\} = \sum_{n \in U} \prod_{\ell=1}^{K} \frac{\rho_\ell^{n_\ell}}{n_\ell!} e^{-\rho_\ell}$$

for $U = T_k$ and $U = S$

confidence interval?

Monte Carlo summation

For blocking probabilities

$$B_k = \frac{\sum_{n \in T_k} \prod_{\ell=1}^{K} \frac{\rho_\ell^{n_\ell}}{n_\ell!}}{\sum_{n \in S} \prod_{\ell=1}^{K} \frac{\rho_\ell^{n_\ell}}{n_\ell!}} \frac{e^{-\rho_\ell}}{e^{-\rho_\ell}}$$

Estimate enumerator and denominator via Monte Carlo summation:

confidence interval? Harvey-Hills method (acceptance rejection method HH)

$$B_k = \frac{\Pr\{X(\rho) \in T_k\}}{\Pr\{X(\rho) \in S\}} = \Pr\{X(\rho) \in T_k \mid X(\rho) \in S\}$$

draw V_i from Poisson(ρ), $i = 1, \ldots, n$ iid

if sample $\notin S$ ignore

if sample $\in S$ count

if sample also $\in T_k$ count as succes

unbiased estimator!

$$g(V_i) = 1(V_i \in T_k \mid V_i \in S)$$

$$Eg(V) = B_k$$

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sections 5.1, 5.2, 5.5 (reduced load method)

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