MQSN 2020 exercises – Second set

4. Consider a system consisting of a CPU with $J-1$ clients that send jobs to the CPU. The CPU equally divides its computing time over the jobs present at the CPU. Each client has a Poisson arrival process of new jobs with rate λ_i for client i, $i = 1, \dots, J-1$. A job that arrives at a client first needs to be handled by the client (e.g. obtaines a time stamp). To this end, the client has a single server handling the jobs, and jobs wait for their turn in a queue of unlimited capacity (are treated First In First Out). The service time of a job at the client is exponential with rate ν_i for client i, $i = 1, \dots, J-1$. Upon completion of this activity, the job is sent to the CPU. Here the required amount of service is exponential with rate μ. Upon completion of the job at the CPU, the job returns to the client it originated from to be handled a second time. To this end, it again joins the tail of the queue, waits for its turn (FIFO) and has again exponential service time with rate ν_i to e.g. obtain a second time stamp. Upon this second service completion the job leaves the system.

   (a) Model this system as a network of queues and give a complete description of the Markov chain that records the number of customers in each queue, i.e., state space, transition rates.

   (b) Give the stability condition for this network.

   (c) Give the equilibrium distribution for this network and prove correctness.

   (d) Give the mean sojourn time in the system of a job that arrives at client 1.

   (e) Relate the network of this exercise to that of exercise 3. Elaborate on the difference in the models and behaviour of these networks.

5. Consider an open network of J general[1] quasi reversible queues to which customers of type u arrive according to independent Poisson processes with rate $\mu_0(u)$, $u = 1, \dots, U$. A customer that leaves queue i moves to queue j with probability p_{ij}, $i, j = 0, \dots, J$, $\sum_{j=0}^{\infty} p_{ij} = 1$, $i = 0, \dots, J$.[2]
   (a) Give a complete description of the Markov chain that records the states of the queues, i.e., state space, transition rates.
   (b) Give the equilibrium distribution and prove correctness of the equilibrium distribution.

6. Consider a symmetric queue to which customers of type u arrive according to independent Poisson processes with rate $\mu_0(u)$, $u = 1, \dots, U$. Let the service requirement of customers of type u have a phase-type distribution $F(u, x) = \sum_{k=1}^{\infty} p_k(u) Erl(k, \nu(u))(x)$, $x > 0$, $\sum_{k=1}^{\infty} p_k(u) = 1$, with mean $\tau(u)$, $u = 1, \dots, U$.
   (a) Give a complete description of the Markov chain that records the number of customers in the queue, including their types and remaining number of phases), i.e., states, state space, transition rates.
   (b) Give the equilibrium distribution and prove correctness of the equilibrium distribution.
   (c) Show that the queue is quasi-reversible.
   (d) Give the equilibrium distribution for the total number of customers in the queue and prove correctness of the equilibrium distribution, i.e., show that the equilibrium distribution of the total number of customers depends on the distribution of the service requirements only through its mean.

[1] General: the state of the queue is $\mathbf{n}$ without further structure as considered in the lectures on quasi-reversible queues.

[2] So we do not consider fixed routes but Markov routing.

7. Consider an open tandem network of J FIFO queues to which customers of type u arrive according to independent Poisson processes with rate $\mu_0(u)$, $u = 1, \ldots, U$. Let the service requirement of customers of type u in queue j have a negative exponential distribution with mean τ_j, $u = 1, \ldots, U$, $j = 1, \ldots, J$. Assume that queue j has finite capacity c_j, $j = 1, \ldots, J$. Let the network operate under the push-out protocol: a customer that arrives to a saturated queue pushes the customer in service out of the queue and enters the tail of the queue.

   (a) Give a complete description of the Markov chain that records the number of customers in the queue, i.e., state space, transition rates.

   (b) Show that a queue that operates under the push-out protocol is quasi-reversible.

   (c) Give the equilibrium distribution and prove correctness of the equilibrium distribution.

   (d) Give the distribution of the number of customers seen by a customer arriving to queue j, $j = 1, \ldots, J$.

   (e) Prove or dispute the following statement: *The departure process of customers of type u from queue j (including customers pushed out from the queue) is a Poisson process with rate $\mu_0(u)$.*

8. Consider a $J \times I$ grid (the points $\{(1,1), (1,2), \ldots, (1,J), (2,1), \ldots, (I,J)\}$. Each node in the grid is a queue. Customers arrive according to a Poisson process to the nodes on the edge of the grid (source nodes), and follow a path through the grid to a node on the edge of the grid (sink nodes), e.g., from node (1,2) to node $(J-1, J)$. The customer-type uniquely defines its source and sink nodes. In a path, a customer may route to the horizontal or vertical neighbouring nodes, only, i.e., from node (i,j) to $(i-1,j), (i+1,j), (i,j-1), (i,j+1)$. Let $1/\mu(i,j)$ denote the mean service time of a customer at node (i,j) and assume that the service times of all customers are exponentially distributed. Let node (i,j) operate under the (κ, γ, δ) protocol, i.e., the service discipline is described by three functions $\kappa_{(i,j)}(n) = 1(n > 0)$, $\gamma_{(i,j)}(k,n)$, and $\delta_{(i,j)}(k,n)$..

   (a) Give an optimisation problem to optimally select routes for each customer type such that the total sojourn time for all types is minimised.

   (b) Consider the 6×6 grid with the following source to destination pairs: (1,3) to (3,6); (1,4) to (6,4); (1,5) to (5,1); (2,6) to (6,1); (4,6) to (4,1); (5,6) to (1,2). Solve the optimisation problem in the case that all queues are single server FIFO queues with $\lambda_t = \lambda$ for all t, $t = 1, \ldots, 6$, and $\mu(i,j) = \mu$ for all i,j, $i,j = 1, \ldots, 6$. Draw conclusion on the structure of the optimal routes for a general grid with source and destination nodes.