Addendum to 5.8.4:
Kelly System for Investing and Kuhn-Tucker Conditions

In this addendum to Section 5.8.4 of the book *Operations Research, Introduction to Models and Methods* by Boucherie et.al, an algorithm is derived for the Kelly system for gambling and investing with multiple betting/investment objects.[1] To do so, we first give an elementary discussion of the Kuhn-Tucker conditions for nonlinear optimization problems.

Kuhn-Tucker conditions for nonlinear programming

The most general form of a nonlinear programming problem is

$$\begin{aligned} &\max \quad f(\mathbf{x}) \\ &\text{subject to} \quad g_i(\mathbf{x}) \leq b_i \quad \text{for } i = 1, 2, \ldots, m, \end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ represents the (decision) variables, m is the number of constraints for the variables and the b_i are given constants. The set of feasible solutions is defined by

$$D = \{\mathbf{x} \in \mathbb{R}^n \,:\, g_i(\mathbf{x}) \leq b_i \text{ for } i = 1, 2, \ldots, m\}.$$

It is noted that the set D is convex if the $g_i(\mathbf{x})$'s are convex functions.[2] A feasible solution $\mathbf{x}^* \in D$ is said to be an *optimal solution* (global optimum) for the nonlinear programming problem if $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x} \in D$.

For any $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$, define the *Lagrange function*

$$\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i (g_i(\mathbf{x}) - b_i).$$

A very useful result is:

Theorem 1. *Suppose that $\mathbf{x}^*$ and λ^* satisfy*

$$\begin{aligned} &(i) \quad g_i(\mathbf{x}^*) \leq b_i \text{ for } i = 1, 2, \ldots, m \\ &(ii) \quad \mathcal{L}(\mathbf{x}^*, \lambda^*) \geq \mathcal{L}(\mathbf{x}, \lambda^*) \text{ for all } x \in \mathbb{R}^n \\ &(iii) \quad \lambda_i^* \geq 0 \text{ for } i = 1, 2, \ldots, m \\ &(iv) \quad \lambda_i^* \{g_i(\mathbf{x}^*) - b_i\} = 0 \text{ for } i = 1, \ldots, m. \end{aligned}$$

Then $\mathbf{x}^*$ is an optimal solution for the nonlinear programming problem.

[1] The Kelly system is named after the physicist John Kelly Jr. Working at Bell Labs, he published in 1956 a paper titled *A New Interpretation of Information Rate* in the Bell System Technical Journal. Virtually no one took much note of the article when it first appeared. Nowadays it is widely used in gambling and investing. In the paper Kelly posited a scenario in which a horse-race better has an edge: a 'private wire' of somewhat reliable, but not perfect tips from inside information. How should he bet? Wager too little, and the advantage is squandered. Too much, and ruin beckons. The Kelly bet size is found by maximizing the expected value of the logarithm of wealth, which is equivalent to maximizing the expected geometric growth rate. In the context of the St. Petersburg paradox, it was already suggested by Daniel Bernoulli in 1738 that a gambler should not maximize expected return but rather logarithmic utility.

[2] A nice treatment of convexity and concavity of functions of several variables can be found in https://mjo.osborne.economics.utoronto.ca/index.php/tutorial/index/1/cvn/t

Proof. The proof is simple. The first condition says that $\mathbf{x}^*$ is a feasible solution. Take any other feasible solution $\mathbf{x}$. Then.

$$f(\mathbf{x}^*) \overset{\text{(iv)}}{=} f(\mathbf{x}^*) - \sum_{i=1}^{m} \lambda_i^* \Big(g_i(\mathbf{x}^*) - b_i\Big) \overset{\text{(ii)}}{\geq} f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i^* \Big(g_i(\mathbf{x}) - b_i\Big) \overset{\text{(iii)}}{\geq} f(\mathbf{x}).$$

The last inequality also uses that $g_i(\mathbf{x}) - b_i \leq 0$ for all i.

The λ_i are called the *Lagrange multipliers* and condition (iv) is called the complementary slackness condition. In fact, Theorem 1 suggests a relaxation approach in which you try to solve the difficult nonlinear programming problem by solving the unconstrained optimization problem max $\{\mathcal{L}(\mathbf{x}, \lambda) : \mathbf{x} \in \mathbb{R}^n\}$ for given non-negative values of the Lagrange multipliers, where you try to choose λ in such way the corresponding optimal solution $\mathbf{x}^*$ satisfies the conditions (i) and (iv) in Theorem 1. However, this very computationally intensive approach is not practically useful for most problems.

Suppose now that the function $f(\mathbf{x})$ is differentiable and concave, and the functions $g_i(\mathbf{x})$ for $i = 1, \ldots, m$ are differentiable and convex. Then, the set of feasible solutions is a convex set and for any given non-negative Lagrange multipliers λ_i, the Lagrange function $\mathcal{L}(\mathbf{x}, \lambda)$ is concave as function of $\mathbf{x}$ (verify!). In this case condition (ii) is equivalent with

$$\nabla_x \mathcal{L}(\mathbf{x}^*, \lambda) = 0.$$

As consequence of Theorem 1, we now obtain the following important main theorem in nonlinear programming:

Theorem 2. *Suppose that the function $f(\mathbf{x})$ is differentiable and concave on the set of feasible solutions, and the functions $g_i(\mathbf{x})$ are differentiable and convex for $i = 1, \ldots, m$. If $\mathbf{x}^*$ and λ^* satisfy*

$$\begin{array}{ll} (i) & g_i(\mathbf{x}^*) \leq b_i \quad \text{for } i = 1, 2, \ldots, m \\ (ii) & \dfrac{\partial f(\mathbf{x}^*)}{\partial x_j} - \displaystyle\sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(\mathbf{x}^*)}{\partial x_j} = 0 \quad \text{for } j = 1, \ldots, n \\ (iii) & \lambda_i^* \geq 0 \quad \text{for } i = 1, 2, \ldots, m \\ (iv) & \lambda_i^* \{g_i(\mathbf{x}^*) - b_i\} = 0 \quad \text{for } i = 1, \ldots, m. \end{array}$$

Then $\mathbf{x}^*$ is an optimal solution for the nonlinear programming problem.

The four conditions in Theorem 2 are called *Kuhn-Tucker conditions.* These conditions generalize the optimality conditions for linear programming, where the λ_i's play the role of the dual variables.

Special case of linear equality constraints. We now consider an optimization problem with a nonlinear criterion function, linear equality constraints and non-negativity constraints for the variables:

$$\begin{aligned}
&\max \quad f(x_1, \ldots, x_n) \\
&\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{for } i = 1, 2 \ldots, m, \\
&x_j \geq 0 \quad \text{for } j = 1, \ldots, n
\end{aligned}$$

The following Kuhn-Tucker conditions then apply:

Theorem 3. *Suppose the function $f(x_1, \ldots, x_n)$ is concave on the convex set of feasible solutions. A feasible solution $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ is an optimal solution for the above optimization problem if multipliers $\lambda_1^*, \ldots, \lambda_m^* \in \mathbb{R}$ exist such that*

$$\begin{aligned}
&\frac{\partial f(x^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* a_{ij} \leq 0 \quad \text{for } j = 1, \ldots, n \\
&x_j^* \Big[\frac{\partial f(x^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* a_{ij} \Big] = 0 \quad \text{for } j = 1, \ldots, n \\
&\sum_{j=1}^{n} a_{ij} x_j^* = b_i \quad \text{for } i = 1, \ldots, m \\
&x_j^* \geq 0 \quad \text{for } j = 1, \ldots, n.
\end{aligned}$$

It is not difficult to derive Theorem 3 from Theorem 2 by considering each non-negativity requirement $x_j \geq 0$ as an inequality $g_{m+j}(\mathbf{x}) \leq 0$ with $g_{m+j}(\mathbf{x}) = -x_j$, and replacing each linear equality $\sum_{j=1}^{n} a_{ij} x_j = b_i$ by the two inequalities $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ and $\sum_{j=1}^{n} -a_{ij} x_j \leq -b_i$. The details of the derivation are left to the reader. In the next section we will use Theorem 3 to analyze the Kelly betting system.

Remark. The Kuhn-Tucker conditions are the very foundation for several computational methods for solving nonlinear programming problems, such as the quadratic programming problem with a quadratic criterion function and linear constraints. The Kuhn-Tucker conditions for quadratic programming problems have a simple form that can make solutions considerably easier to obtain than for general linear programming problems. Choosing an algorithm for a nonlinear programming problem is often difficult because no one algorithm can be expected to work for every kind of nonlinear programming problems. Software platforms as AIMMS, AMPL, GAMS, Python and **R** include a number of optimizers for nonlinear programming problems with the hope that one of these methods will suffice for the given problem. It should be pointed out that in many problems it is difficult to determine whether the objective function is concave in the feasible region and hence it is difficult to guarantee convergence to a global optimum.

The Kelly betting system

In investment situations and in sport events such as soccer matches and horse races multiple investments or bets can be simultaneously done. Imagine that opportunities to bet or invest arise at successive times $t = 1, 2, \ldots$. There are n betting objects $j = 1, \ldots, n$, where $n \geq 2$. You can simultaneously bet on one or more of these objects.

Assumption: (**a**). *At any betting opportunity, only one betting object can be successful (e.g. in a horse race only one horse can win), where object j will be successful with a given probability p_j and non-successful with probability $1 - p_j$, independently of what happened at earlier betting opportunities. Hereby*

$$0 < p_j < 1 \text{ for all } j \quad \text{and} \quad \sum_{j=1}^{n} p_j = 1.$$

(**b**). *At any betting opportunity, a stake on each non-successful object j is lost, while $f_j > 0$ dollars are added to your bankroll for every dollar staked on the successful object j. The payoffs f_j are such that $p_j f_j > 1$ for at least one object j and $\sum_{j=1}^{n} 1/f_j \geq 1$.*

The probabilities p_j are typically subjective probabilities being different for each person. For example, in horse racing you can imagine that your personal estimates of the win probability of the horses are different from the bookmaker's estimates. In the Assumption, the requirement $\sum_{j=1}^{n} p_j = 1$ can be relaxed to $\sum_{j=1}^{n} p_j \leq 1$ (introduce then an auxiliary investment object $n + 1$ with f_{n+1} very close to 0 and $p_{n+1} = 1 - \sum_{i=1}^{n} p_i$).

You start with a certain capital. At any investment opportunity, you can invest any amount up to the size of your current capital, where earlier profits can be reinvested.

Question: How to (re)invest each time your current capital in order to maximize the growth rate of your capital in the long run?

It makes no sense to invest each time your whole capital in the object j with the largest value of $p_j f_j$. It is true that this maximizes the *expected* reward, but the law of large numbers tells you that you will go broke at a certain moment when you invest many times your whole capital in that object. The simple but powerful idea of Kelly is to invest each time a *same fixed proportion* of your current capital if you want to maximize the long-run growth rate of your capital. A *Kelly strategy* is characterized by parameters $\alpha_1, \ldots, \alpha_n$ with $\alpha_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i \leq 1$. Under this strategy you invest each time the same fraction α_i of your current capital in object i, while you keep in reserve a fraction

$$\beta = 1 - \sum_{i=1}^{n} \alpha_i$$

of your current capital.

For a given Kelly strategy $(\alpha_1, \ldots, \alpha_n)$, we first determine the long-run growth rate of your capital. Next we use the Kuhn-Tucker condition from Theorem 3 to find the values of $\alpha_1, \ldots, \alpha_n$ for which the long-run growth rate of your capital is maximum.

Derivation of the growth rate. Let V_0 be your initial capital. Define the random variable V_n as

$$V_m = \text{the size of your capital after } m \text{ investment periods.}$$

Then, letting the random variable R_{ki} be equal to f_i if in the kth investment period the investment in object i is successful and be 0 otherwise,

$$V_m = \big(\beta + \sum_{i=1}^{n} \alpha_i R_{1i}\big) \times \ldots \times \big(\beta + \sum_{i=1}^{n} \alpha_i R_{mi}\big) V_0 \quad \text{for } m = 1, 2, \ldots.$$

The growth factor G_m is defined by

$$G_m = \frac{1}{m} \ln \Big(\frac{V_m}{V_0}\Big) \quad \text{for } m = 1, 2, \ldots.$$

Then,

$$G_m = \frac{1}{m} \Big[\ln \big(\beta + \sum_{i=1}^{n} \alpha_i R_{1i}\big) + \ldots + \ln \big(\beta + \sum_{i=1}^{n} \alpha_i R_{mi}\big) \Big].$$

For any fixed i, the random variables R_{ki} for $k = 1, 2, \ldots$ are independent and identically distributed. By the law of large numbers,

$$\lim_{m \to \infty} G_m = E\Big[\ln \big(\beta + \sum_{i=1}^{n} \alpha_i R_i\big) \Big] \quad \text{with probability 1},$$

where random vector $(R_1, \ldots, R_n)$ has the joint distribution

$$P(R_i = f_i,\ R_j = 0 \text{ for } j \neq i) = p_i \quad \text{for } i = 1, \ldots, n.$$

Thus

$$\lim_{m \to \infty} G_m = \sum_{i=1}^{n} p_i [\ln(\beta + f_i \alpha_i)] \quad \text{with probability 1.}$$

This formula gives the asymptotic growth factor of your capital when each time you invest the same fixed fraction α_i of your current capital in object i for $i = 1, \ldots, n$.

Kuhn-Tucker analysis for the optimal Kelly strategy. The goal is to find the values for the α_i's such that the long-run growth rate of your capital is maximum. Therefore you must solve the optimization problem

$$\begin{array}{lll} \text{Maximize} & f(\beta, \alpha_1, \ldots, \alpha_n) & = \sum_{i=1}^{n} p_i \ln(\beta + f_i \alpha_i) \\ \text{subject to} & \beta + \displaystyle\sum_{i=1}^{n} \alpha_i & = 1 \\ & \beta, \alpha_1, \ldots, \alpha_n & \geq 0. \end{array}$$

The criterion function $f(\beta, \alpha_1, \ldots, \alpha_n)$ is concave on the convex set of feasible solutions.[3] By the Kuhn-Tucker conditions in Theorem 3, $\beta, \alpha_1, \ldots, \alpha_n \geq 0$ are optimal values for the optimization problem if for some real number λ,

$$
\begin{aligned}
\frac{p_i f_i}{\beta + f_i \alpha_i} - \lambda &\leq 0 \quad \text{for } i = 1, \ldots, n, && (5.1) \\
\sum_{i=1}^{n} \frac{p_i}{\beta + f_i \alpha_i} - \lambda &\leq 0, && (5.2) \\
\alpha_i \Big[\frac{p_i f_i}{\beta + f_i \alpha_i} - \lambda \Big] &= 0 \quad \text{for } i = 1, \ldots, n, && (5.3) \\
\beta \Big[\sum_{i=1}^{n} \frac{p_i}{\beta + f_i \alpha_i} - \lambda \Big] &= 0, && (5.4) \\
\beta + \sum_{i=1}^{n} \alpha_i = 1. & && (5.5)
\end{aligned}
$$

Observe that (5.1) and (5.3) give

$$\frac{p_i f_i}{\beta} - \lambda \leq 0 \text{ if } \alpha_i = 0 \quad \text{and} \quad \frac{p_i f_i}{\beta + f_i \alpha_i} - \lambda = 0 \text{ if } \alpha_i > 0.$$

In view of $p_j < 1$ for all j, is reasonable to expect that $\beta > 0$ in the optimal solution. Let us take this for granted for the moment and check it later. Then, condition (5.4) reduces to

$$\sum_{i=1}^{n} \frac{p_i}{\beta + f_i \alpha_i} = \lambda, \tag{5.6}$$

and so condition (5.2) also holds. Let the set V be defined by

$$V = \{ i \mid \alpha_i > 0 \}.$$

Using the above observations, the above Kuhn-Tucker conditions can now be rewritten as

$$
\begin{aligned}
\frac{p_i f_i}{\beta + f_i \alpha_i} &= \lambda \quad \text{for } i \in V && (5.7) \\
\frac{p_i f_i}{\beta} &\leq \lambda \quad \text{for } i \notin V && (5.8) \\
\sum_{i=1}^{n} \frac{p_i}{\beta + f_i \alpha_i} &= \lambda && (5.9) \\
\beta + \sum_{i=1}^{n} \alpha_i &= 1. && (5.10)
\end{aligned}
$$

[3] Letting $f(x, y) = \ln(x + cy)$ for variables $x, y > 0$ and constant $c > 0$, it follows that $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2 = 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, and so $f(x, y)$ is concave in the two variables $x, y > 0$. Using this result and the basic definition of concavity, it is now readily verified that $\sum_{i=1}^{n} p_i \ln(\beta + f_i \alpha_i)$ is concave on the convex set of feasible solutions.

The next step is to solve β, the α_i's and λ from this system of conditions. Equation (5.7) can be rewritten as

$$\alpha_i = \frac{p_i}{\lambda} - \frac{\beta}{f_i} \quad \text{for } i \in V. \tag{5.11}$$

Note $\lambda \neq 0$ under condition (5.9). By the definition of V, condition (5.10) can be written as $\sum_{i \in V} \alpha_i = 1 - \beta$. Substituting (5.11) in this equality, we get

$$\beta = \frac{1 - \sum_{i \in V} p_i/\lambda}{1 - \sum_{i \in V} 1/f_i}. \tag{5.12}$$

Next we verify that $\lambda = 1$. Replacing $\sum_{i \in V} p_i$ by $1 - \sum_{i \notin V} p_i$, we can rewrite (5.12) as

$$\sum_{i \notin V} p_i = 1 - \lambda + \lambda\beta\Big(1 - \sum_{i \in V} 1/f_i\Big).$$

Using this equality and splitting $\sum_{i=1}^{n}$ into $\sum_{i \in V}$ and $\sum_{i \notin V}$ in (5.9), equation (5.9) can be rewritten as

$$\sum_{i \in V} \frac{p_i}{\beta + f_i \alpha_i} + \frac{1 - \lambda}{\beta} = \lambda \sum_{i \in V} 1/f_i.$$

Substituting (5.7) in this equation gives,

$$\sum_{i \in V} \frac{\lambda}{f_i} + \frac{1 - \lambda}{\beta} = \lambda \sum_{i \in V} \frac{1}{f_i}.$$

showing that under the conditions (5.7)$-$(5.10),

$$\lambda = 1.$$

Summarizing, we must find non-negative $\alpha_1, \ldots, \alpha_n$ and $V = \{i \mid \alpha_i > 0\}$ such that

(**a**) $\sum_{i \in V} 1/f_i < 1$

(**b**) $p_i f_i > \beta$ for $i \in V$ and $p_i f_i \leq \beta$ for $i \notin V$.

(**c**) $\beta = 1 - \sum_{v \in V} \alpha_v > 0$.

Then the Kuhn-Tucker conditions (5.7)$-$(5.10) with $\lambda = 1$ are satisfied. Note that the second part of (**b**) is condition (5.8).

In order to satisfy (**a**)$-$(**c**), it is no restriction to assume that the investment objects are (re)numbered such that

$$p_1 f_1 \geq p_2 f_2 \geq \ldots \geq p_n f_n.$$

This ordering implies the following result for the algorithm for computing the optimal solution:

$$p_j f_j \leq B(s) \quad \text{for all } j \geq s + 1 \text{ if } p_{s+1} f_{s+1} \leq B(s),$$

where $B(k)$ is defined by

$$B(k) = \frac{1 - \sum_{j=1}^{k} p_j}{1 - \sum_{j=1}^{k} 1/f_j} \quad \text{for } k = 1, \ldots, n.$$

This leads to the following algorithm.

Algorithm

Step 0. Arrange that $p_1 f_1 \geq p_2 f_2 \geq \ldots \geq p_n f_n$.

Step 1. Determine r as the largest integer k for which $\sum_{j=1}^{k} 1/f_j < 1$.[4]

Step 2. Calculate $B(k)$ over the indexes $k = 1, \ldots, r$. Stop at the first index k for which $p_{k+1} f_{k+1} \leq B(k)$ (and then $p_j f_j \leq B(k)$ for all $j > k$). Let s be this index and let $\beta = B(s)$.

Step 3. Set $\alpha_i = p_i - \beta/f_i$ for $i = 1, \ldots, s$ and $\alpha_i = 0$ for $i > s$.

It is now easily seen that the algorithm satisfies the conditions (**a**)−(**c**). Condition (**c**) is satisfied: $B(s) > 0$, by $s \leq r < n$, and so $\alpha_i < p_i$ for all i, which implies that $\sum_{i \in V} \alpha_i < \sum_{i=1}^{n} p_i = 1$. Thus the algorithm produces the optimal values for the α_i's.[5]

Next we give two numerical examples to illustrate the algorithm.

Numerical examples. The Kelly strategy has been developed for situations in which many betting opportunities repeat themselves under identical conditions. However, the Kelly strategy provides also a useful heuristic guideline for situations with only one opportunity to bet.

Example 1 (Soccer). Suppose that the soccer club Manchester United is hosting a match against Liverpool, and that a bookmaker is paying out 4.5 times the stake if Liverpool wins, 4.5 times the stake if the match ends in a draw, and 1.75 times the stake if Manchester United wins. You estimate Liverpool's chance of winning at 25%, the chance of the game ending in a draw at 25%, and the chance of Manchester winning at 50%. If you are prepared to bet 100 pounds, how should you bet on this match? The Kelly betting model with $n = 3$ betting objects applies, where

$$p_1 = 0.25 \text{ (win for Liverpool)},\ p_2 = 0.25 \text{ (draw)},\ p_3 = 0.50 \text{ (win for United)}$$
$$f_1 = f_2 = 4.5 \text{ and } f_3 = 1.75.$$

Since $p_1 f_1 = p_2 f_2 = 1.125$ and $p_3 f_3 = 0.875$, the condition $p_1 f_1 \geq p_2 f_2 \geq p_3 f_3$ is satisfied. The algorithm goes as follows:

[4] Note that $r < n$, by the second part of Assumption 2.

[5] In the analysis we have assumed that $n \geq 2$. For the case of a single investment/betting object ($n = 1$) with win probability p and payment factor f such that $0 < p < 1$ and $pf > 1$, the algorithm boils down to the optimal Kelly betting fraction $\alpha = (pf - 1)/(f - 1)$. This betting fraction satisfies $0 < \alpha < 1$.

Step 1. Since $1/f_1 = \frac{10}{45}$, $1/f_1 + 1/f_2 = \frac{20}{45}$ and $1/f_1 + 1/f_2 + 1/f_3 > 1$, the index $r = 2$.
Step 2. $B(1) = \frac{27}{28}$, $B(2) = \frac{9}{10}$ and $p_2 f_2 = 1.125 > B(1)$. This gives $s = 2$ with $\beta = B(s) = 0.9$.
Step 3. $\alpha_1 = \alpha_2 = 0.25 - \frac{0.9}{4.5} = 0.05$ and $\alpha_3 = 0$.

Thus the Kelly strategy proposes that you stake 5% of your bankroll of 100 pounds on a win for Liverpool, 5% on a draw, and 0% on a win for Manchester United. For this strategy, the subjective expected value of your bankroll after the match is equal to $100 - 10 + 0.25 \times 4.5 \times 5 + 0.25 \times 4.5 \times 5 = 101.25$ pounds. The expected percentage increase of your bankroll is 1.25%. It is interesting to note that the two concurrent bets on the soccer match act as a partial hedge for each other, reducing the overall level of risk.[6]

Example 2 (Horse race). In a horse race there are seven horses A, B, C , D , E, F and G with respective win probabilities 40%, 25%, 20%, 7%, 4%, 3% and 1% and payoff odds 1.625:1, 2.9:1, 4.5:1, 9:1, 14:1. 17:1 and 49:1. Payoff odds a:1 means that in case of a win you will receive your stake plus a pounds for each pound staked. Numbering the horses A, B, C, D, E, F, and G as 1 (= C), 2 (= A), 3 (= B), 4 (= D), 5 (= E), 6 (= F), and 7 (= G), the Kelly model applies with

$$p_1 = 0.2,\ p_2 = 0.4,\ p_3 = 0.25,\ p_4 = 0.07,\ p_5 = 0.04,\ p_6 = 0.03,\ p_7 = 0.01,$$
$$f_1 = 5.5,\ f_2 = 2.625,\ f_3 = 3.9, f_4 = 10,\ f_5 = 15,\ f_6 = 18,\ f_7 = 50$$

satisfying the condition of decreasing values of the $p_i f_i$'s:

$$p_1 f_1 = 1.1,\ \ p_2 f_2 = 1.05,\ \ p_3 f_3 = 0.975,\ \ p_4 f_4 = 0.7,$$
$$p_5 f_5 = 0.6,\ \ p_6 f_6 = 0.54,\ \ p_7 f_7 = 0.50.$$

The algorithm goes as follows:

Step 1. The index $r = 5$ is the largest value of k for which $\sum_{j=1}^{k} 1/f_j < 1$.
Step 2. $B(1) = 0.97778$, $B(2) = 0.91485$, $B(3) = 0.82956$, $B(4) = 0.98986$ and $B(5) = 2.82635$. Also, $p_2 f_2 > B(1)$, $p_3 f_3 > B(2)$, but $p_4 f_4 \leq B(3)$. This gives $s = 3$ with $\beta = B(s) = 0.82956$.
Step 3. $\alpha_1 = 0.0492$, $\alpha_2 = 0.0840$, $\alpha_3 = 0.0373$, and $\alpha_j = 0$ for $j > 3$.

Thus you bet 8.4% of your bankroll on horse A, 3.7% on horse B, 4.9% on horse C and nothing on the other horses. It is noteworthy that horse B is included in your bet, even though a bet on horse B alone is not favorable ($p_3 f_3 < 1$). The expected value of the percentage increase in your bankroll is $100 \times \sum_{j=1}^{3} (p_i f_i \alpha_i - \alpha_i) = 4.6\%$.

[6] An interesting project would be to derive an algorithm for the case of simultaneous betting options, where the outcomes of the bets are *independent* of each other and thus more than one bet can be successful at the same time. Think of betting on a number of soccer matches that are played at the same time. How should a gambler allocate the stakes when the Kelly criterion of maximizing log-utility is used? This question is addressed in C. Whitrow. Algorithms for Optimal Location of Bets on Many Simultaneous Events, *Journal of the Statistical Royal Society Series C*, Vol. 66, No 5 (2007), pp. 607-623.