Mathemagics

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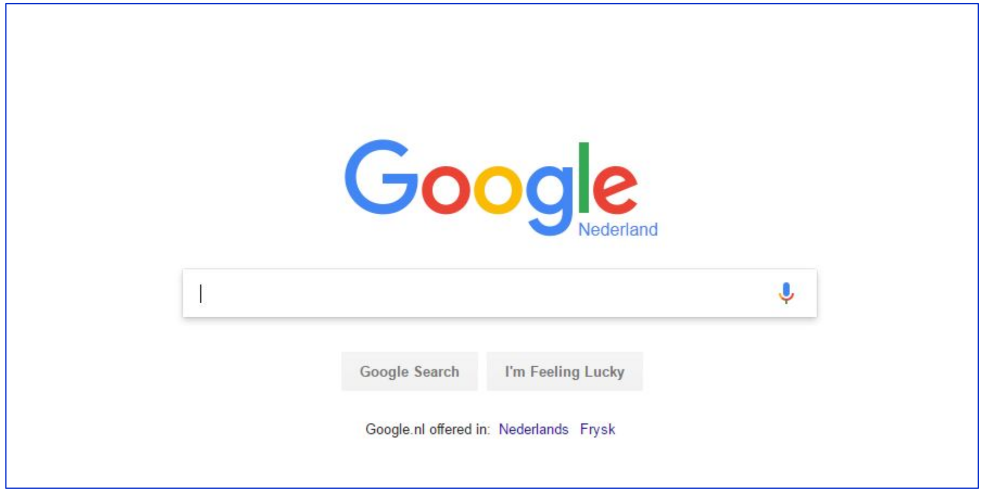
Internet

Internet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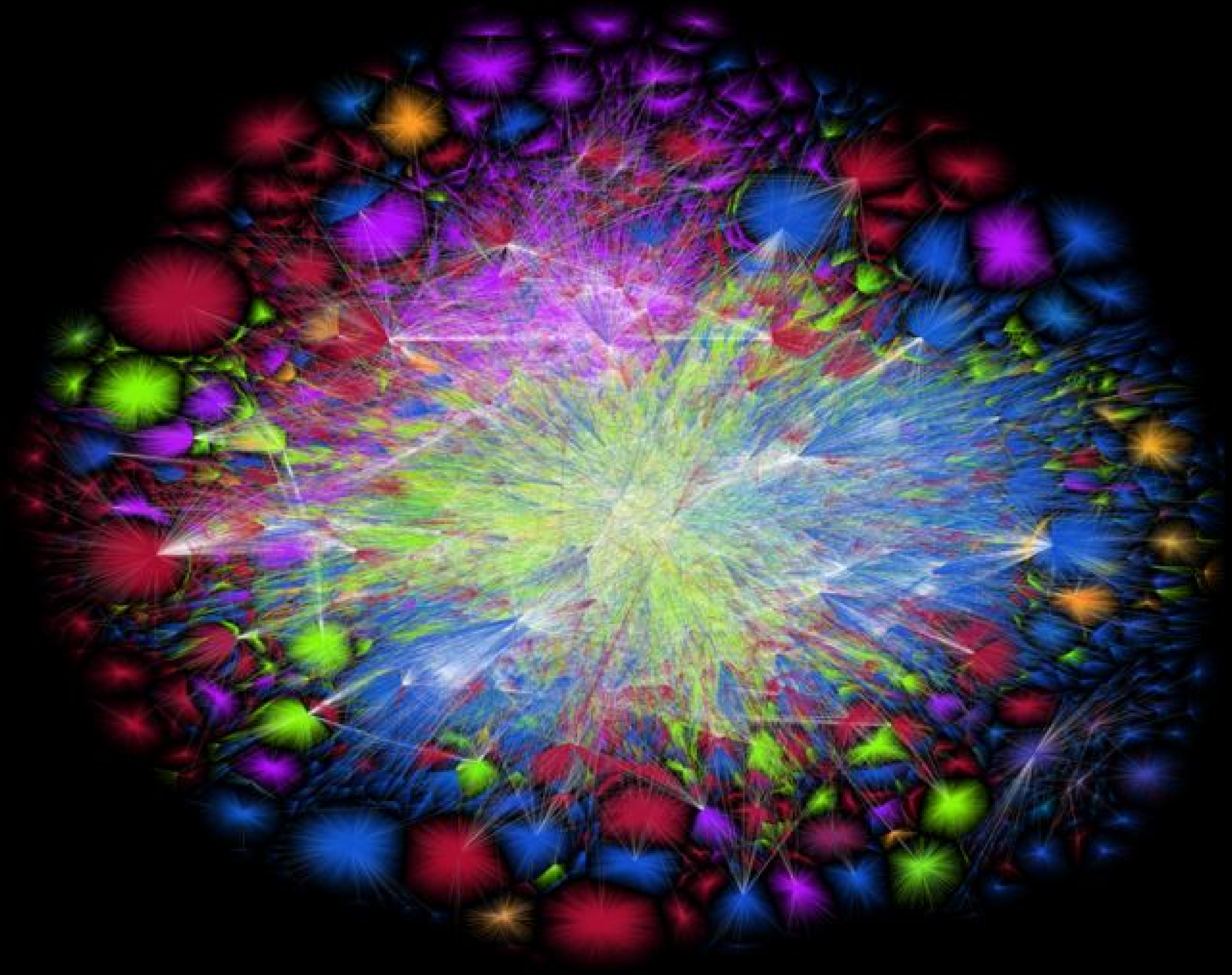
Internet

Will the Internet stay connected?

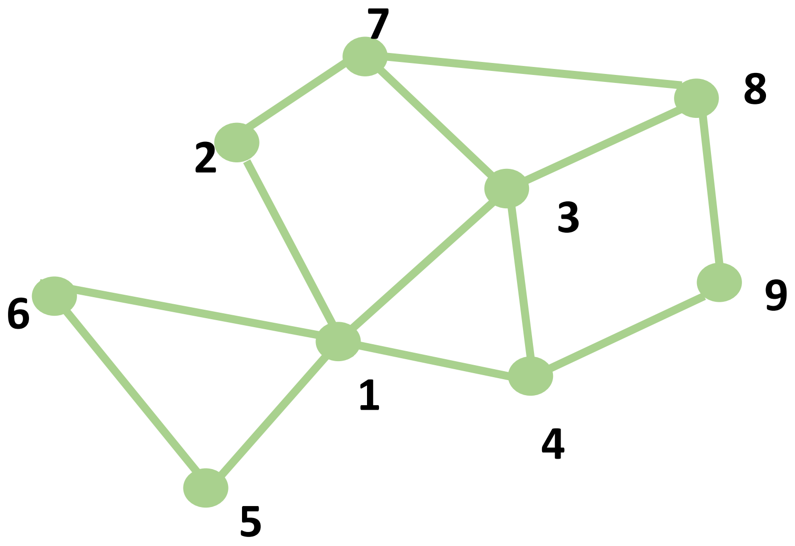
Internet as a graph

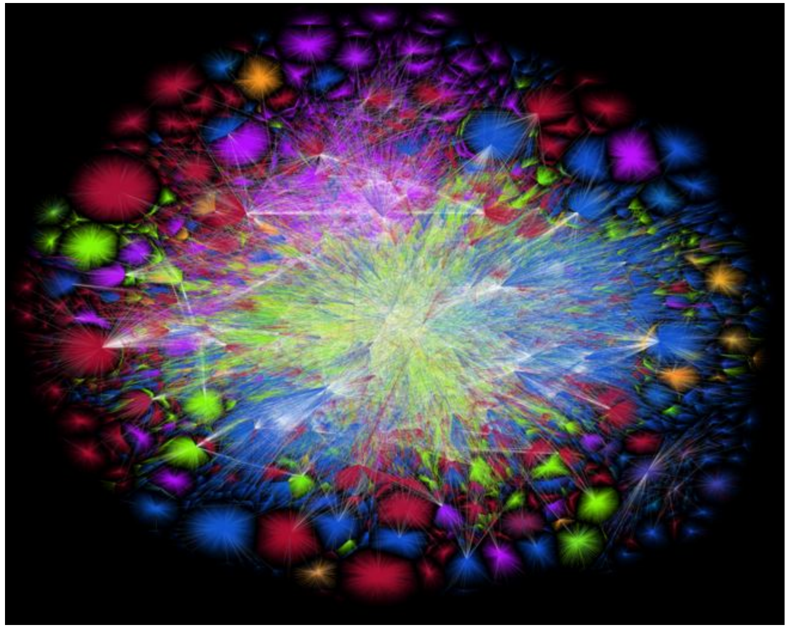
- Servers/computers = vertices
- Connections = edges
- How will this graph look like?



Barrett Lyon www.opte.org

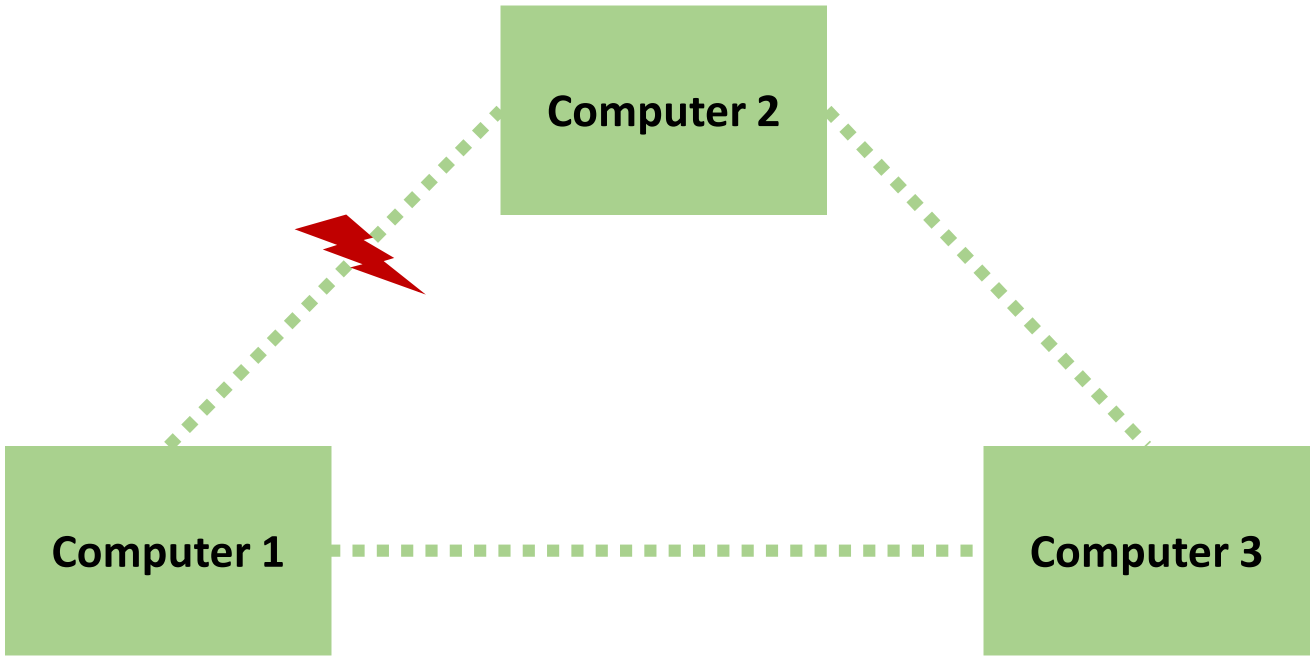
- **Connected graph:**
  There is path along edges from any vertex to any other vertex





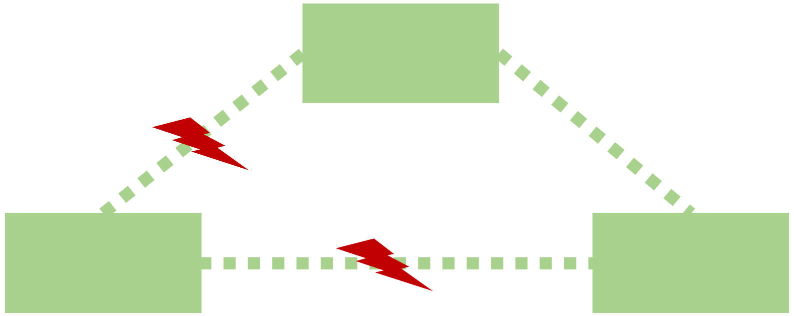
- Will the Internet stay connected under failures, overloads, attacks?

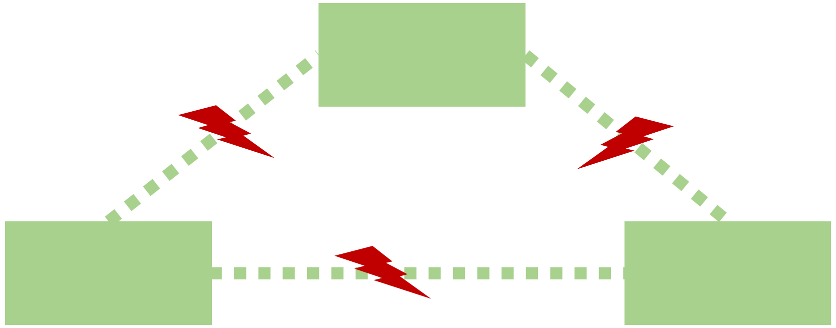
Mini-Internet



- A channel is available with probability ***p*, *0<p<1***
- A channel is unavailable with probability ***1-p***

Probability to disconnect the network





$3p(1\text{-}p)^2$ $+$ $(1\text{-}p)^3$

- When ***(1-p)*** is small, ***(1-p)> 3p(1-p)²+ (1-p)³***
- The network is more robust than one channel!
- **What about large networks?**

Erdös-Rényi random graph (1959)

- ***n*** vertices
- An edge between two vertices exists with probability ***p***
- Independently of other edges
- Take ***p=p(n)***
- **Theorem (Erdös-Rényi).**
  - If **$p>\ln(n)/n$**, then with high probability the network is **connected**
  - If **$p<\ln(n)/n$**, then with high probability the network is **disconnected**
  - If **$p=\ln(n)/n$**, then the network is **disconnected with probability, which converges to e^{-1}**

Phase transition

Ice turning to water at 0°C

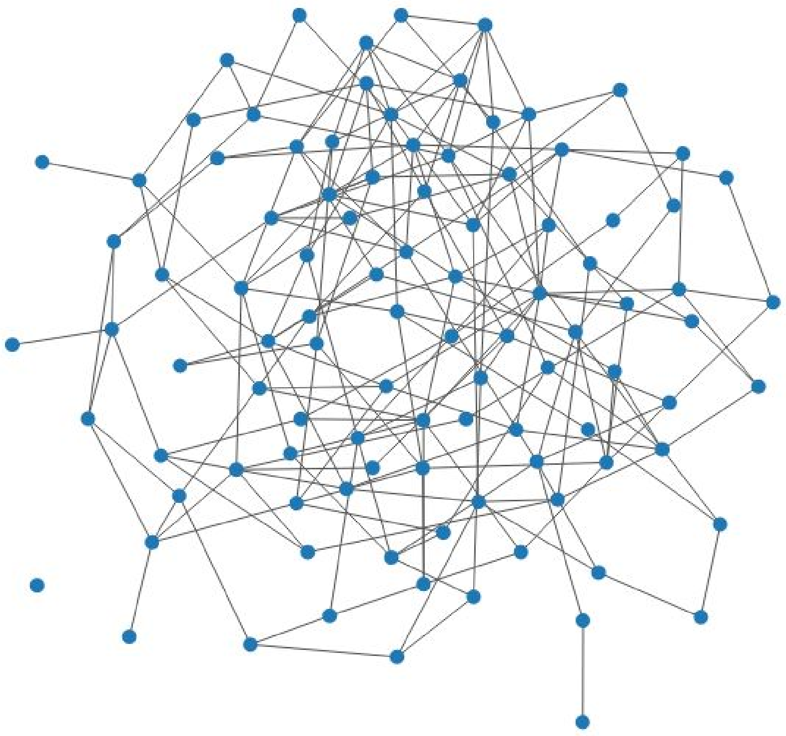
Phase transition

- **Theorem (Erdös-Rényi).**
  - If **$p>\ln(n)/n$**, then with high probability the network is **connected**
  - If **$p<\ln(n)/n$**, then with high probability the network is **disconnected**
  - If **$p=\ln(n)/n$**, then the network is **disconnected with probability, which converges to e^{-1}**

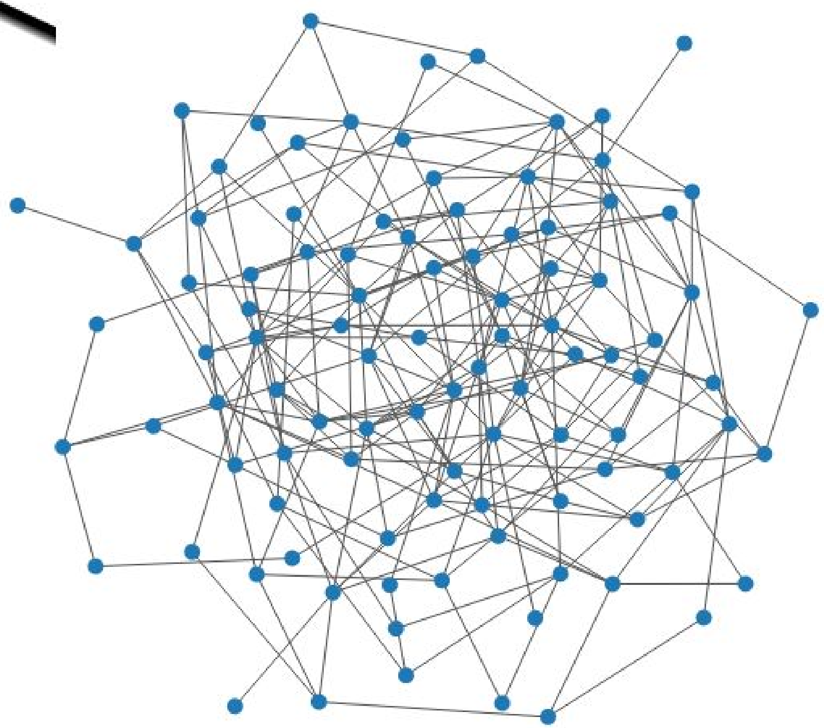
- Critical probability **$p=\ln(n)/n$**
- Decreases with ***n***
- Again, larger networks are more robust

Example

- ***n=100, ln(n)/n≈0.046***



p=0.04



p=0.05

Magic revealed

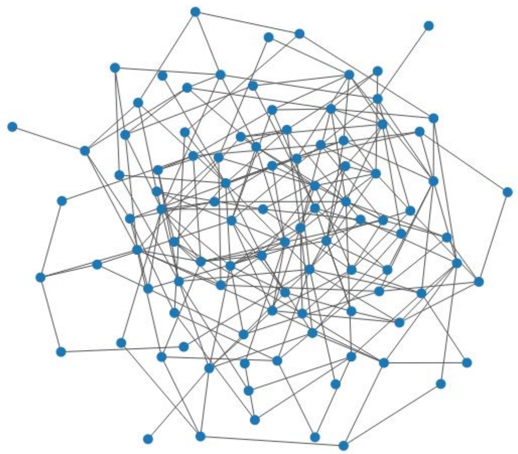
- Most likely way to get the network disconnected: completely disconnect at least one of the vertices
  - It is more difficult to disconnect a group of vertices
- ***P***(one vertex is disconnected) $= (1 - p(n))^{n-1}$
- Average number of disconnected vertices $= n\,(1 - p(n))^{n-1}$
- Substitute $p(n) = c \ln(n)/n$

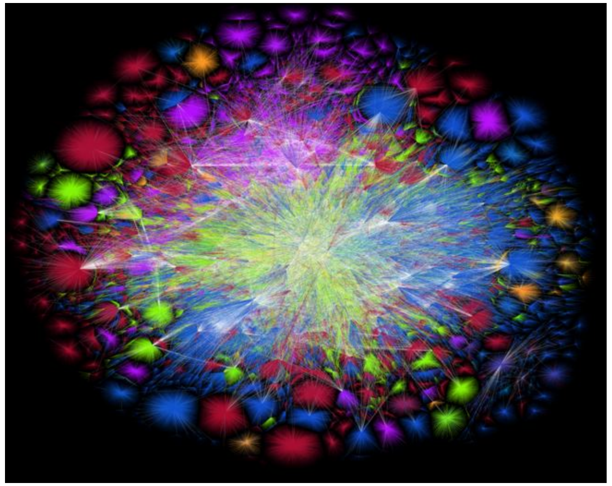
$$\lim_{n\to\infty} n\left(1-\frac{c\ln(n)}{n}\right)^{n-1} = \lim_{n\to\infty} ne^{-c\ln(n)} = \lim_{n\to\infty} n^{1-c}.$$

- If ***c<1*** then the average number of disconnected vertices goes to infinity
- If ***c>1*** then the average number of disconnected vertices goes to zero
- The actual number of disconnected vertices is close to its average
- If ***c=1***, the number of disconnected vertices converges to a Poisson(1) distribution, and P(no disconnected vertices)=e^{-1}

Back to the Internet

- Erdös-Rényi random graph is not a realistic model for the Internet





- Hubs, backbone, bandwidth
- A lot of research on robustness of the Internet
- However, even the simplest model given important insights:
  - Robustness of large network
  - Phase transition

Mathemagics forever!

N. Litvak, A. Raigorodskii. *Who needs mathematics? A clear book about how the digital world is arranged.* MIF publishers, Moscow.

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