Optimal Control

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Lecture 1

- Organization
- What is OC about?
- Appendix B: $\dot{x}(t) = f(x(t))$
  - existence
  - Lipschitz
  - stability & equilibrium
  - Lyapunov

Organization

- 12 lectures, 4 tutorials
- Canvas
- Book “A Course on Optimal Control”
- Videos (a bit outdated)
- GM & HZ
- Exam
- Schedule (in short)
  - Appendix B: Differential Equations & Lyapunov functions
  - Chapter 1: Calculus of Variations
  - Chapter 2: Optimal Control: Minimum Principle
  - Chapter 3: Optimal Control: Dynamic Programming
  - Chapter 4: Optimal Control: Linear Quadratic Control (LQ)

What is OC about?

Example (Classic optimization (what you know already))

$$\min_{x\in\mathbb{R}} J(x)$$

In this course we minimize not over numbers, but over functions:

Example (Calculus of Variations)

Let $\mathrm{x}(t)$ be the position of a car at time t (of, say, the "solar challenge car").

$$\min_{\mathrm{x}:[0,T]\to\mathbb{R}} \text{"used energy"}$$

for given initial $\mathrm{x}(0)$ and final $\mathrm{x}(T)$.

Example (Calculus of Variations (previous example))

Let $\mathtt{x}(t)$ be the position of a car at time t (of, say, the "solar challenge car").

$$\min_{\mathtt{x}:[0,T]\to\mathbb{R}} \text{"used energy"}$$

for given initial $\mathtt{x}(0)$ and final $\mathtt{x}(T)$.

In reality car position $\mathtt{x}$ is not free to choose: dynamical constraint:

Example (Optimal Control)

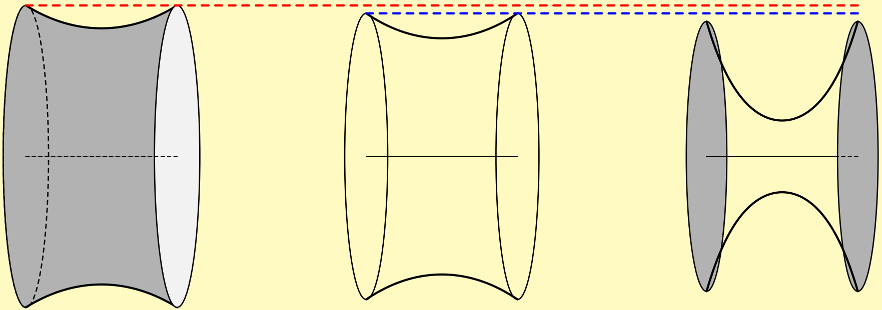
Let u be throttle opening of the car.

$$\min_{\mathtt{u}:[0,T]\to[0,1]} \text{"used energy"}$$

subject to $\dot{\mathtt{x}}(t) = f(\mathtt{x}(t), \mathtt{u}(t))$ and given $\mathtt{x}(0)$ and/or $\mathtt{x}(T)$.

Notice, also $\mathtt{u}(t)$ may be constrained

Calculus of Variations example: catenoid



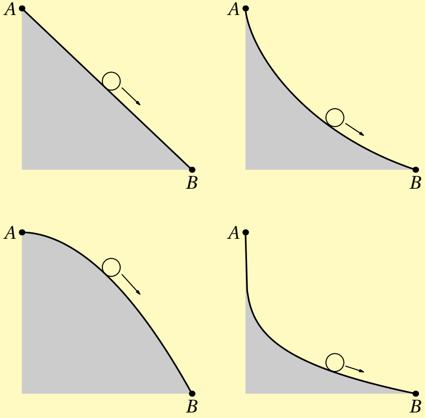
catenoid is minimal

catenoid surface is equal to two-disc

two-disc is minimal

Calculus of Variations example: brachistochrone (1696)

Four paths from A to B:



Which is fastest?

- Organization
- What is OC about?
- Appendix B: DE's $\dot{\mathtt{x}}(t) = f(\mathtt{x}(t))$
    - existence and uniqueness of solution $\mathtt{x}(t)$
    - Lipschitz continuity
    - stability & equilibrium
    - Lyapunov functions for stability

Part I

Appendix B — DE's and Lyapunov Functions

Overview

B.1 Existence and Uniqueness of Solutions

$$\begin{cases} \dot{\mathtt{x}}_1(t) &= f_1(\mathtt{x}_1(t),\ldots,\mathtt{x}_n(t)), & \mathtt{x}_1(0) = x_{01} \\ \dot{\mathtt{x}}_2(t) &= f_2(\mathtt{x}_1(t),\ldots,\mathtt{x}_n(t)), & \mathtt{x}_2(0) = x_{02} \\ & \vdots & \\ \dot{\mathtt{x}}_n(t) &= f_n(\mathtt{x}_1(t),\ldots,\mathtt{x}_n(t)), & \mathtt{x}_n(0) = x_{0n} \end{cases}$$

$$\dot{\mathtt{x}}(t) = f(\mathtt{x}(t)), \qquad \mathtt{x}(0) = x_0 \in \mathbb{R}^n, \qquad t \geq 0$$

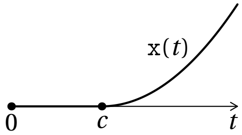
We write $\mathtt{x}$ or $\mathtt{x}(t)$ or $\mathtt{x}(t; x_0)$

The DE

$$\dot{\mathtt{x}}(t) = \sqrt{\mathtt{x}(t)}, \qquad \mathtt{x}(0) = 0$$

has many solutions:

$$\mathtt{x}(t) = \begin{cases} 0 & t \in [0, c] \\ \frac{1}{4}(t-c)^2 & t > c \end{cases}$$

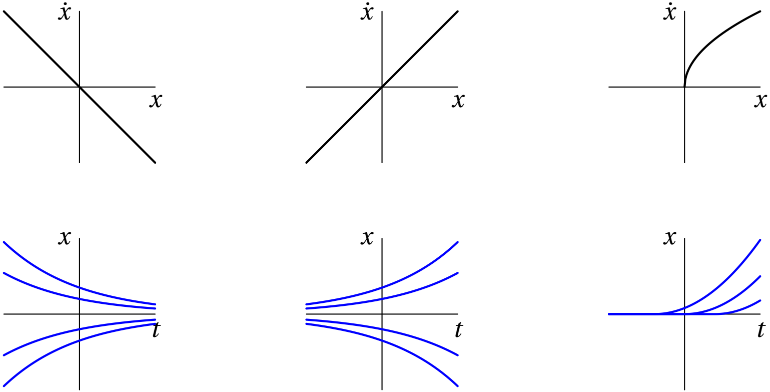


Every $c \geq 0$ yields a solution.
So the solution of this DE is not unique! Weird.

$$\dot{\mathtt{x}}(t) = -\mathtt{x}(t)$$

$$\dot{\mathtt{x}}(t) = +\mathtt{x}(t)$$

$$\dot{\mathtt{x}}(t) = \sqrt{\mathtt{x}(t)}$$



Definition (Lipschitz continuity)

Let $\Omega \subset \mathbb{R}^n$ and let $\|\cdot\|$ be some norm on $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}^n$ is Lipschitz-continuous on Ω if a Lipschitz constant $K \geq 0$ exists such that

$$\|f(x) - f(z)\| \leq K\|x - z\| \qquad \forall x, z \in \Omega.$$

It is Lipschitz-continuous at x_0 if it is LC on some neighborhood Ω of x_0.
It is locally Lipschitz continuous if it is LC at every $x_0 \in \mathbb{R}^n$.

Example

$f(x) = \sqrt{x}$ is not LC at $x = 0$

Example

$f(x) = kx$ has (global) Lipschitz constant $K = |k|$

Then solution of $\dot{\mathrm{x}}(t) = f(\mathrm{x}(t))$ grows at most exponentially fast: $|\mathrm{x}(t)| \leq |\mathrm{x}(0)|\mathrm{e}^{Kt}$

Example

If $f(x)$ is C^1 at x then it is LC at x

Classic result:

Theorem (Existence and uniqueness of solution)

Let $x_0 \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. If f is LC at x_0 then, for some $T > 0$, the DE

$$\dot{\mathtt{x}}(t) = f(\mathtt{x}(t)), \qquad \mathtt{x}(0) = x_0, \qquad t \geq 0$$

has a unique solution $\mathtt{x}(t; x_0)$ for all $t \in [0, T)$.

Moreover, for every fixed $t \in [0, T)$, the solution $\mathtt{x}(t; x_0)$ is continuous at x_0. Specifically, if ...., then

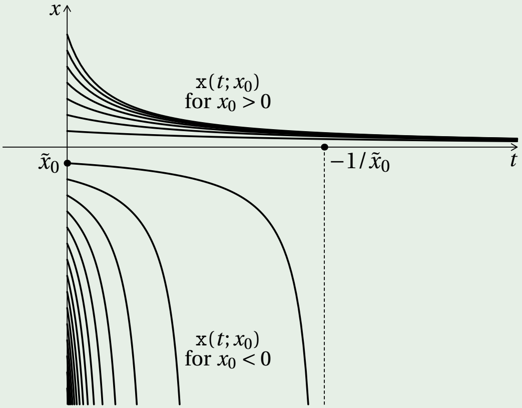
$$\|\mathtt{x}(t; x_0) - \mathtt{x}(t; z_0)\| \leq \|x_0 - z_0\| \mathrm{e}^{Kt} \qquad \forall t \in [0, T).$$

Hence, for C^1 functions $f(x)$ the solution can be extended indefinitely, but ...

... but the solution can still escape in finite time:

Example ($f(x) = -x^2$)

The solution of $\dot{\mathtt{x}}(t) = -\mathtt{x}^2(t),\ \mathtt{x}(0) = x_0$ is $\mathtt{x}(t; x_0) = \dfrac{x_0}{t x_0 + 1}$:



Theorem (Escape time)

Suppose $f:\mathbb{R}^n \to \mathbb{R}^n$ is locally LC. Then for every $\mathtt{x}(0) = x_0$ there is a unique escape time $t(x_0) > 0$ (possibly $t(x_0) = \infty$) such that $\mathtt{x}(t; x_0)$ exists and is unique on $[0, t(x_0))$ but does not exist for $t > t(x_0)$.

Moreover, if $t(x_0) < \infty$ then $\lim_{t \uparrow t(x_0)} \|\mathtt{x}(t; x_0)\| = \infty$.

If f is globally LC then $t(x_0) = \infty$, i.e., the solution $\mathtt{x}(t; x_0)$ then exists and is unique for all $t \geq 0$.

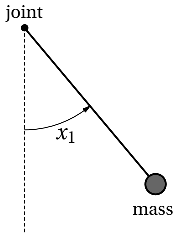
B.2 Definitions of stability (& equilibrium)

A constant function $\mathrm{x}(t) = \bar{x}$ satisfies $\dot{\mathrm{x}}(t) = f(\mathrm{x}(t))$ iff $0 = f(\bar{x})$.

Definition (Equilibrium)

$\bar{x} \in \mathbb{R}^n$ is an equilibrium (point) if $f(\bar{x}) = 0$.

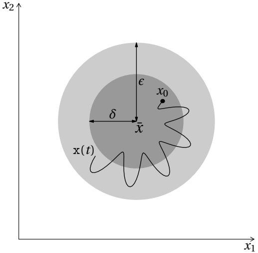
So the equilibrium points are the constant solutions.



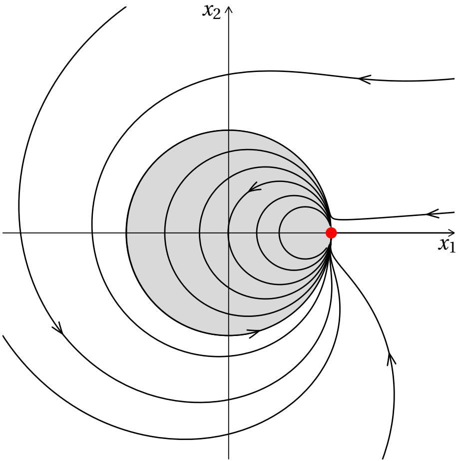
Definition (Stable and unstable equilibria)

An equilibrium point $\bar{x}$ of a DE $\dot{\mathrm{x}}(t) = f(\mathrm{x}(t)), \mathrm{x}(0) = x_0$ is called

1. stable if $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $\|x_0 - \bar{x}\| < \delta \implies \|\mathrm{x}(t; x_0) - \bar{x}\| < \epsilon\ \forall t \geq 0$.
2. attractive if $\exists \delta_1 > 0$ s.t. $\|x_0 - \bar{x}\| < \delta_1 \implies \lim_{t\to\infty} \mathrm{x}(t; x_0) = \bar{x}$.
3. asymptotically stable if it is stable and attractive.
4. globally attractive if $\lim_{t\to\infty} \mathrm{x}(t; x_0) = \bar{x}$ for every $x_0 \in \mathbb{R}^n$.
5. globally asymptotically stable if it is stable and globally attractive.
6. unstable if $\bar{x}$ is not stable. This means $\exists \epsilon > 0$ such that $\forall \delta > 0$ an x_0 and a $t_1 > 0$ exists for which $\|x_0 - \bar{x}\| < \delta$ yet $\|\mathrm{x}(t_1; x_0) - \bar{x}\| \geq \epsilon$.



An exercise of Appendix B: globally attractive yet not stable:



B.3 Lyapunov Functions

Here we can “see” that $\bar{x} := (0,0)$ is asymptotically stable without knowing the solution:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -x_2(t) \\ x_1(t) \end{bmatrix} - \epsilon \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \epsilon > 0.$$

How to generalize (without resorting to “plots”)?

Another example where we can "see" the stability from plots

Example

Consider $\dot{x}(t) = -x^3(t)$.

We claim that $V(x(t)) \to 0$ as $t \to \infty$:

$$\dot{V}(x(t)) = \frac{dx^2(t)}{dt}$$
$$=$$
$$=$$

Awesome: to verify that $\dot{V}(x(t)) \leq 0$ we don't need to know the solution $x(t)$.
This idea generalizes nicely:

$$
\begin{aligned}
\dot V(\mathtt{x}(t)) &= \frac{\mathrm{d}V(\mathtt{x}(t))}{\mathrm{d}t} \\
&= \frac{\partial V(\mathtt{x}(t))}{\partial x_1}\dot{\mathtt{x}}_1(t) + \cdots + \frac{\partial V(\mathtt{x}(t))}{\partial x_n}\dot{\mathtt{x}}_n(t) \\
&= \frac{\partial V(\mathtt{x}(t))}{\partial x_1} f_1(\mathtt{x}(t)) + \cdots + \frac{\partial V(\mathtt{x}(t))}{\partial x_n} f_n(\mathtt{x}(t)) \\
&= \frac{\partial V(x)}{\partial x^{\mathrm{T}}} f(x)\Big|_{x=\mathtt{x}(t)}.
\end{aligned}
$$

So $\dot V(\mathtt{x}(t)) \leq 0$ for all solutions $\mathtt{x}(t)$ and all t iff

$$
\frac{\partial V(x)}{\partial x^{\mathrm{T}}} f(x) \leq 0 \qquad \forall x \in \mathbb{R}^n
$$

Notice: final condition does not require solutions of the DE! In fact, t is gone. Awesome.

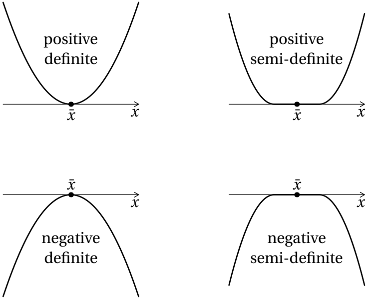
Definition (Positive and negative (semi) definite)

Let $\Omega \subseteq \mathbb{R}^n$ and assume it is a neighborhood of some $\bar{x} \in \mathbb{R}^n$.
A function $V : \Omega \to \mathbb{R}$ is positive definite on Ω relative to $\bar{x}$ if

$$V(\bar{x}) = 0 \text{ while } V(x) > 0 \text{ for all } x \in \Omega \setminus \{\bar{x}\}.$$

It is positive semi-definite if $V(\bar{x}) = 0$ and $V(x) \geq 0$ for all other $x \in \Omega$.

We say V is negative (semi) definite if $-V$ is positive (semi) definite.



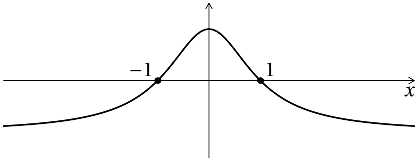
Theorem (Lyapunov's second stability theorem)

Consider $\dot{\mathrm{x}}(t) = f(\mathrm{x}(t))$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz continuous. Let $\bar{x}$ be an equilibrium of this DE. If there is a neighborhood Ω of $\bar{x}$ and a function $V : \Omega \to \mathbb{R}$ such that on Ω:

1. V is continuously differentiable,
2. V is positive definite relative to $\bar{x}$,
3. $\dot{V}$ is negative semi-definite relative to $\bar{x}$,

then $\bar{x}$ is a stable equilibrium, and we call V a Lyapunov function.

If in addition $\dot{V}$ is negative definite (so *not* just negative *semi*-definite) then $\bar{x}$ is asymptotically stable, and we call V a strong Lyapunov function.



Example (First order system)

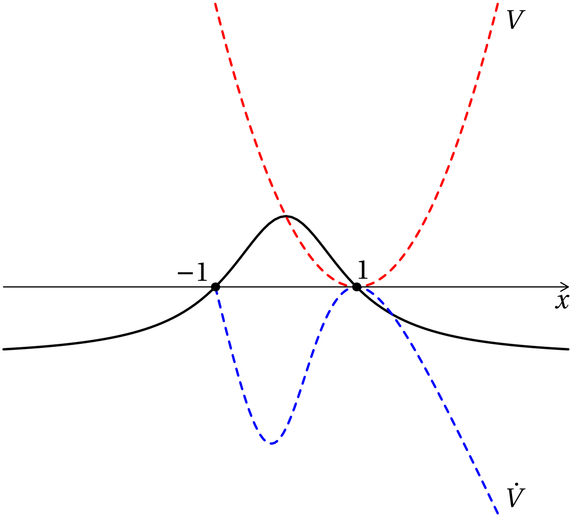
Consider $\dot{\mathrm{x}}(t) = \dfrac{1-\mathrm{x}^2(t)}{1+\mathrm{x}^2(t)}$. For equilibrium $\bar{x} = 1$ we propose

$$V(x) = (x-1)^2.$$

It is C^1 (on $\mathbb{R}$), it is positive definite (relative to $\bar{x}$), and

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x) = 2(x-1)\frac{1-x^2}{1+x^2} = -2\frac{(1-x)^2(1+x)}{1+x^2} \leq 0 \qquad \forall x \geq -1$$

Actually $\dot{V}(x) < 0$ for all $x \in (-1,\infty) \setminus \{\bar{x}\}$ so it is a *strong* Lyapunov function on $\Omega := (-1,\infty)$ and, hence, the equilibrium $\bar{x} = 1$ is asymptotically stable.



V
-1
1
x
$\dot{V}$

Example

$$\begin{bmatrix} \dot{\mathrm{x}}_1(t) \\ \dot{\mathrm{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -x_2(t) \\ \mathrm{x}_1(t) \end{bmatrix} - \epsilon \begin{bmatrix} \mathrm{x}_1(t) \\ \mathrm{x}_2(t) \end{bmatrix} \qquad \epsilon > 0.$$

Consider equilibrium $\bar{x} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = (0,0)$. We propose this Lyapunov function:

$$V(x) = x_1^2 + x_2^2$$

Then (on $\Omega = \mathbb{R}^2$):

- V is C^1 (trivial)
- V is positive definite (trivial)
- $\dot{V}$ is negative definite, because:

So V is strong Lyapunov function. Hence $(0,0)$ is asymptotically stable

Lecture 2

- Summary
- Lyapunov
  - Theorem
  - Example
  - PROOF!
  - Global
- Appendix A.1

$$
\begin{aligned}
\dot{\mathrm{x}}(t) &= f(\mathrm{x}(t)), \qquad \mathrm{x}(0) = x_0 \\
0 &= f(\bar{x}) \\
\text{stable:} \quad & \forall \epsilon > 0\ \exists \delta > 0 : \|x_0 - \bar{x}\| < \delta \implies \|\mathrm{x}(t; x_0) - \bar{x}\| < \epsilon\ \forall t \geq 0 \\
\text{asymptotically stable:} \quad & \text{stable } \&\ \exists \delta_1 > 0 : \|x_0 - \bar{x}\| < \delta_1 \implies \lim_{t\to\infty} \mathrm{x}(t; x_0) = \bar{x} \\
\text{globally asymptotically stable:} \quad & \text{stable } \&\ \forall x_0 \in \mathbb{R}^n \implies \lim_{t\to\infty} \mathrm{x}(t; x_0) = \bar{x}
\end{aligned}
$$

$$
\dot{V}(\mathrm{x}(t)) \leq 0 \ \ {\color{red}\forall \mathrm{x}(t)\ \forall t} \qquad \iff \qquad \frac{\partial V(x)}{\partial x^{\mathrm{T}}} f(x) \leq 0 \ \ {\color{red}\forall x \in \mathbb{R}^n}
$$

Example

$$\begin{bmatrix} \dot{\mathrm{x}}_1(t) \\ \dot{\mathrm{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -\mathrm{x}_2(t) \\ \mathrm{x}_1(t) \end{bmatrix} - \epsilon \begin{bmatrix} \mathrm{x}_1(t) \\ \mathrm{x}_2(t) \end{bmatrix} \qquad \epsilon > 0$$

$$V(x) = x_1^2 + x_2^2$$

V is C^1 (on $\Omega = \mathbb{R}^2$)

V is positive definite (trivial)

V is negative definite: $\dot{V}(x) = -2\epsilon(x_1^2 + x_2^2)$

So $\bar{x} = (0,0)$ is as.stable & V a strong Lyapunov function

The main result from previous lecture:

Theorem (Lyapunov's second stability theorem)

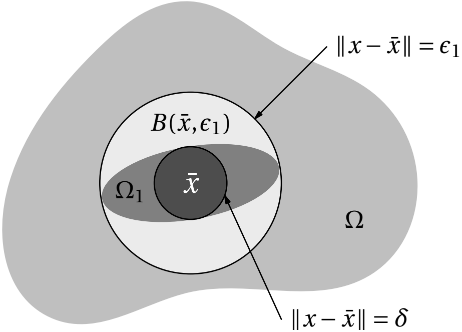
Consider $\dot{\mathrm{x}}(t) = f(\mathrm{x}(t))$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz continuous. Let $\bar{x}$ be an equilibrium of this DE. If there is a neighborhood Ω of $\bar{x}$ and a function $V : \Omega \to \mathbb{R}$ such that on Ω:

1. V is continuously differentiable,
2. V is positive definite relative to $\bar{x}$,
3. $\dot{V}$ is negative semi-definite relative to $\bar{x}$,

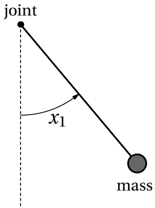
then $\bar{x}$ is a stable equilibrium, and we call V a Lyapunov function.

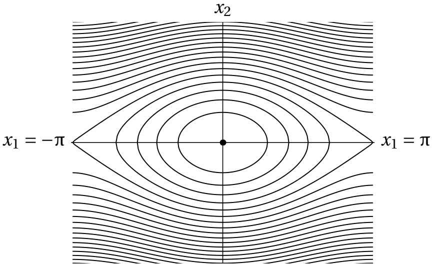
If in addition $\dot{V}$ is negative definite (so *not* just negative *semi*-definite) then $\bar{x}$ is asymptotically stable, and we call V a strong Lyapunov function.

PROOF:



$\|x-\bar{x}\| = \epsilon_1$
$B(\bar{x}, \epsilon_1)$
Ω_1
$\bar{x}$
Ω
$\|x-\bar{x}\| = \delta$





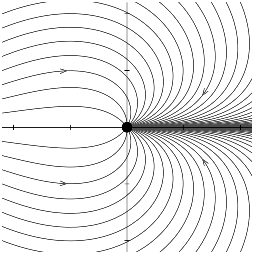
Example (Pendulum)

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\tfrac{g}{\ell}\sin(x_1(t))$$

$$V(x) = \frac{1}{2}m\ell^2 x_2^2 + mg\ell\big(1-\cos(x_1)\big), \qquad \Omega = \{x \in \mathbb{R}^2 \mid -2\pi < x_1 < 2\pi\}$$

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x_1} f_1(x) + \frac{\partial V(x)}{\partial x_2} f_2(x) = mg\ell\sin(x_1)x_2 - m\ell^2 x_2 \frac{g}{\ell}\sin(x_1) = 0$$

Stable. (Not asymptotically stable: why?)



Example (Global asymptotic stability)

$$\dot{\mathtt{x}}_1(t) = -\mathtt{x}_1(t) + \mathtt{x}_2^2(t)$$
$$\dot{\mathtt{x}}_2(t) = -\mathtt{x}_2(t)\mathtt{x}_1(t) - x_2(t)$$

$$V(x) = x_1^2 + x_2^2$$

$$\dot{V}(x) = 2x_1(-x_1 + x_2^2) + 2x_2(-x_2x_1 - x_2) = -2(x_1^2 + x_2^2) < 0 \qquad \forall x \neq 0$$

Asymptotically stable, but phase portrait suggests globally asymptotically stable.

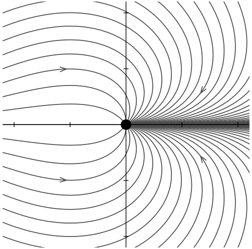
Theorem (Global asymptotic stability)

Suppose all conditions of Lyapunov theorem are met with $\Omega = \mathbb{R}^n$.
If $V : \mathbb{R}^n \to \mathbb{R}$ is a strong Lyapunov function, and, in addition,

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty,$$

then the system is globally asymptotically stable.

This property is known as radial unboundedness



Example (Global asymptotic stability)

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t)$$
$$\dot{x}_2(t) = -x_2(t)x_1(t) - x_2(t)$$

$$V(x) = x_1^2 + x_2^2$$

$$\dot{V}(x) = 2x_1(-x_1 + x_2^2) + 2x_2(-x_2x_1 - x_2) = -2(x_1^2 + x_2^2) < 0 \qquad \forall x \neq 0$$

... & radially unbounded, so $\bar{x} = (0,0)$ is globally asymptotically stable

A.1 Positive Definite Functions and Matrices

Example

$V(x) = x_1^2 + x_2^2$ is positive definite. Easy.

Example

But what if

$$\begin{aligned} V(x) &= x_1^2 + x_1 x_2 + x_2^2 \\ &= \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0 & \qquad \forall x \neq 0 \\
\iff a x_1^2 + 2ab x_1 x_2 + c x_2^2 > 0 & \qquad \forall x \neq 0 \\
\iff a(x_1 + (b/a)x_2)^2 + (c - b^2/a)x_2^2 > 0 & \qquad \forall x \neq 0 \\
\iff a > 0 \ \ \& \ \ c - b^2/a > 0 & \qquad \forall x \neq 0 \\
\iff a > 0 \ \ \& \ \ ac - b^2 > 0 & \\
\iff P_{11} > 0 \ \ \& \ \ \det(P) > 0 &
\end{aligned}
$$

Theorem

$P = P^{\mathrm{T}} > 0$ iff $\det(P_{1:k,1:k}) > 0$ for all $k = 1, \ldots, n$

$\det(P_{1:k,1:k})$ = “leading principle minors”

Example

$$P = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} > 0?$$

Example

$$P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} > 0?$$

Lecture 3

- Summary
  - Appendix A.1
  - Lyapunov theorem (by example)
- B.4 LaSalle's Invariance Principle
  - Invariant set
  - Theorem
- B.5 Cost-to-go
- B.6 Linearization [read it yourself]

Example

Let

$$V(x) = x^{\mathrm{T}} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & c \end{bmatrix} x$$

It is positive definite (relative to $x = (0,0,0)$) iff three inequalities hold:

1. $2 > 0$
2. $\det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3 > 0$
3. $\det(P) = c \det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + 1 \det \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = 3c - 2 > 0$

So positive definite iff $c > 2/3$

Notation: $V > 0$

LaSalle's Invariance Principle

Lyapunov theorem is awesome, but is not the end of the story:

Example (Pendulum with friction)

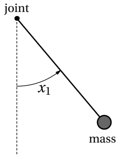
$$\dot{x}_1(t) = x_2(t)$$

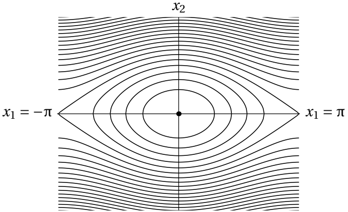
$$\dot{x}_2(t) = -\frac{g}{\ell}\sin(x_1(t)) - \frac{d}{m}x_2(t),$$

$$V(x) = \tfrac{1}{2}m\ell^2 x_2^2 + mg\ell(1-\cos(x_1))$$

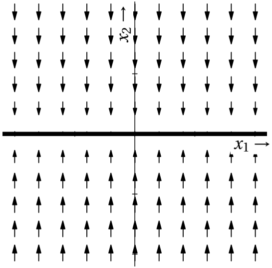
$$\dot{V}(x) = -d\ell^2 x_2^2 \leq 0$$

A Lyapunov function, not strong! But we feel it is asymptotically stable nonetheless





First guess: if $\dot{V}(x) < 0$ for "almost all" x, then..? Not correct:

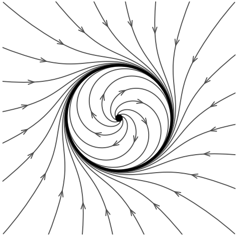


Example

$$\dot{\mathrm{x}}_1(t) = 0$$

$$\dot{\mathrm{x}}_2(t) = -\mathrm{x}_2(t)$$

with $V(x) = x_1^2 + x_2^2$. Then $\dot{V}(x) = -2x_2^2$ is negative "almost everywhere" but system is not asymptotically stable



Example (Nice example to illustrate invariant sets)

$$\dot{x}_1 = x_2 + x_1\left(1 - x_1^2 - x_2^2\right)$$
$$\dot{x}_2 = -x_1 + x_2\left(1 - x_1^2 - x_2^2\right)$$

$$r := x_1^2 + x_2^2 \quad \Longrightarrow \quad \dot{r} = 2r(1-r)$$

Definition (Invariant set)

A set $\mathcal{G} \subseteq \mathbb{R}^n$ is a *(forward)* invariant set of some given DE, if

$$x_0 \in \mathcal{G} \quad \Longrightarrow \quad x(t; x_0) \in \mathcal{G} \quad \forall\, t > 0$$

Theorem (LaSalle's Invariance Principle)

Let $\bar{x}$ be an equilibrium of a locally Lipschitz-continuous $\dot{\mathrm{x}}(t) = f(\mathrm{x}(t))$.
Suppose V is a Lyapunov function on a neighborhood Ω of $\bar{x}$.
Then Ω contains a (nonempty) closed and bounded invariant neighborhood $\mathscr{K}$ of $\bar{x}$, and then

$$x_0 \in \mathscr{K} \quad \implies \quad \mathrm{x}(t; x_0) \to \mathscr{G} := \{x \in \mathscr{K} \mid \dot{V}(\mathrm{x}(t; x)) = 0 \ \forall t \geq 0\} \ \text{ as } t \to \infty.$$

In particular, if $\mathscr{G} = \{\bar{x}\}$ then $\bar{x}$ is an asymptotically stable equilibrium.

Existence of such $\mathscr{K}$ is guaranteed by the theorem.
And $\mathscr{G}$ is nonempty (it always contains $\bar{x}$).

Example

$$\dot{\mathtt{x}}_1(t) = \mathtt{x}_2^3(t)$$
$$\dot{\mathtt{x}}_2(t) = -\mathtt{x}_1^3(t) - \mathtt{x}_2(t)$$

$$V(x) = x_1^4 + x_2^4$$
$$\dot{V}(x) = -4x_2^4 \leq 0$$

$$\begin{aligned}\mathscr{G} &= \{x \mid \dot{V}(\mathtt{x}(t;x)) = 0 \forall t > 0\} \\ &= \{x \mid \mathtt{x}_2(t) = 0 \forall t > 0 \text{ and satisfies DE}\} \\ &= \{x \mid \mathtt{x}_2(t) = 0, \mathtt{x}_1(t) = 0 \forall t > 0\} \\ &= \{(0,0)\}\end{aligned}$$

B.5 Cost-to-go Lyapunov functions

- How to find $V(x)$?
- Energy?
- Inspired by phase portrait?
- ....?
- or: cost-to-go

Suppose we have to pay "rent" $L(x)$ per unit time when we are at x.
The total payment over all future time is the cost-to-go:

$$V(x_0) := \int_0^\infty L(\mathrm{x}(t; x_0))\,\mathrm{d}t$$

$$V(\mathrm{x}(t)) = \int_t^\infty L(\mathrm{x}(\tau))\,\mathrm{d}\tau$$

Theorem

If cost-to-go $V(x)$ converges, then

$$\dot{V}(x) = -L(x)$$

in words:
"the current cost-to-go minus the cost-to-go from tomorrow onwards, is what we pay today"

Proof.

Let $g(t) = L(\mathrm{x}(t))$. Then $V(\mathrm{x}(t)) = \int_t^\infty g(\tau)\,\mathrm{d}\tau \ldots$

$\square$

Theorem (Lyapunov equation)

Let $A \in \mathbb{R}^{n\times n}$ and consider $\dot{\mathrm{x}}(t) = A\mathrm{x}(t)$ with equilibrium $\bar{x} = 0 \in \mathbb{R}^n$. Suppose $Q \in \mathbb{R}^{n\times n}$ is positive definite and consider cost-to-go

$$V(x_0) := \int_0^\infty x^{\mathrm{T}}(t) Q \mathrm{x}(t)\,\mathrm{d}t$$

in which $\mathrm{x}(0) = x_0$. The following four statements are equivalent.

1. $\bar{x} = 0$ is a globally asymptotically stable equilibrium of $\dot{\mathrm{x}}(t) = A\mathrm{x}(t)$.
2. $\bar{x} = 0$ is an asymptotically stable equilibrium of $\dot{\mathrm{x}}(t) = A\mathrm{x}(t)$.
3. $V(x)$ exists for every $x \in \mathbb{R}^n$, and is a strong Lyapunov function. In fact $V(x) = x^{\mathrm{T}} P x$, with $P \in \mathbb{R}^{n\times n}$ the positive definite matrix
$$P := \int_0^\infty \mathrm{e}^{A^{\mathrm{T}} t} Q \mathrm{e}^{At}\,\mathrm{d}t.$$
4. The Lyapunov equation $A^{\mathrm{T}} P + PA = -Q$ has a unique solution P, and this P is symmetric positive definite.

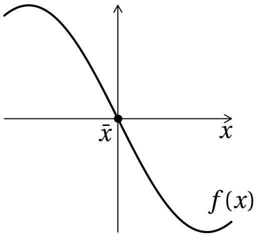
In that case the P of items 3 and 4 are the same.

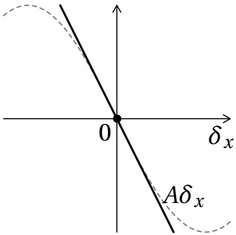
Example

$$\dot{\mathrm{x}}(t) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \mathrm{x}(t)$$

B.6 Lyapunov's First Method (Linearization)

Read B.6 yourself (you have seen it before in other courses)





Part II

Chapter 1 — Calculus of Variations

Overview

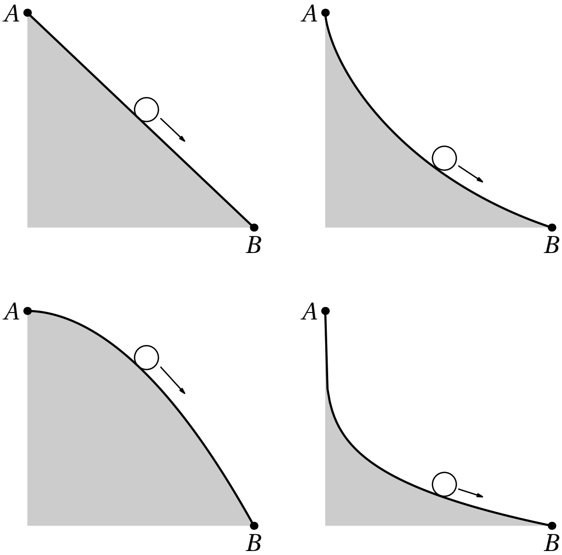
Lecture 4

- Brachistochrone (history)
- Shortest path
- SPITCOV
- Euler-Lagrange
- Examples

1.1 Introduction

Johann Bernoulli (1667–1748)

1696:


A
B
A
B
A
B
A
B

Example (Brachistochrone)

$$\frac{1}{2}mv^2 - mgy = c$$

$$v = \sqrt{2gy}$$

$$dt = \frac{ds}{v} = \sqrt{\frac{1+\dot{y}^2(x)}{2gy(x)}}dx$$

$$T = \int_0^T 1\,dt = \int_{x_0}^{x_1} \sqrt{\frac{1+\dot{y}^2(x)}{2gy(x)}}\,dx$$

$$J(y) = \int_{x_0}^{x_1} F(x, y(x), \dot{y}(x))\,dx.$$

We need to minimize the integral over all functions $y : [x_0, x_1] \to \mathbb{R}$ subject to $y(x_0) = y_0 = 0$ and $y(x_1) = y_1$.

J is known as the cost (function), and F is the running cost.

Example (Shortest path)

$$\mathrm{d}s = \sqrt{1 + \dot{y}^2(x)}\,\mathrm{d}x$$

$$J(y) = \int_{x_0}^{x_1} \sqrt{1 + \dot{y}^2(x)}\,\mathrm{d}x$$

$$J(y) = \int_{x_0}^{x_1} F(x, y(x), \dot{y}(x))\,\mathrm{d}x.$$

This has to be minimized over al functions $y : [x_0, x_1] \to \mathbb{R}$, subject to

$$y(x_0) = y_0, \qquad y(x_1) = y_1.$$

Usually other variable names:

$$J(x) = \int_0^T F(t, x(t), \dot{x}(t))\,\mathrm{d}t, \qquad x(0) = x_0, \quad x(T) = x_T.$$

1.2 Euler-Lagrange Equation

Definition (Simplest Problem in the Calculus of Variations (SPITCOV))

Given a final time $T > 0$ and a function $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and states $x_0, x_T \in \mathbb{R}^n$, the SPITCOV is to minimize the cost

$$J(\mathrm{x}) := \int_0^T F(t, \mathrm{x}(t), \dot{\mathrm{x}}(t))\,\mathrm{d}t$$

over all functions $\mathrm{x} : [0, T] \to \mathbb{R}^n$ that satisfy given boundary conditions

$$\mathrm{x}(0) = x_0, \qquad \mathrm{x}(T) = x_T.$$

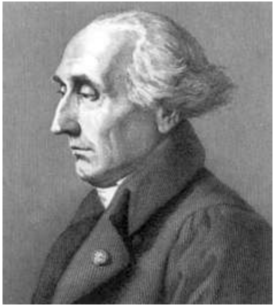
Theorem (Euler-Lagrange equation — necessary 1st-order condition)

Suppose that F is C^1. Necessary for a C^1 function $\mathtt{x}_*$ to solve SPITCOV is that it satisfies the DE

$$\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) F(t, \mathtt{x}_*(t), \dot{\mathtt{x}}_*(t)) = 0 \qquad \text{for all } t \in [0, T].$$

About these derivatives: suppose $F(t, x, \dot{x}) = t\sin(x) + \dot{x}^2$. Then:

Leonhard Euler (1707–1783)



Joseph-Louis Lagrange (1736–1813)

- Euler-Lagrange is a “first order condition”
- To prepare for the proof of EL: consider classic 1st-order condition in standard optimization:
  - Let $J : \mathbb{R}^2 \to \mathbb{R}$ be some “smooth enough” function
  - At solution $x_* \in \mathbb{R}^2$ of $\min_{x\in\mathbb{R}^2} J(x)$ we have

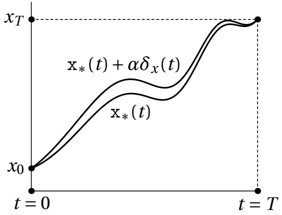
$$\frac{\partial J(x_*)}{\partial x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, equivalently, all directional derivatives are zero:

$$\forall \delta_x \in \mathbb{R}^2 : \qquad \frac{\mathrm{d}\bar{J}(x_* + \alpha\delta_x)}{\mathrm{d}\alpha} = 0$$

Consider perturbation of $\mathtt{x}_*$:

$$\mathtt{x}(t) = \mathtt{x}_*(t) + \alpha\delta_x(t):$$



and realize that $\delta_x(0) = \delta_x(T) = 0$.

$$\bar{J}(\alpha) := J(\mathtt{x}_* + \alpha\delta_x)$$

At $\mathtt{x}_*$ and every δ_x we must have $\bar{J}'(0) = 0$

Example

$$J(\mathrm{x}) = \int_{-1}^{1} \alpha^2 \mathrm{x}^2(t) + \dot{\mathrm{x}}^2(t)\,\mathrm{d}t, \quad \mathrm{x}(-1) = 1, \quad \mathrm{x}(1) = 1.$$

$$\text{EL:}\ \ 0 = \left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right)(\alpha^2 \mathrm{x}^2(t) + \dot{\mathrm{x}}^2(t))$$

$$= 2\alpha^2 \mathrm{x}(t) - \frac{\mathrm{d}}{\mathrm{d}t}(2\dot{\mathrm{x}}(t)) = 2\alpha^2 \mathrm{x}(t) - 2\ddot{\mathrm{x}}(t)$$

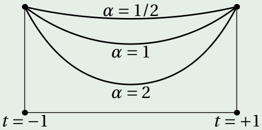
$$\ddot{\mathrm{x}}(t) = \alpha^2 \mathrm{x}(t)$$

$$\mathrm{x}(t) = c\mathrm{e}^{\alpha t} + d\mathrm{e}^{-\alpha t}$$

$$1 = \mathrm{x}(-1) = c\mathrm{e}^{-\alpha} + d\mathrm{e}^{+\alpha},$$

$$1 = \mathrm{x}(1) = c\mathrm{e}^{+\alpha} + d\mathrm{e}^{-\alpha}$$

$$\mathrm{x}_*(t) = \frac{\mathrm{e}^{\alpha t} + \mathrm{e}^{-\alpha t}}{\mathrm{e}^{\alpha} + \mathrm{e}^{-\alpha}}$$



Hyperbolic cosine. How does it depend on α?

For later: $F(t, x, \dot{x}) = \alpha^2 x^2 + \dot{x}^2$ and this does not depend on t

1.3 Beltrami Identity

Often $F(t, x, \dot{x})$ does not depend on time: $F(x, \dot{x})$

Theorem (Beltrami Identity)

If $F(t, x, \dot{x})$ does not depend on t, then solutions $\mathtt{x}_*$ of SPITCOV satisfy the Beltrami Identity

$$F(\mathtt{x}_*(t), \dot{\mathtt{x}}_*(t)) - \dot{\mathtt{x}}_*^{\mathrm{T}}(t) \frac{\partial F(\mathtt{x}_*(t), \dot{\mathtt{x}}_*(t))}{\partial \dot{x}} = C \qquad \forall\, t \in [0, T]$$

Proof.

EL $\implies$ Beltrami.

□

Example (Brachistochrone – cycloid)

$$F(y,\dot{y}) = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}$$

$$C = F(y,\dot{y}) - \dot{y}\frac{\partial F(y,\dot{y})}{\partial \dot{y}}$$

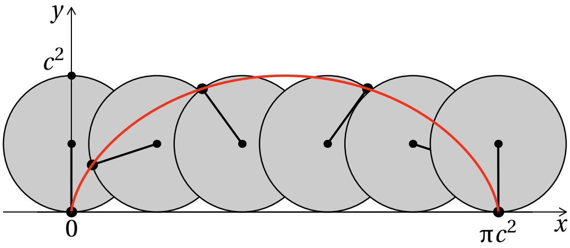
$$= \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}} - \frac{\dot{y}^2}{\sqrt{2gy(1+\dot{y}^2)}}$$

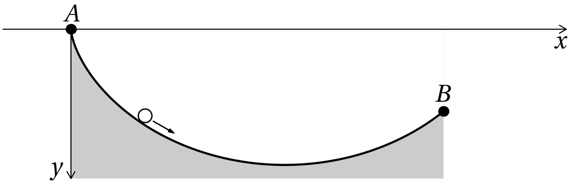
$$= \frac{1}{\sqrt{2gy(1+\dot{y}^2)}}$$

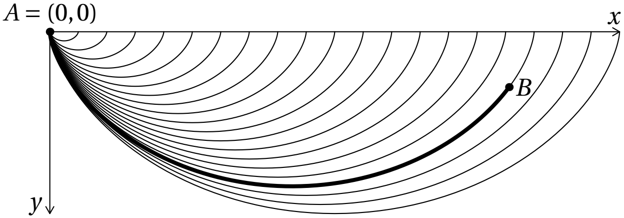
$$c^2 = y(1+\dot{y}^2), \qquad c^2 := \frac{1}{2gC^2}.$$

$$x(\phi) = \frac{c^2}{2}(\phi - \sin(\phi)) \qquad y(\phi) = \frac{c^2}{2}(1-\cos(\phi))$$

$$x(\phi) = \frac{c^2}{2}(\phi - \sin(\phi)), \qquad y(\phi) = \frac{c^2}{2}(1 - \cos(\phi)):$$






$A = (0,0)$
x
B
y

Lecture 5

- Recap: EL & Beltrami
- Example (catenoid)
- Higher-order EL
- Relaxed boundary conditions
- Legendre (2nd-order necesarry condition)

- Given a final time $T > 0$ and a function $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and states $x_0, x_T \in \mathbb{R}^n$, the SPITCOV is to minimize the cost

$$J(\mathrm{x}) := \int_0^T F(t, \mathrm{x}(t), \dot{\mathrm{x}}(t))\, \mathrm{d}t$$

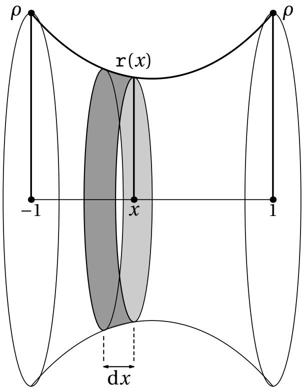
  over all $\mathrm{x} : [0, T] \to \mathbb{R}^n$ that satisfy $\mathrm{x}(0) = x_0, \mathrm{x}(T) = x_T$.

- Suppose F is C^1. Necessary for a C^1 function x_* to solve SPITCOV is that it satisfies the EL equation

$$\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) F(t, \mathrm{x}_*(t), \dot{\mathrm{x}}_*(t)) = 0 \qquad \text{for all } t \in [0, T].$$

- If "$F(x, \dot{x})$" then solutions of SPITCOV satisfy the Beltrami Identity

$$F(\mathrm{x}_*(t), \dot{\mathrm{x}}_*(t)) - \dot{\mathrm{x}}_*^{\mathrm{T}}(t) \frac{\partial F(\mathrm{x}_*(t), \dot{\mathrm{x}}_*(t))}{\partial \dot{x}} = C \qquad \forall t \in [0, T]$$



Example (Minimal surface area: catenoid & Goldschmidt two-disc)

$$J(\mathrm{y}) = \int_{-1}^{1} 2\pi \mathtt{r}(x)\sqrt{1+\dot{\mathtt{r}}^2(x)}\,\mathrm{d}x, \qquad \mathtt{r}(\pm 1) = \rho.$$

Assumes $\mathtt{r}(x) \geq 0$

Example (Catenoid — continued)

$F(x,\mathtt{r},\dot{\mathtt{r}}) = 2\pi\mathtt{r}\sqrt{1+\dot{\mathtt{r}}^2}$. Beltrami:

$$2\pi\mathtt{r}\sqrt{1+\dot{\mathtt{r}}^2} - \dot{\mathtt{r}}2\pi\mathtt{r}\frac{\dot{\mathtt{r}}}{\sqrt{1+\dot{\mathtt{r}}^2}} = C$$

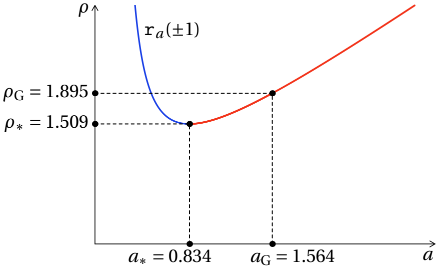
$$\mathtt{r}(1+\dot{\mathtt{r}}^2) - \mathtt{r}\dot{\mathtt{r}}^2 = \frac{C}{2\pi}\sqrt{1+\dot{\mathtt{r}}^2}$$

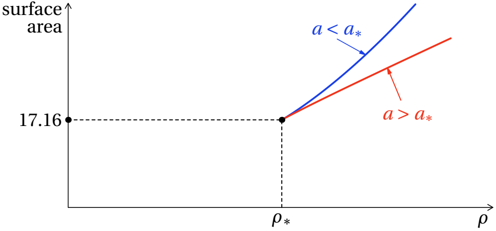
$$\mathtt{r} = \frac{C}{2\pi}\sqrt{1+\dot{\mathtt{r}}^2}$$

$$\mathtt{r}^2 = a^2(1+\dot{\mathtt{r}}^2)$$

$$\mathtt{r}_a(x) := a\cosh(x/a), \qquad a \geq 0.$$

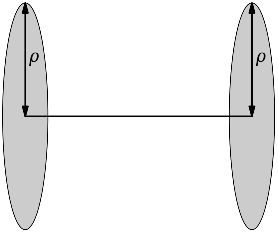
$\mathtt{r}_a(x) := a\cosh(x/a)$:

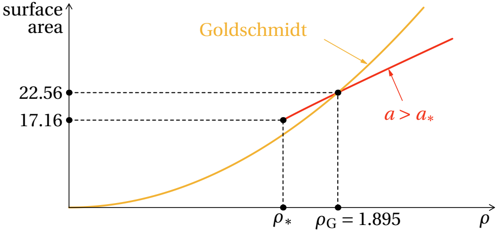




Conclusion:

- for $\rho < \rho_*$ catenoid is not optimal!
- for $\rho > \rho_*$ there are two catenoids ...
- ... and the catenoid with $a \geq a_*$ achieves minimal area
- what about this a_G:





Out of these two (catenoid & Goldschmidt):

- catenoid is optimal if $\rho \geq \rho_G$
- Goldschmidt is optimal if $\rho \leq \rho_G$

Deeper analysis: these are the true optimal solutions

1.4 Higher-Order Euler-Lagrange Equation

Theorem (Higher-order Euler-Lagrange equation)

Consider the problem of minimizing the integral

$$J(\mathrm{x}) = \int_0^T F(t, \mathrm{x}(t), \dot{\mathrm{x}}(t), \ddot{\mathrm{x}}(t))\, \mathrm{d}t$$

over the functions $\mathrm{x} : [0, T] \to \mathbb{R}^n$ that satisfy the boundary conditions

$$\mathrm{x}(0) = x_0, \quad \dot{\mathrm{x}}(0) = x_0^d, \qquad \mathrm{x}(T) = x_T, \quad \dot{\mathrm{x}}(T) = x_T^d$$

for given values x_0, x_0^d and x_T, x_T^d. Suppose F is C^2. Necessary for a C^2 function x_* to minimize $J(\mathrm{x})$ subject to these boundary condtions is that

$$\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial}{\partial \ddot{x}}\right) F(t, \mathrm{x}_*(t), \dot{\mathrm{x}}_*(t), \ddot{\mathrm{x}}_*(t)) = 0 \qquad \forall\, t \in [0, T].$$

Proof is very similar to that of “SPITCOV”

Proof: recall $\mathtt{x}(0), \mathtt{x}(T), \dot{\mathtt{x}}(0), \dot{\mathtt{x}}(T)$ are fixed. Follow EL proof:

$$
\begin{aligned}
0 &= \frac{\mathrm{d}J(\mathtt{x}_* + \alpha\delta_x)}{\mathrm{d}\alpha}\bigg|_{\alpha=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\alpha}\int_0^T F(t, \mathtt{x}_* + \alpha\delta_x, \dot{\mathtt{x}}_* + \alpha\dot{\delta}_x, \ddot{\mathtt{x}}_* + \alpha\ddot{\delta}_x)\,\mathrm{d}t \qquad \bigg|_{\alpha=0} \\
&= \int_0^T \frac{\partial F}{\partial x^{\mathrm{T}}}\delta_x + \frac{\partial F}{\partial \dot{x}^{\mathrm{T}}}\dot{\delta}_x + \frac{\partial F}{\partial \ddot{x}^{\mathrm{T}}}\ddot{\delta}_x\,\mathrm{d}t \\
&= \underbrace{\left[\frac{\partial F}{\partial \dot{x}^{\mathrm{T}}}\delta_x\right]_0^T}_{=0} + \underbrace{\left[\frac{\partial F}{\partial \ddot{x}^{\mathrm{T}}}\dot{\delta}_x\right]_0^T}_{=0} + \underbrace{\left[-\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \ddot{x}^{\mathrm{T}}}\right)\delta_x\right]_0^T}_{=0} \quad + \quad \int_0^T \left(\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{x}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial F}{\partial \ddot{x}}\right)^{\mathrm{T}}\delta_x\,\mathrm{d}t
\end{aligned}
$$

1.5 Relaxed Boundary Conditions

How does "EL" change if we relax some boundary conditions, say

$$\mathrm{x}(0) = \begin{bmatrix} \text{free} \\ \text{fixed} \\ \text{fixed} \end{bmatrix}, \qquad \mathrm{x}(T) = \begin{bmatrix} \text{fixed} \\ \text{fixed} \\ \text{free} \end{bmatrix}.$$

So we have more "degrees of freedom". Implies that EL still holds:

The variational argument, $\mathtt{x} = \mathtt{x}_* + \alpha\delta_x$, gave 1st-order condition:

$$\frac{\partial F(t, \mathtt{x}_*(t), \dot{\mathtt{x}}_*(t))}{\partial \dot{x}^{\mathrm{T}}}\delta_x(t)\bigg|_0^T + \int_0^T \left(\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right)F(\cdots)\right)^{\mathrm{T}} \delta_x(t)\,\mathrm{d}t = 0 \qquad \forall \delta_x$$

In particular for the special ones with $\delta_x(0) = \delta_x(T) = 0$, which led to EL.
So the above 1st-order condition holds iff

$$\frac{\partial F(t, \mathtt{x}_*(t), \dot{\mathtt{x}}_*(t))}{\partial \dot{x}^{\mathrm{T}}}\delta_x(t)\bigg|_0^T = 0 \qquad \& \qquad \text{EL holds}$$

$\forall \delta_x$:

$$\mathrm{x}(0) = \begin{bmatrix} \text{free} \\ \text{fixed} \\ \text{fixed} \end{bmatrix}, \qquad \mathrm{x}(T) = \begin{bmatrix} \text{fixed} \\ \text{fixed} \\ \text{free} \end{bmatrix}.$$

$$\delta_x(0) = \begin{bmatrix} \text{free} \\ 0 \\ 0 \end{bmatrix}, \qquad \delta_x(T) = \begin{bmatrix} 0 \\ 0 \\ \text{free} \end{bmatrix}.$$

The condition $\frac{\partial F(t,\mathrm{x}(t),\dot{\mathrm{x}}(t))}{\partial \dot{x}^{\mathrm{T}}}\delta_x(t)\Big|_0^T = 0$ thus holds for all possible δ_x iff:

$$\frac{\partial F(0,\mathrm{x}(0),\dot{x}(0))}{\partial \dot{x}} = \begin{bmatrix} 0 \\ \text{free} \\ \text{free} \end{bmatrix}, \qquad \frac{\partial F(T,\mathrm{x}(T),\dot{\mathrm{x}}(T))}{\partial \dot{x}} = \begin{bmatrix} \text{free} \\ \text{free} \\ 0 \end{bmatrix}.$$

In words: “to every x_i that is free to choose there corresponds a condition on $\partial F/\partial \dot{x}_i$”

$$J(\mathtt{x}) = \int_0^T F(t, \mathtt{x}(t), \dot{\mathtt{x}}(t))\,\mathrm{d}t + G(\mathtt{x}(0)) + K(\mathtt{x}(T))$$

Theorem (Relaxed boundary conditions)

Let $T > 0$. Suppose $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is C^1 & $K, G : \mathbb{R}^n \to \mathbb{R}$ are C^1. Let $\mathbb{I}_0, \mathbb{I}_T$ be subsets of $\{1, \ldots, n\}$, and consider $\mathtt{x} : [0, T] \to \mathbb{R}^n$ whose initial $\mathtt{x}(0)$ and final $\mathtt{x}(T)$ are fixed except for

$$x_i(0) = \text{free} \quad \forall i \in \mathbb{I}_0 \qquad \text{and} \qquad x_j(T) = \text{free} \quad \forall j \in \mathbb{I}_T.$$

A C^1 function $\mathtt{x}_*$ is a stationary solution of $J(x)$ iff EL holds and

$$\frac{\partial F(0, \mathtt{x}_*(0), \dot{\mathtt{x}}_*(0))}{\partial \dot{x}_i} - \frac{\partial G(\mathtt{x}_*(0))}{\partial x_i} = 0 \qquad \forall i \in \mathbb{I}_0,$$

$$\frac{\partial F(T, \mathtt{x}_*(T), \dot{\mathtt{x}}_*(T))}{\partial \dot{x}_j} + \frac{\partial K(\mathtt{x}_*(T))}{\partial x_j} = 0 \qquad \forall j \in \mathbb{I}_T.$$

Important special free endpoint case: if $\mathtt{x}(0) = x_0$ while $\mathtt{x}(T) = \text{free}$, then

$$\frac{\partial F(T, \mathtt{x}_*(T), \dot{\mathtt{x}}_*(T))}{\partial \dot{x}} - \frac{\partial K(\mathtt{x}_*(T))}{\partial x} = 0 \in \mathbb{R}^n$$

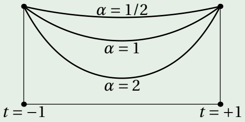
Example (Classic SPITCOV & free endpoint)

We considered this SPITCOV problem

$$J(\mathrm{x}) = \int_{-1}^{1} \alpha^2 \mathrm{x}^2(t) + \dot{\mathrm{x}}^2(t)\,\mathrm{d}t, \quad \mathrm{x}(-1) = 1, \quad \mathrm{x}(1) = 1.$$

$$\text{EL:} \quad \mathrm{x}(t) = c\mathrm{e}^{\alpha t} + d\mathrm{e}^{-\alpha t}$$

$$\mathrm{x}_*(t) = \frac{\mathrm{e}^{\alpha t} + \mathrm{e}^{-\alpha t}}{\mathrm{e}^{\alpha} + \mathrm{e}^{-\alpha}}$$

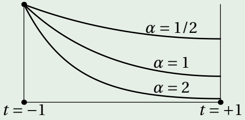


Now free endpoint problem: $\mathrm{x}(-1) = 1$, $\mathrm{x}(1) =$free. Satisfies EL and

$$\text{initial condition:} \quad 1 = \mathrm{x}(-1) = c\mathrm{e}^{-\alpha} + d\mathrm{e}^{\alpha}$$

$$\text{free endpoint condition:} \quad 0 = \frac{\partial \alpha^2 \mathrm{x}^2(t) + \dot{\mathrm{x}}^2(t)}{\partial \dot{x}}\bigg|_{t=1} = 2\dot{\mathrm{x}}(1) = 2(c\alpha\mathrm{e}^{\alpha} - d\alpha\mathrm{e}^{-\alpha})$$

$$\mathrm{x}_*^{\text{free}}(t) = \frac{\mathrm{e}^{\alpha(t-1)} + \mathrm{e}^{-\alpha(t-1)}}{\mathrm{e}^{2\alpha} + \mathrm{e}^{-2\alpha}}$$



1.6 Second-Order Conditions for Minimality

So far we considered necessary 1st-order conditions:

- EL $\iff \left.\frac{\mathrm{d}J(\mathtt{x}_*+\alpha\delta)}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0$ for every allowable δ
- $J(\mathtt{x}_* + \alpha\delta_x) = J(\mathtt{x}_*) + \mathfrak{o}(\alpha)$ ...
- ... where $\mathfrak{o}(\alpha)/\alpha \to 0$ as $\alpha \to 0$.

Now 2nd-order conditions:

- Legendre (necessary for optimality)
- Convexity (sufficient for optimality; next time)

We need 2nd-order Taylor series for fixed t:

$$
\begin{aligned}
F(t, x+\delta_x, y+\delta_y) = F(t,x,y) \\
&+ \begin{bmatrix} \dfrac{\partial F(t,x,y)}{\partial x^{\mathrm{T}}} & \dfrac{\partial F(t,x,y)}{\partial y^{\mathrm{T}}} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \\
&+ \frac{1}{2} \begin{bmatrix} \delta_x^{\mathrm{T}} & \delta_y^{\mathrm{T}} \end{bmatrix} \underbrace{\begin{bmatrix} \dfrac{\partial^2 F(t,x,y)}{\partial x \partial x^{\mathrm{T}}} & \dfrac{\partial^2 F(t,x,y)}{\partial x \partial y^{\mathrm{T}}} \\ \dfrac{\partial^2 F(t,x,y)}{\partial y \partial x^{\mathrm{T}}} & \dfrac{\partial^2 F(t,x,y)}{\partial y \partial y^{\mathrm{T}}} \end{bmatrix}}_{\text{Hessian of } F} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \\
&+ \mathfrak{o}\left(\left\| \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \right\|^2 \right).
\end{aligned}
$$

We assume F is C^2 so Hessian is symmetric.

Theorem (Legendre condition — 2nd-order necessary condition)

Consider SPITCOV. Assume F is C^2 and that solution $\mathtt{x}_*$ of EL that satisfies boundary conditions is C^2. Necessary for $\mathtt{x}_*$ to be minimizing is that

$$\frac{\partial^2 F(t, \mathtt{x}_*(t), \dot{\mathtt{x}}_*(t))}{\partial \dot{x} \partial \dot{x}^{\mathrm{T}}} \geq 0 \qquad \forall t \in [0, T].$$

Meaning: the above $n \times n$ matrix is positive semi-definite $\forall t \in [0, T]$.

Proof.

Analyze the second derivative of $J(\mathtt{x}_* + \alpha\delta_x)$ with respect to α... □

Adrien-Marie Legendre (1752–1833)

Example (Three cases)

- Let $F(t,x,\dot{x}) = \alpha^2 x^2 + \dot{x}^2$. Then $\frac{\partial^2 F(t,x,\dot{x})}{\partial \dot{x}^2} = 2$.
  So for sure $\frac{\partial^2 F(t,x_*(t),\dot{x}_*(t))}{\partial \dot{x}^2} \geq 0$ for all t. Legendre holds!
- Let $F(t,x,\dot{x}) = \sqrt{1+\dot{x}^2}$. Then $\frac{\partial^2 F(t,x,\dot{x})}{\partial \dot{x}^2} = \frac{1}{(1+\dot{x}^2)^{3/2}}$.
  So for sure $\frac{\partial^2 F(t,x_*(t),\dot{x}_*(t))}{\partial \dot{x}^2} \geq 0$ for all t. Legendre holds!
- Let $F(t,x,\dot{x}) = (\dot{x}/(2\pi))^2 - x^2$, with $x(0) = x(1) = 0$. Then $\frac{\partial^2 F(t,x,\dot{x})}{\partial \dot{x}^2} = \frac{2}{(2\pi)^2} > 0$, so Legendre holds, but...

Legendre's condition is necessary but not sufficient for optimality

Lecture 6

- Recap SPITCOV & EL & Legendre
- § 1.6 Hessian & convexity
- § 1.7 Integral contraints
- § 2.1 Intro to OC

- Given a final time $T > 0$ and a function $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and states $x_0, x_T \in \mathbb{R}^n$, the SPITCOV is to minimize the cost

$$J(\mathrm{x}) := \int_0^T F(t, \mathrm{x}(t), \dot{\mathrm{x}}(t))\,\mathrm{d}t$$

  over all $\mathrm{x} : [0,T] \to \mathbb{R}^n$ that satisfy $\mathrm{x}(0) = x_0, \mathrm{x}(T) = x_T$.
- Suppose F is C^1. Necessary for a C^1 function x_* to solve SPITCOV is that it satisfies the EL equation

$$\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) F(t, \mathrm{x}_*(t), \dot{\mathrm{x}}_*(t)) = 0 \qquad \text{for all } t \in [0,T].$$

- Legendre condition. Consider SPITCOV. Assume F is C^2 and that x_* is C^2 and satisfies EL & boundary conditions. Necessary for x_* to solve SPITCOV (i.e., minimizing) is that

$$\frac{\partial^2 F(t, \mathrm{x}_*(t), \dot{\mathrm{x}}_*(t))}{\partial \dot{x} \partial \dot{x}^{\mathrm{T}}} \geq 0 \qquad \forall t \in [0,T].$$

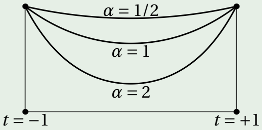
- ... nothing sufficient yet (be patient)

Example

$$J(\mathrm{x}) = \int_{-1}^{1} \alpha^2 \mathrm{x}^2(t) + \dot{\mathrm{x}}^2(t)\,\mathrm{d}t, \qquad \mathrm{x}(-1) = 1, \quad \mathrm{x}(1) = 1.$$

$$0 = \left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right)(\alpha^2 \mathrm{x}^2(t) + \dot{\mathrm{x}}^2(t))$$

$$\mathrm{x}(t) = c\mathrm{e}^{\alpha t} + d\mathrm{e}^{-\alpha t}$$



$$\mathrm{x}_*(t) = \frac{\mathrm{e}^{\alpha t} + \mathrm{e}^{-\alpha t}}{\mathrm{e}^{\alpha} + \mathrm{e}^{-\alpha}}$$

But we cannot (yet) conclude that this x_* is optimal :(
(Soon we can :-)

Classic optimization:

- $J : \mathbb{R} \to \mathbb{R}$ is convex [see Appendix A.7] if...
- if J is C^2 then J is convex iff $J''(x) \geq 0 \, \forall x$
- convex: 1st-order condition $\iff$ optimal!

Sloppy proof-by-picture:

Recall the 2nd-order Taylor series (at fixed t):

$$F(t, x+\delta_x, y+\delta_y) \quad = \quad F(t,x,y) + \left[\frac{\partial F(t,x,y)}{\partial x^{\mathrm{T}}} \; \frac{\partial F(t,x,y)}{\partial y^{\mathrm{T}}} \right] \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \delta_x^{\mathrm{T}} & \delta_y^{\mathrm{T}} \end{bmatrix} H(t,x,y) \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} + \mathfrak{o}(\cdot)$$

where H is the Hessian

$$H(t,x,y) = \begin{bmatrix} \dfrac{\partial^2 F(t,x,y)}{\partial x \partial x^{\mathrm{T}}} & \dfrac{\partial^2 F(t,x,y)}{\partial x \partial y^{\mathrm{T}}} \\ \dfrac{\partial^2 F(t,x,y)}{\partial y \partial x^{\mathrm{T}}} & \dfrac{\partial^2 F(t,x,y)}{\partial y \partial y^{\mathrm{T}}} \end{bmatrix}$$

Then $F(t,\cdot,\cdot)$ is convex (at t) iff $H(t,x,y) \geq 0$ for all x, y.

Theorem (Convexity — optimal solutions)

Consider SPITCOV, and suppose F is C^2.

If the Hessian $H(t,x,y)$ is positive semi-definite for all $x, y \in \mathbb{R}^n$ and all $t \in [0,T]$ then every solution $\mathtt{x}_*$ of EL that meets the boundary conditions is a optimal solution of SPITCOV.

If Hessian > 0 for all $x, y \in \mathbb{R}^n$ and all $t \in [0,T]$ then $\mathtt{x}_*$ is the unique optimal solution of SPITCOV.

Example (Two (three?) cases)

$$F(t,x,\dot{x}) = \alpha^2 x^2 + \dot{x}^2$$

$$H(t,x,\dot{x}) = \begin{bmatrix} 2\alpha^2 & 0 \\ 0 & 2 \end{bmatrix} > 0$$

$$F(t,x,\dot{x}) = \sqrt{1+\dot{x}^2}$$

$$H(t,x,\dot{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1/(1+\dot{x}^2)^{3/2} \end{bmatrix} \geq 0$$

"Brachistochrone"?

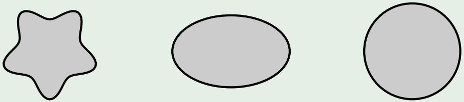

Legendre (1752–1833)


Jacobi (1804–1851)


Weierstrass (1815–1897)

1.7 Integral Constraints

Example (Queen Dido's isoperimetric problem)



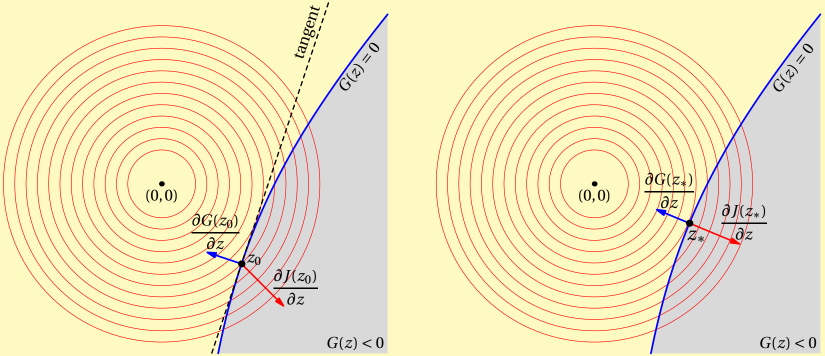
Minimize

$$\int_0^T F(t, \mathrm{x}(t), \dot{\mathrm{x}}(t))\,\mathrm{d}t$$

subject to boundary conditions and an integral contraint

$$\mathrm{x}(0) = x_0, \qquad \mathrm{x}(T) = x_T, \qquad \int_0^T M(t, \mathrm{x}(t), \dot{\mathrm{x}}(t))\,\mathrm{d}t = c_0.$$

Quick summary of Lagrange multipliers (§ 2.2 & § A.8)



$$\begin{cases} z_* = \arg\min_{z\in\mathbb{R}^n} \; J(z) \\ \text{subject to } G(z)=0 \end{cases} \quad \Longrightarrow \quad \frac{\partial J(z_*)}{\partial z^{\mathrm{T}}} + \mu^{\mathrm{T}} \frac{\partial G(z_*)}{\partial z^{\mathrm{T}}} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad G(z_*)=0$$

But these are the 1st-order conditions of the unconstrained $\mathfrak{J}(z,\mu) := J(z) + \mu^{\mathrm{T}} G(z)$.

Theorem (Euler-Lagrange Eqn for integral-constrained minimization)

Let $c_0 \in \mathbb{R}$. Suppose F and M are C^1 in all its components, and that x_* minimizes

$$\int_0^T F(t, \mathrm{x}(t), \dot{\mathrm{x}}(t))\,\mathrm{d}t$$

subject to $\mathrm{x}(0) = x_0, \mathrm{x}(T) = x_T$ and integral constraint

$$\int_0^T M(t, \mathrm{x}(t), \dot{\mathrm{x}}(t))\,\mathrm{d}t = c_0,$$

and that x_* is C^2. Then either there is a Lagrange multiplier $\mu_* \in \mathbb{R}$ s.t.

$$\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right)(F(t, \mathrm{x}_*(t), \dot{\mathrm{x}}_*(t)) + \mu_* M(t, \mathrm{x}_*(t), \dot{\mathrm{x}}_*(t))) = 0$$

for all $t \in [0, T]$, or M satisfies the Euler-Lagrange equation itself,

$$\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) M(t, \mathrm{x}_*(t), \dot{\mathrm{x}}_*(t)) = 0 \qquad \forall t \in [0, T].$$

Example (Normal and abnormal)

$$J(\mathrm{x}) = \int_0^1 \mathrm{x}(t)\,\mathrm{d}t, \qquad \mathrm{x}(0)=0, \mathrm{x}(1)=1, \quad \int_0^1 \dot{\mathrm{x}}^2(t)\,\mathrm{d}t = C$$

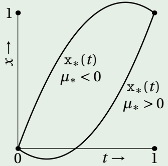
if normal: $0 = \left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right)(\mathrm{x}(t) + \mu\dot{\mathrm{x}}^2(t)) = 1 - \frac{\mathrm{d}}{\mathrm{d}t}\big(2\mu\dot{x}(t)\big) = 1 - 2\mu\ddot{\mathrm{x}}(t)$

$$\mathrm{x}_*(t) = \frac{1}{4\mu}t^2 + bt + c = \tfrac{1}{4\mu}t^2 + (1 - \tfrac{1}{4\mu})t$$

$$C = \int_0^1 \dot{\mathrm{x}}_*^2(t)\,\mathrm{d}t = \int_0^1 (\tfrac{1}{2\mu}t + 1 - \tfrac{1}{4\mu})^2\,\mathrm{d}t = 1 + \frac{1}{48\mu^2}$$

$$\mu_* = \frac{\pm 1}{\sqrt{48(C-1)}}$$

assumes $C > 1$



Example (... continued (now abnormal))

recall: $$J(\mathrm{x}) = \int_0^1 \mathrm{x}(t)\,\mathrm{d}t, \qquad \mathrm{x}(0) = 0, \quad \mathrm{x}(1) = 1, \quad \int_0^1 \dot{\mathrm{x}}^2(t)\,\mathrm{d}t = C$$

if abnormal: $$0 = \left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right)\dot{\mathrm{x}}^2(t) = -2\ddot{\mathrm{x}}(t)$$

$$\mathrm{x}(t) = bt + c$$

$$\mathrm{x}_*(t) = t$$

$$C = \int_0^1 \dot{\mathrm{x}}_*^2(t)\,\mathrm{d}t = 1$$

Corresponds to $\mu = \infty$. It is the case where the integral constraint together with the boundary conditions is tight. There are, so to say, no degrees of freedom left to shape the function. In particular, there is no feasible variation, $\mathrm{x}(t) = \mathrm{x}_*(t) + \alpha\delta_x(t)$...

Part III

Chapter 2 — Minimum Principle

Overview

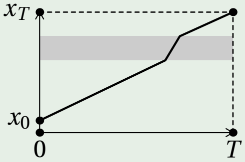
Lecture 7

- Recap OC problem & discontinuous controls
- Recap Lagrange multipliers (§ 2.2)
- § 2.3 First-order conditions
- § 2.4 Towards the Minimum Principle
- § 2.5 Pontryagin's MINIMUM PRINCIPLE
- Examples
- Constancy of the Hamiltonian

2.1 Optimal Control

Example (Limitation of SPITCOV)

$$\min_{\mathtt{x}:[0,T]\to\mathbb{R}} \text{“used energy”}$$



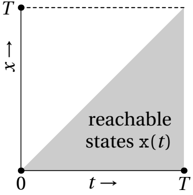
But position $\mathtt{x}(t)$ of the car is not free to choose. For instance its acceleration is bounded.

SPITCOV does not take dynamicals constraints into account! OPTIMAL CONTROL (OC) does:

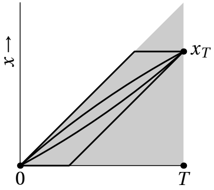
Definition (Optimal Control problem (OC problem))

Consider dynamical constraint $\dot{\mathtt{x}}(t) = f(\mathtt{x}(t), \mathtt{u}(t)),\ \mathtt{x}(0) = x_0$. The OC problem is:

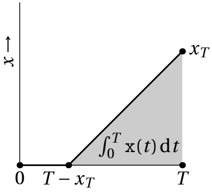
$$\min_{\mathtt{u}:[0,T]\to\mathbb{U}} \underbrace{\int_0^T L(\mathtt{x}(t),\mathtt{u}(t))\,\mathrm{d}t + K(\mathtt{x}(T))}_{J(\mathtt{u})}$$



(a)



(b)



(c)

Example (A bang-bang control example)

consider	$\dot{\mathrm{x}}(t) = \mathrm{u}(t), \qquad \mathrm{x}(0) = 0,$	$\mathbb{U} = [0, 1],$
with cost	$J(\mathrm{u}) := \int_0^T \mathrm{x}(t)\,\mathrm{d}t - \mathrm{x}(T)$	for some $T \geq 1$.
For fixed x_T:	$J(\mathrm{u}_*) = \frac{1}{2}x_T^2 - x_T$	minimal if $x_T = 1$:
For free x_T:	$J(\mathrm{u}_*) = -\frac{1}{2}$	$\mathrm{u}_*(t) = \begin{cases} 0 & \text{if } t < T-1, \\ 1 & \text{if } t > T-1. \end{cases}$

So u_* may be discontinuous! Awesome: “Pontryagin” can handle this!

2.2 Quick Summary: Lagrange Multiplier

A typical problem in classic optimization:

$$\text{constrained:} \quad \min_{z \in \mathbb{R}^n} J(z) \text{ subject to } G(z) = 0$$

$$\text{augmented cost:} \quad \mathfrak{J}(z, p) := p^{\mathsf{T}} G(z) + J(z)$$

Geometric argument of previous lecture suggests:

solutions of constrained minimization problem are
stationary solutions of (unconstrained) augmented cost

2.3 1st-order Conditions for Unbounded Smooth OC's

To get an idea of the (solution of the) OC problem

$$J(\mathtt{u}) = \int_0^T L(\mathtt{x}(t), \mathtt{u}(t))\,\mathrm{d}t + K(\mathtt{x}(T)), \qquad \dot{\mathtt{x}}(t) = f(\mathtt{x}(t), \mathtt{u}(t)), \quad \mathtt{x}(0) = x_0,$$

analyze first the stationary solutions $\mathtt{q} := (\mathtt{x}, \mathtt{p}, \mathtt{u})$ of the augmented cost

$$\mathfrak{J}(\mathtt{q}) := \int_0^T \mathtt{p}^{\mathrm{T}}(t)\big(f(\mathtt{x}(t), \mathtt{u}(t)) - \dot{\mathtt{x}}(t)\big) + L(\mathtt{x}(t), \mathtt{u}(t))\,\mathrm{d}t + K(\mathtt{x}(T))$$

$$= \int_0^T \mathfrak{L}(\mathtt{q}(t), \dot{\mathtt{q}}(t))\,\mathrm{d}t + K(\mathtt{x}(T))$$

$$\mathfrak{L}(q, \dot{q}) := p^{\mathrm{T}}(f(x, u) - \dot{x}) + L(x, u)$$

That is a calculus-of-variations problem in $q := (x, p, u)$ if $\mathbb{U} = \mathbb{R}^m$

$$\mathfrak{L}(q,\dot{q}) := p^{\mathrm{T}}(f(x,u)-\dot{x}) + L(x,u)$$

This does not depend on time. So the Beltrami identity holds for every stationary solution $\mathtt{q} := (\mathtt{x},\mathtt{p},\mathtt{u})$ of $\mathfrak{J}(\mathtt{q})$:

$$\mathfrak{L}(\mathtt{q}(t),\dot{\mathtt{q}}(t)) - \dot{\mathtt{q}}^{\mathrm{T}}(t)\frac{\partial \mathfrak{L}(\mathtt{q}(t),\dot{\mathtt{q}}(t))}{\partial \dot{q}} = C.$$

For our $\mathfrak{L}$ we have:

$$\begin{aligned}
\mathfrak{L}(q,\dot{q}) - \dot{q}^{\mathrm{T}}\frac{\partial \mathfrak{L}(q,\dot{q})}{\partial \dot{q}} &= \mathfrak{L}(q,\dot{q}) - \left(\dot{x}^{\mathrm{T}}\frac{\partial \mathfrak{L}(q,\dot{q})}{\partial \dot{x}} + \dot{p}^{\mathrm{T}}\frac{\partial \mathfrak{L}(q,\dot{q})}{\partial \dot{p}} + \dot{u}^{\mathrm{T}}\frac{\partial \mathfrak{L}(q,\dot{q})}{\partial \dot{u}}\right) \\
&= p^{\mathrm{T}}(f(x,u)-\dot{x}) + L(x,u) - (-\dot{x}^{\mathrm{T}}p + 0 + 0) \\
&= p^{\mathrm{T}}f(x,u) + L(x,u) \\
&=: H(x,p,u) \qquad \text{(optimal control) Hamiltonian}
\end{aligned}$$

Lemma (Hamiltonian equations for smooth unbounded controls)

Let $\mathbb{U} = \mathbb{R}^m, x_0 \in \mathbb{R}^n$. Then smooth-enough functions $\mathtt{x}_*, \mathtt{p}_*, \mathtt{u}_*$ are stationary solutions of cost $\mathfrak{J}(\mathtt{q})$ with $\mathtt{x}_*(0) = x_0$, if-and-only-if

$$\dot{\mathtt{x}}_*(t) = \frac{\partial H(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t))}{\partial p}, \qquad \mathtt{x}_*(0) = x_0,$$

$$\dot{\mathtt{p}}_*(t) = -\frac{\partial H(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t))}{\partial x}, \qquad \mathtt{p}_*(T) = \frac{\partial K(\mathtt{x}_*(T))}{\partial x},$$

$$0 = \frac{\partial H(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t))}{\partial u}.$$

Here $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$ is called the (optimal control) Hamiltonian and it is defined as

$$H(x, p, u) = p^{\mathrm{T}} f(x, u) + L(x, u).$$

2.4 Towards the Minimum Principle

Example (Legendre condition in terms of Hamiltonians)

Consider this calculus of variations problem with free endpoint:

$$\min_{\mathtt{x}:[0,T]\to\mathbb{R}^n} \int_0^T F(\mathtt{x}(t), \dot{\mathtt{x}}(t))\,\mathrm{d}t, \qquad \mathtt{x}(0) = x_0.$$

This an optimal control problem with

$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \qquad \mathtt{x}(0) = x_0, \qquad \mathbb{U} = \mathbb{R}^n, \qquad L(x,u) = F(x,u).$$

The Hamiltonian in this case is $H(x,p,u) = p^{\mathrm{T}} u + F(x,u)$.

$$\text{Legendre:} \quad \frac{\partial^2 F(\mathtt{x}_*(t), \dot{\mathtt{x}}_*(t))}{\partial \dot{x} \partial \dot{x}^{\mathrm{T}}} \geq 0 \qquad \forall t \in [0,T]$$

$$\iff \text{Legendre:} \quad \frac{\partial^2 H(\mathtt{x}_*(t), p(t), \mathtt{u}_*(t))}{\partial u \partial u^{\mathrm{T}}} \geq 0 \qquad \forall t \in [0,T]$$

Very interesting if we take $p(t) = \mathtt{p}_*(t)$!! Could it be? YES:

2.5 Minimum Principle

Theorem (Pontryagin's Minimum Principle)

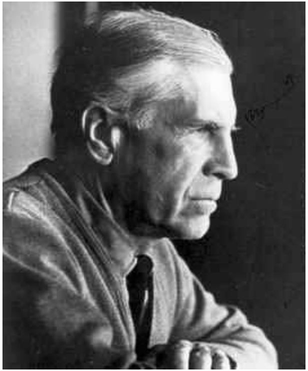
Assume $f(x,u)$ & $\partial f(x,u)/\partial x$ & $L(x,u)$ & $\partial L(x,u)/\partial x$ are continuous in x and u, and that $K(x)$ and $\partial K(x)/\partial x$ are continuous in x.
Suppose $\mathtt{u}_*:[0,T]\to\mathbb{U}$ is a solution of the OC problem, and assume it is piecewise continuous, and let $\mathtt{x}_*:[0,T]\to\mathbb{R}^n$ be the resulting optimal state. Then there is a unique costate $\mathtt{p}_*:[0,T]\to\mathbb{R}^n$ such that

$$\dot{\mathtt{x}}_*(t)=\frac{\partial H(\mathtt{x}_*(t),\mathtt{p}_*(t),\mathtt{u}_*(t))}{\partial p},\qquad \mathtt{x}_*(0)=x_0,$$
$$\dot{\mathtt{p}}_*(t)=-\frac{\partial H(\mathtt{x}_*(t),\mathtt{p}_*(t),\mathtt{u}_*(t))}{\partial x},\qquad \mathtt{p}_*(T)=\frac{\partial K(\mathtt{x}_*(T))}{\partial x},$$

and the input $\mathtt{u}_*(t)$ minimizes the Hamiltonian,

$$H(\mathtt{x}_*(t),\mathtt{p}_*(t),\mathtt{u}_*(t))=\min_{u\in\mathbb{U}}H(\mathtt{x}_*(t),\mathtt{p}_*(t),u),$$

at every $t\in[0,T]$ where $\mathtt{u}_*(t)$ is continuous.



Lev Semenovich Pontryagin (1908–1988).
He lost his eyesight at the age of 14.
He is famous for his Maximum Principle (1956).
It was presented to “the west” at the 1st IFAC World Congress (1960) in Moscow.

Example (Switching inputs)

consider $\dot{\mathtt{x}}(t) = \mathtt{u}(t)$ $\quad \mathtt{x}(0) = x_0 \quad \mathbb{U} = [-1, 1]$

and $J(\mathtt{u}) = \int_0^1 \mathtt{x}(t)\,\mathrm{d}t - \frac{1}{2}\mathtt{x}(1)$

$$\implies H(x, p, u) = pu + x \qquad K(x) = -\tfrac{1}{2}x$$

$$\implies \dot{\mathtt{p}}(t) = -1 \qquad \mathtt{p}(1) = \frac{\partial K(\mathtt{x}_*(1))}{\partial x} = -\tfrac{1}{2}$$

$$\implies \mathtt{p}_*(t) = \frac{1}{2} - t$$

$$\implies \mathtt{u}_*(t) = \begin{cases} -1 & \text{if } 0 \le t < \frac{1}{2} \\ +1 & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

Assuming an optimal control exists, it must be this one. So optimal is to move $\mathtt{x}(t)$ down as fast as possible over the 1st half of the time interval and then back up as fast as possible over the 2nd half!

Example (Optimal reinvestment)

$$\dot{\mathtt{x}}(t) = \alpha \mathtt{u}(t)\mathtt{x}(t) \qquad \mathtt{x}(0) = x_0 > 0 \quad \mathtt{u}(t) \in [0,1]$$

$$J(\mathtt{u}) = \int_0^T (\mathtt{u}(t) - 1)\mathtt{x}(t)\,\mathrm{d}t$$

$$H(x,p,u) = p\alpha ux + (u-1)x = ux(1+\alpha p) - x$$

$$\dot{\mathtt{x}}(t) = \alpha \mathtt{u}(t)\mathtt{x}(t) \qquad \mathtt{x}(0) = x_0$$

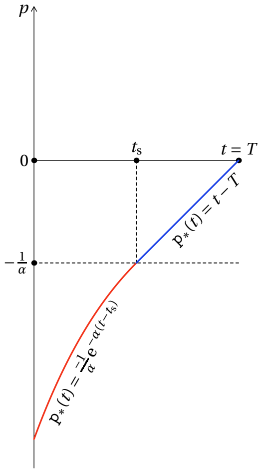
$$\dot{\mathtt{p}}(t) = (1 - \mathtt{u}(t)) - \mathtt{p}(t)\alpha\mathtt{u}(t) \qquad \mathtt{p}(T) = 0$$

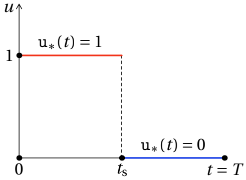
$$\mathtt{u}_*(t) = \begin{cases} 0 & \text{if } 1 + \alpha\mathtt{p}_*(t) > 0 \\ 1 & \text{if } 1 + \alpha\mathtt{p}_*(t) < 0 \end{cases}$$

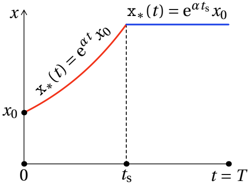
$$p_{\text{switch}} = -1/\alpha$$

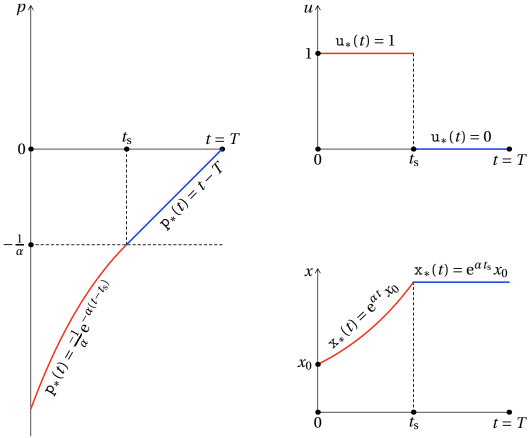
$$\dot{x}(t) = \alpha u(t) x(t) \qquad x(0) = x_0$$

$$\dot{p}(t) = (1 - u(t)) - p(t)\alpha u(t) \qquad p(T) = 0$$









$$H(x_*(t), p_*(t), u_*(t)) = p_*(t)\alpha u_*(t) x_*(t) + (u_*(t) - 1)x_*(t) = \begin{cases} p_*(t)\alpha x_*(t) & \text{if } u_*(t) = 1 \\ -x_*(t) & \text{if } u_*(t) = 0 \end{cases} = -e^{\alpha T - 1} x_0$$

Lemma (Constancy of the Hamiltonian)

Let all the assumptions of MP thm be satisfied, and assume that $\frac{\partial f(x,u)}{\partial u}, \frac{\partial L(x,u)}{\partial u}$ are continuous. Suppose that $\mathtt{u}_*$ is C^1 at all but finitely many $t \in [0,T]$. Then a constant H_* exists such that

$$H(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t)) = H_*$$

at every t where $\mathtt{u}_*(t)$ is continuous.

For the smooth unbounded case this is Beltrami:

Proof for the smooth unbounded case.

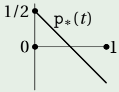
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t)) &= \frac{\partial H}{\partial x^{\mathrm{T}}}\dot{\mathtt{x}}_* + \frac{\partial H}{\partial p^{\mathrm{T}}}\dot{\mathtt{p}}_* + \frac{\partial H}{\partial u^{\mathrm{T}}}\dot{\mathtt{u}}_* \\ &= \frac{\partial H}{\partial x^{\mathrm{T}}}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p^{\mathrm{T}}}\left(-\frac{\partial H}{\partial x}\right) + \frac{\partial H}{\partial u^{\mathrm{T}}}\dot{\mathtt{u}}_* = \frac{\partial H}{\partial u^{\mathrm{T}}}\dot{\mathtt{u}}_* = 0.\end{aligned}$$

But it also holds for non-smooth bounded case!! □

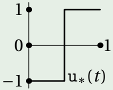
Example (Switching example — continued)

We considered $\dot{\mathtt{x}}(t) = \mathtt{u}(t), \mathtt{x}(0) = x_0, \mathbb{U} = [-1, 1]$ & $J(\mathtt{u}) = \int_0^1 \mathtt{x}(t)\,\mathrm{d}t - \frac{1}{2}\mathtt{x}(1)$:

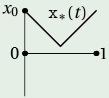
$$\mathtt{p}_*(t) = \frac{1}{2} - t$$



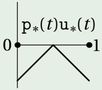
$$\mathtt{u}_*(t) = \begin{cases} -1 & \text{if } 0 \le t < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$



$$\mathtt{x}_*(t) = \begin{cases} x_0 - t & \text{for } 0 \le t \le \frac{1}{2} \\ x_0 - 1 + t & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$



$$\mathtt{p}_*(t)\mathtt{u}_*(t) = \begin{cases} -(\frac{1}{2} - t) & \text{if } 0 \le t < \frac{1}{2} \\ (\frac{1}{2} - t) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$



So $H(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t)) = \mathtt{p}_*(t)\mathtt{u}_*(t) + \mathtt{x}_*(t)$ indeed is constant!

Lecture 8

- Recap Minimum Principle! & Constancy of Hamiltonians
- Proof...
- § 2.6 OC with final constraints
- § 2.7 OC with free final time
- “Car parking problem”

$J(\mathtt{u}) = \int_0^T L(\mathtt{x}(t), \mathtt{u}(t))\, \mathrm{d}t + K(\mathtt{x}(T)), \qquad \dot{\mathtt{x}}(t) = f(\mathtt{x}(t), \mathtt{u}(t)), \qquad \mathtt{x}(0) = x_0, \qquad \mathtt{u}(t) \in \mathbb{U}$

Theorem (Pontryagin's Minimum Principle)

Assume $f(x,u)$ & $\partial f(x,u)/\partial x$ & $L(x,u)$ & $\partial L(x,u)/\partial x$ are continuous in x and u, and that $K(x)$ and $\partial K(x)/\partial x$ are continuous in x. Suppose $\mathtt{u}_* : [0,T] \to \mathbb{U}$ is a solution of the OC problem, and assume it is piecewise continuous, and let $\mathtt{x}_* : [0,T] \to \mathbb{R}^n$ be the resulting optimal state. Then there is a unique costate $\mathtt{p}_* : [0,T] \to \mathbb{R}^n$ such that

$$\dot{\mathtt{x}}_*(t) = \frac{\partial H(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t))}{\partial p}, \qquad \mathtt{x}_*(0) = x_0,$$

$$\dot{\mathtt{p}}_*(t) = -\frac{\partial H(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t))}{\partial x}, \qquad \mathtt{p}_*(T) = \frac{\partial K(\mathtt{x}_*(T))}{\partial x},$$

and the input $\mathtt{u}_*(t)$ minimizes the Hamiltonian,

$$H(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t)) = \min_{u \in \mathbb{U}} H(\mathtt{x}_*(t), \mathtt{p}_*(t), u),$$

at every $t \in [0,T]$ where $\mathtt{u}_*(t)$ is continuous. Here, $H(x,p,u) := p^{\mathrm{T}} f(x,u) + L(x,u)$

Moreover, $H(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t))$ is constant as a function of t

Example (Linear system with quadratic cost)

$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \qquad \mathtt{x}(0) = x_0, \quad \mathbb{U} = \mathbb{R} \qquad\qquad J(\mathtt{u}) = \int_0^1 \mathtt{x}^2(t) + \mathtt{u}^2(t)\,\mathrm{d}t$$

$$H(x,p,u) = pu + x^2 + u^2$$

$$\implies \quad \mathtt{u}_*(t) = -\tfrac{1}{2}\mathtt{p}_*(t)$$

$$\dot{\mathtt{x}}_*(t) = -\tfrac{1}{2}\mathtt{p}_*(t) \qquad\qquad \mathtt{x}_*(0) = x_0$$

$$\dot{\mathtt{p}}_*(t) = -2\mathtt{x}_*(t) \qquad\qquad \mathtt{p}_*(1) = 0$$

$$\implies \quad \ddot{\mathtt{p}}_*(t) = \mathtt{p}_*(t) \qquad\qquad \implies \quad \mathtt{p}_*(t) = c_1\mathrm{e}^{t} + c_2\mathrm{e}^{-t}$$

$$\implies \quad \mathtt{x}_*(t) = -\tfrac{1}{2}c_1\mathrm{e}^{t} + \tfrac{1}{2}c_2\mathrm{e}^{-t}$$

The two constants c_1, c_2 follow uniquely from the two boundary conditions $\mathtt{x}_*(0) = x_0,\ \mathtt{p}_*(1) = 0$

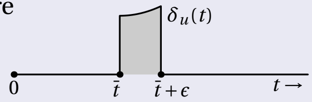
Proof.

$$\dot{\mathtt{p}}_*(t) = A(t)\mathtt{p}_*(t) + b(t), \quad \mathtt{p}_*(T) = \partial K(\mathtt{x}_*(T))/\partial x$$

suppose $\exists \bar{t}, \hat{u}$: $\quad c = H(\mathtt{x}_*(\bar{t}), \mathtt{p}_*(\bar{t}), \hat{u}) - H(\mathtt{x}_*(\bar{t}), \mathtt{p}_*(\bar{t}), \mathtt{u}_*(\bar{t})) < 0$

$$\bar{\mathtt{u}}(t) = \begin{cases} \hat{u} & \text{if } t \in [\bar{t}, \bar{t}+\epsilon], \\ \mathtt{u}_*(t) & \text{elsewhere} \end{cases}$$

$\bar{\mathtt{u}}(t) = \mathtt{u}_*(t) + \delta_u(t)$:



$$\int_0^T H(\mathtt{x}_*(t), \mathtt{p}_*(t), \bar{\mathtt{u}}(t)) - H(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t))\, \mathrm{d}t = c\epsilon + \mathfrak{o}(\epsilon)$$

$$\mathtt{x} = \mathtt{x}_* + \delta_x$$

$$\dot{\delta}_x = (\dot{\mathtt{x}}_* + \dot{\delta}_x) - \dot{\mathtt{x}}_* \quad = \quad f(\mathtt{x}_* + \delta_x, \mathtt{u}_* + \delta_u) - f(\mathtt{x}_*, \mathtt{u}_*)$$

Proof.

$$
\begin{aligned}
\Delta &= J(\mathrm{u}_* + \delta_u) - J(\mathrm{u}_*) \\
&= K(\mathrm{x}_*(T) + \delta_x(T)) - K(\mathrm{x}_*(T)) + \int_0^T L(\mathrm{x}_* + \delta_x, \mathrm{u}_* + \delta_u) - L(\mathrm{x}_*, \mathrm{u}_*)\,\mathrm{d}t \\
&= \frac{\partial K(\mathrm{x}_*(T))}{\partial x^{\mathrm{T}}}\delta_x(T) + \int_0^T L(\mathrm{x}_* + \delta_x, \mathrm{u}_* + \delta_u) - L(\mathrm{x}_*, \mathrm{u}_*)\,\mathrm{d}t + \mathfrak{o}(\epsilon) \\
&= \mathrm{p}_*^{\mathrm{T}}(T)\delta_x(T) + \int_0^T -\mathrm{p}_*^{\mathrm{T}}[f(\mathrm{x}_* + \delta_x, \mathrm{u}_* + \delta_u) - f(\mathrm{x}_*, \mathrm{u}_*)] + H(\mathrm{x}_* + \delta_x, \mathrm{p}_*, \mathrm{u}_* + \delta_u) - H(\mathrm{x}_*, \mathrm{p}_*, \mathrm{u}_*)\,\mathrm{d}t + \mathfrak{o}(\epsilon) \\
&= \mathrm{p}_*^{\mathrm{T}}(T)\delta_x(T) + \int_0^T -\mathrm{p}_*^{\mathrm{T}}\dot{\delta}_x + H(\mathrm{x}_* + \delta_x, \mathrm{p}_*, \mathrm{u}_* + \delta_u) - H(\mathrm{x}_*, \mathrm{p}_*, \mathrm{u}_* + \delta_u)\,\mathrm{d}t \\
&\qquad + \int_0^T H(\mathrm{x}_*, \mathrm{p}_*, \mathrm{u}_* + \delta_u) - H(\mathrm{x}_*, \mathrm{p}_*, \mathrm{u}_*)\,\mathrm{d}t + \mathfrak{o}(\epsilon) \\
&= \mathrm{p}_*^{\mathrm{T}}(T)\delta_x(T) + \int_0^T -\mathrm{p}_*^{\mathrm{T}}\dot{\delta}_x + \frac{\partial H(\mathrm{x}_*, \mathrm{p}_*, \mathrm{u}_* + \delta_u)}{\partial x^{\mathrm{T}}}\delta_x\,\mathrm{d}t + c\epsilon + \mathfrak{o}(\epsilon) \\
&= \mathrm{p}_*^{\mathrm{T}}(T)\delta_x(T) + \int_0^T -\mathrm{p}_*^{\mathrm{T}}\dot{\delta}_x - \dot{\mathrm{p}}_*^{\mathrm{T}}\delta_x\,\mathrm{d}t + c\epsilon + \mathfrak{o}(\epsilon) = \mathrm{p}_*^{\mathrm{T}}(T)\delta_x(T) + \Big[-\mathrm{p}_*^{\mathrm{T}}(t)\delta_x(t)\Big]_0^T + c\epsilon + \mathfrak{o}(\epsilon) = c\epsilon + \mathfrak{o}(\epsilon)
\end{aligned}
$$

2.6 Optimal Control with Final Constraints

So far final state $\mathtt{x}(T)$ was free. Now suppose some final components are fixed:

$$\dot{\mathtt{x}}(t) = f(\mathtt{x}(t), \mathtt{u}(t)), \qquad \mathtt{x}(0) = x_0, \qquad \mathtt{x}_i(T) = \hat{x}_i, \quad i = 1, \dots, r.$$

Then the final conditions on the corresponding costate components are absent:

$$\mathtt{p}_i(T) = \frac{\partial K(\mathtt{x}_*(T))}{\partial x_i}, \quad i = r+1, \dots, n.$$

Complication:

Example (OC without feasible perturbation)

$\dot{\mathtt{x}}(t) = \mathtt{u}^2(t), \quad \mathtt{x}(0) = 0, \quad \mathtt{x}(T) = 0.$ Uniquely determines $\mathtt{u}(t)$. So every perturbation is infeasible!

MP proof breaks down.. Leads to normal/abnormal case
(similar to the normal/abnormal EL for integral constrained SPITCOV):

$$H_\lambda(x, p, u) = p^{\mathrm{T}} f(x, u) + \lambda L(x, u) = \begin{cases} p^{\mathrm{T}} f(x, u) + L(x, u) & \text{if } \lambda = 1 \text{ "normal"} \\ p^{\mathrm{T}} f(x, u) & \text{if } \lambda = 0 \text{ "abnormal"} \end{cases}$$

Theorem (Minimum Principle for constrained final state)

Consider "constrained OC". Assume $f(x,u)$, $\frac{\partial f(x,u)}{\partial x}$, $L(x,u)$, $\frac{\partial L(x,u)}{\partial x}$ are continuous in x and u, and $K(x)$ and $\frac{\partial K(x)}{\partial x}$ continuous in x.

Suppose $\mathtt{u}_*: [0,T] \to \mathbb{U}$ solves the OC problem, and assume it is piecewise continuous, and let $\mathtt{x}_*: [0,T] \to \mathbb{R}^n$ be the resulting optimal state. Then there is a $\mathtt{p}_*: [0,T] \to \mathbb{R}^n$ and a constant $\lambda \in \{0,1\}$ such that $(\lambda_*, \mathtt{p}_*(t)) \neq (0,0)$ for all $t \in [0,T]$, and

$$\dot{\mathtt{x}}_*(t) = \frac{\partial H_\lambda(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t))}{\partial p}, \qquad \mathtt{x}_*(0) = x_0, \quad \mathtt{x}_{*i}(T) = \hat{x}_i, \; i = 1, \dots, r,$$

$$\dot{\mathtt{p}}_*(t) = -\frac{\partial H_\lambda(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t))}{\partial x}, \qquad \mathtt{p}_{*i}(T) = \frac{\partial K(\mathtt{x}_*(T))}{\partial x_i}, \; i = r+1, \dots, n,$$

and along the solution $\mathtt{x}_*(t), \mathtt{p}_*(t)$, the input $\mathtt{u}_*(t)$ minimizes the modified Hamiltonian,

$$H_\lambda(\mathtt{x}_*(t), \mathtt{p}_*(t), \mathtt{u}_*(t)) = \min_{u \in \mathbb{U}} H_\lambda(\mathtt{x}_*(t), \mathtt{p}_*(t), u),$$

at every $t \in [0,T]$ where $\mathtt{u}_*(t)$ is continuous.

Usually we have the normal case:

Example (Integrator system with fixed initial and final state)

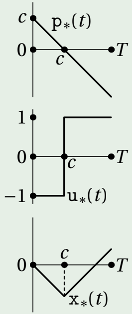
$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \quad \mathtt{x}(0) = \mathtt{x}(T) = 0, \quad \mathbb{U} = [-1, 1], \qquad J(\mathtt{u}) = \int_0^T \mathtt{x}(t)\, \mathrm{d}t$$

$$H_1(x, p, u) = pu + x$$

$$\dot{\mathtt{p}}_*(t) = -1, \;\; \mathtt{p}_*(T) = \text{free} \implies \mathtt{p}_*(t) = c - t$$

$$\mathtt{u}_*(t) = \begin{cases} -1 & \text{if } t < c \\ +1 & \text{if } t > c \end{cases}$$

$$\mathtt{x}_*(t) = \begin{cases} -t & \text{if } t < c \\ -2c + t & \text{if } t > c \end{cases}$$



We have $\mathtt{x}(T) = 0$ iff $c = T/2$

2.7 Free Final Time

Let's optimize over T as well: $J(\mathtt{u}, T) = \int_0^T L(\mathtt{x}(t), \mathtt{u}(t))\, \mathrm{d}t + K(\mathtt{x}(T))$.

Theorem (Minimum Principle with free final time)

Suppose $(\mathtt{u}_*, T_*)$ is a solution of the optimal control problem with free final time, and that $\mathtt{u}_*$ is piecewise continuous on $[0, T_*]$, and that $T_* < \infty$. Then all conditions of MP Thm hold (with $T = T_*$), and

$$H_\lambda(\mathtt{x}_*(T_*), \mathtt{p}_*(T_*), \mathtt{u}_*(T_*)) = 0.$$

Proof.

For the standard OC problem (normal case):

$$\begin{aligned} 0 = \frac{\mathrm{d}J(\mathtt{u}_*, T_*)}{\mathrm{d}T} &= \frac{\partial K(\mathtt{x}_*(T_*))}{\partial x^{\mathrm{T}}} \dot{\mathtt{x}}_*(T_*) + L(\mathtt{x}_*(T_*), \mathtt{u}_*(T_*)) \\ &= \mathtt{p}_*^{\mathrm{T}}(T_*) f(\mathtt{x}_*(T_*), \mathtt{u}_*(T_*)) + L(\mathtt{x}_*(T_*), \mathtt{u}_*(T_*)) \\ &= H_1(\mathtt{x}_*(T_*), \mathtt{p}_*(T_*), \mathtt{u}_*(T_*)). \end{aligned}$$

□

Example (Minimal time car parking problem — James Bond)

$$\dot{x}_1(t) = x_2(t) \qquad x_1(0) = x_{01}, \qquad x_1(T) = 0,$$

$$\dot{x}_2(t) = u(t), \qquad x_2(0) = x_{02}, \qquad x_2(T) = 0, \quad u(t) \in [-1, 1]$$

$$J(u) = \int_0^T 1 \, dt$$

$$H_1(x, p, u) = p_1 x_2 + p_2 u + 1$$

$$\dot{p}_1(t) = 0, \qquad p_1(T) = \text{free}$$

$$\dot{p}_2(t) = -p_1(t). \qquad p_2(T) = \text{free}$$

$$p_2(t) = at + b$$

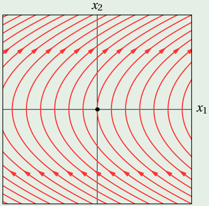
$$u_*(t) = -\operatorname{sgn}(p_2(t)) \qquad \text{it changes sign at most once!}$$

Notice: $(a, b) \neq (0, 0)$ for otherwise $H = 1 \neq 0$.

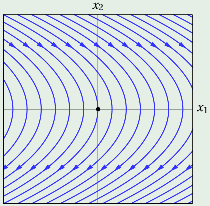
Example (James Bond — continued)

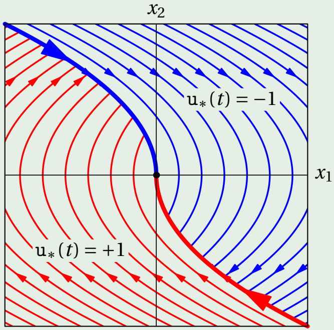
Previous slide: $\dot{\mathtt{x}}_1 = \mathtt{x}_2$, $\dot{\mathtt{x}}_2 = \mathtt{u}$ and $\mathtt{u}(t) = \pm 1$ with at most one switch:

$\mathtt{u}(t) = +1:$

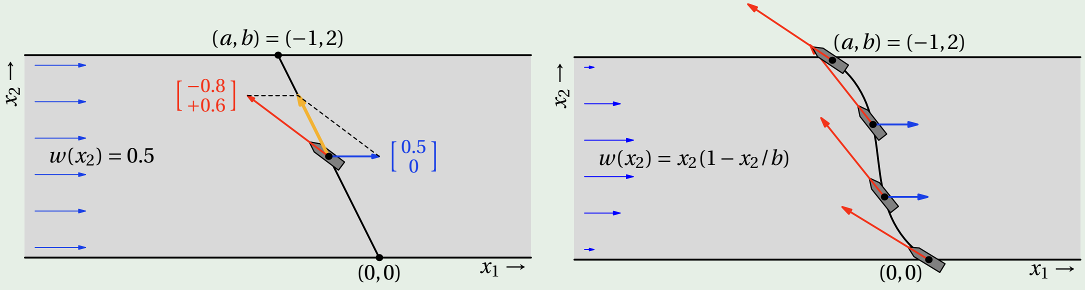


$\mathtt{u}(t) = -1:$





Example (Zermelo – time-optimal control)



$$\begin{aligned}
\dot{\mathtt{x}}_{*1}(t) &= \cos(\mathtt{u}_*(t)) + w(\mathtt{x}_{*2}(t)), & \mathtt{x}_{*1}(0) &= 0,\\
\dot{\mathtt{x}}_{*2}(t) &= \sin(\mathtt{u}_*(t)), & \mathtt{x}_{*2}(0) &= 0,\\
\dot{\mathtt{u}}_*(t) &= -\cos^2(\mathtt{u}_*(t))\, w'(\mathtt{x}_{*2}(t)), & \mathtt{u}_*(0) &= u_0.
\end{aligned}$$

Choose u_0 such that $(\mathtt{x}_{*1}(T), \mathtt{x}_{*2}(T)) = (a,b) := (-1,2)$. Usually requires iteration.

2.8 Convexity

Example (SPITCOV convexity in terms of Hamiltonians)

Consider this calculus of variations problem with free endpoint

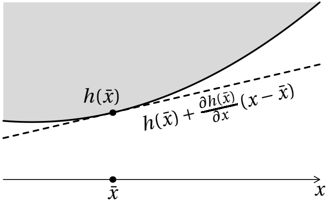
$$\min_{\mathrm{x}:[0,T]\to\mathbb{R}^n} \int_0^T F(\mathrm{x}(t), \dot{\mathrm{x}}(t))\,\mathrm{d}t, \qquad \mathrm{x}(0) = x_0.$$

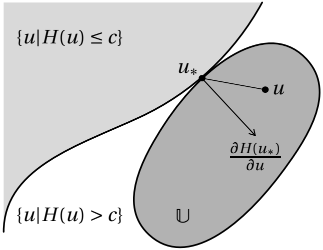
This an optimal control problem with

$$\dot{\mathrm{x}}(t) = \mathrm{u}(t), \qquad \mathrm{x}(0) = x_0, \qquad \mathbb{U} = \mathbb{R}^n, \qquad L(x,u) = F(x,u).$$

The Hamiltonian in this case is $H(x,p,u) = p^{\mathrm{T}} u + F(x,u)$.

$$\begin{aligned} \text{Convexity:} \quad & F(x,\dot{x}) \text{ convex in } x, \dot{x} \in \mathbb{R}^n \\ \iff \quad & H(x,p,u) \text{ convex in } x, u \in \mathbb{R}^n \text{ for whatever } p \end{aligned}$$





Lemma (Mangasarian)

Suppose f, L, K are C^1 and that

- $H(x, \mathtt{p}_*(t), u)$ for every $t \in [0, T]$ is convex in $(x, u) \in (\mathbb{R}^n, \mathbb{U})$,
- $K(x)$ is convex in $x \in \mathbb{R}^n$,
- $\mathbb{U}$ is a convex set,

then MP's $(\mathtt{x}_*, \mathtt{p}_*, \mathtt{u}_*)$ is optimal, in particular $\mathtt{u}_*$ is an optimal control.

All linear f, L, K satisfy...

Proof for the case $K(x) = 0$.

convex: $H(x, \mathrm{p}_*(t), u) \geq H(x_*, \mathrm{p}_*(t), u_*) + \frac{\partial H(x_*, \mathrm{p}_*(t), u_*)}{\partial x^{\mathrm{T}}}(x - x_*) + \frac{\partial H(x_*, \mathrm{p}_*(t), u_*)}{\partial u^{\mathrm{T}}}(u - u_*)$

pointwise optimal: $\frac{\partial H(\mathrm{x}_*(t), \mathrm{p}_*(t), \mathrm{u}_*(t))}{\partial u^{\mathrm{T}}}(u - \mathrm{u}_*(t)) \geq 0 \qquad \forall u \in \mathbb{U}$

combined: $H(x, \mathrm{p}_*(t), u) \geq H(\mathrm{x}_*(t), \mathrm{p}_*(t), \mathrm{u}_*(t)) - \dot{\mathrm{p}}_*^{\mathrm{T}}(t)(x - \mathrm{x}_*(t))$

$$
\begin{aligned}
J(x_0, \mathrm{u}) - J(x_0, \mathrm{u}_*) &= \int_0^T L(\mathrm{x}, \mathrm{u})\,\mathrm{d}t - L(\mathrm{x}_*, \mathrm{u}_*)\,\mathrm{d}t \\
&= \int_0^T (H(\mathrm{x}, \mathrm{p}_*, \mathrm{u}) - \mathrm{p}_*^{\mathrm{T}}\dot{\mathrm{x}}) - (H(\mathrm{x}_*, \mathrm{p}_*, \mathrm{u}_*) - \mathrm{p}_*^{\mathrm{T}}\dot{\mathrm{x}}_*)\,\mathrm{d}t \\
&= \int_0^T (H(\mathrm{x}, \mathrm{p}_*, \mathrm{u}) - H(\mathrm{x}_*, \mathrm{p}_*, \mathrm{u}_*)) - \mathrm{p}_*^{\mathrm{T}}(\dot{\mathrm{x}} - \dot{\mathrm{x}}_*)\,\mathrm{d}t \\
&\geq \int_0^T -\dot{\mathrm{p}}_*^{\mathrm{T}}(\mathrm{x} - \mathrm{x}_*) - \mathrm{p}_*^{\mathrm{T}}(\dot{\mathrm{x}} - \dot{\mathrm{x}}_*)\,\mathrm{d}t \\
&= \left[-\mathrm{p}_*^{\mathrm{T}}(\mathrm{x} - \mathrm{x}_*)\right]_0^T = 0.
\end{aligned}
$$

$\square$

MATLAB script bvp4c developed by L. Shampine and J. Kierzenka can handle mixed boundary conditions (needed in MP). Suppose $\dot{\mathrm{x}} = \mathrm{u}, \mathrm{x}(0) = 1, J(\mathrm{u}) = \int_0^1 \frac{1}{2}\mathrm{x}^2(t) + \frac{1}{2}\mathrm{u}^2(t)\,\mathrm{d}t$. Then $H(x,p,u) = pu + \frac{1}{2}x^2 + \frac{1}{2}u^2$. So the Hamiltonian equations become

$$\begin{bmatrix} \dot{\mathrm{x}}(t) \\ \dot{\mathrm{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathrm{u}(t) \\ -\mathrm{x}(t) \end{bmatrix}, \qquad \begin{bmatrix} \mathrm{x}(0) \\ \mathrm{p}(T) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Define the Hamiltonian equations in Matlab:

```
function xpdot=F(t,xp);           % xp(1)=x and xp(2)=p:
  u=-xp(2);                       % the minimizing here is u = -p (for this problem)
  xpdot=[u;                       % ẋ = u
         -xp(1)];                 % ṗ = -x
```

Define the boundary conditions in Matlab:

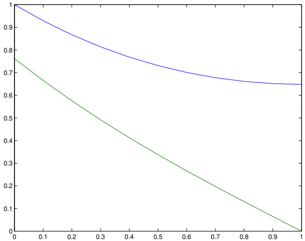
```
function rest=boundaryc(xp0,xpT); % what should be zero at t=0 and t=T:
  rest=[xp0(1)-1;                 % = x(0) - x0 = 0 (because we took x0 = 1)
        xpT(2)-0];                % p(T) = 0
```

Then solve the differential with these boundary conditions:

```
T=1;                               % final time
tgrid=linspace(0,T,25);            % discretized time
% simple constant initial solution: x(t)=1, p=0.5:
solinit=bvpinit(tgrid,[1 0.5]);

sol=bvp4c(@F,@boundaryc,solinit); % SOLVE IT

t=0:.1:T;                          % possibly other discretized time
xp=deval(sol,t);                   % 1st row is state, 2nd is costate
plot(t,xp);                        % plot state and costate
```



Part IV

Chapter 3 — Dynamic Programming

Overview

Lecture 9

- § 3.1 Intro to Dynamic Programming
- § 3.2 Principle of Optimality
- § 3.3 Discrete-time Dynamic Programming [optional]
- § 3.4 Hamilton-Jacobi-Bellman Equation

3.1 Introduction to Dynamic Programming

$$\dot{\mathtt{x}}(t) = f(\mathtt{x}(t), \mathtt{u}(t)), \qquad \mathtt{x}(0) = x_0, \qquad \mathtt{u} : [0, T] \to \mathbb{U} \subseteq \mathbb{R}^m.$$

$$J_{[0,T]}(x_0, \mathtt{u}) := \int_0^T L(\mathtt{x}(t), \mathtt{u}(t))\, \mathrm{d}t + K(\mathtt{x}(T))$$

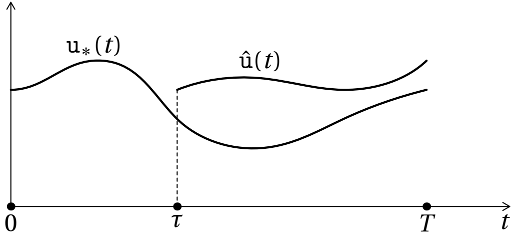
$$J_{[\tau,T]}(x, \mathtt{u}) := \int_\tau^T L(\mathtt{x}(t), \mathtt{u}(t))\, \mathrm{d}t + K(\mathtt{x}(T)), \qquad \mathtt{x}(\tau) = x.$$

Determine minimal $J_{[\tau,T]}(x, \mathtt{u})$ for each “initial” time $\tau \in [0, T]$ and “initial” state $\mathtt{x}(\tau) = x$, and then establish a dynamic relation between these optimal costs (hence the name *dynamic* programming)

Richard Bellman (1920–1984)

3.2 Principle of Optimality

PoO: "every tail of an optimal control is optimal"



If $\mathtt{u}_*$ minimizes $J_{[0,T]}(x_0, \mathtt{u})$ then it also minimizes $J_{[\tau,T]}(\mathtt{x}_*(\tau), \mathtt{u})$ for otherwise

$$J_{[0,T]}(x_0, \bar{\mathtt{u}}) = \int_0^\tau L(\mathtt{x}_*(t), \mathtt{u}_*(t))\,\mathrm{d}t + J_{[\tau,T]}(\mathtt{x}_*(\tau), \hat{\mathtt{u}}) < J_{[0,T]}(x_0, \mathtt{u}_*)$$

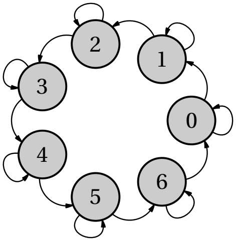
3.3 Discrete-Time Dynamic Programming

$$\mathtt{x}(t+1) = f(\mathtt{x}(t), \mathtt{u}(t)), \qquad \mathtt{x}(0) = x_0, \qquad \mathtt{x}(t) \in \mathbb{X}, \qquad \mathtt{u}(t) \in \mathbb{U}, \qquad t \in \{0, \ldots, T-1\}$$

Example (Naive optimization of a discrete-time OC problem)

$$\mathtt{x}(t+1) = \mathtt{x}(t) + \mathtt{u}(t), \qquad \mathtt{x}(0) = x_0, \qquad \mathbb{X} = \{0, \ldots, 6\}, \qquad \mathbb{U} = \{0, 1\}, \qquad J = \cdots$$

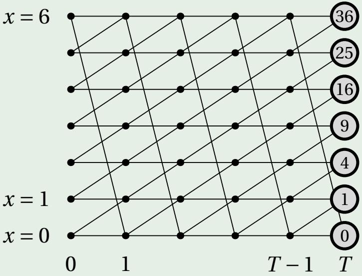
Naive optimization: explore all $\mathtt{x} = (x_0, \mathtt{x}(1), \ldots, \mathtt{x}(T))$ and pick the best.
There are $2^T = |\mathbb{U}|^T$ state trajectories. Computational effort is proportional to $T \times |\mathbb{U}|^T$

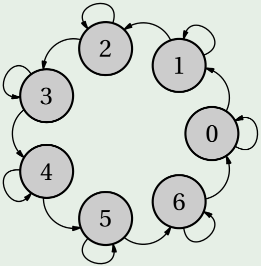


Example (Dynamic programming (1/3))

$$\mathtt{x}(t+1) = \mathtt{x}(t) + \mathtt{u}(t), \quad \mathtt{u}(t) \in \{0,1\}, \quad t \in \{0,\dots,T-1\}, \quad \mathbb{X} = \{0,\dots,6\}$$

$$J_{[0,T]}(x_0, \mathtt{u}) = \sum_{t=0}^{T-1} L(\mathtt{x}(t), \mathtt{u}(t)) + K(\mathtt{x}(T)) \quad \text{with} \quad K(x) = x^2, \quad L(x,u) = u$$



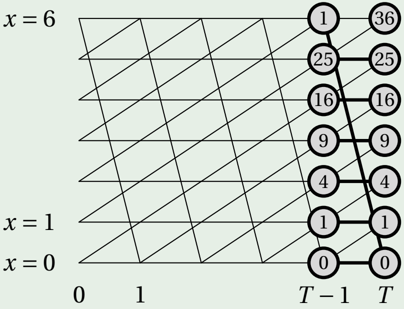


The above shows the final cost $K(x)$ for all states,
(we don't know yet in which final state $\mathtt{x}(T)$ we end up)

Example (Dynamic programming (2/3))

Define optimal cost-to-go: $\mathcal{V}(x, T-1)$ at time $T-1$:

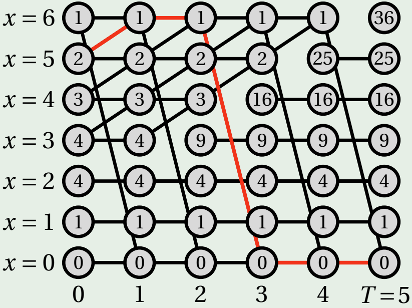
$$\mathcal{V}(x, T-1) = \min_{u \in \{0,1\}} L(x,u) + K(f(x,u))$$



Example (Dynamic programming (3/3))

$$\mathcal{V}(x,t) = \min_{u\in\{0,1\}} \big(L(x,u) + \mathcal{V}(f(x,u), t+1)\big).$$

Solve it backwards in time. For $T=5$ this eventually gives:



$$\mathcal{V}(x,t) = \min_{u\in\mathbb{U}} \big(L(x,u) + \mathcal{V}(f(x,u), t+1)\big) \quad 0 \le t \le T-1, \qquad \mathcal{V}(x,T) = K(x)$$

Number of operations: $T \times |\mathbb{U}| \times |\mathbb{X}|$

3.4 Hamilton-Jacobi-Bellman Equation

$$J_{[\tau,T]}(x,\mathrm{u}) := \int_{\tau}^{T} L(\mathrm{x}(t),\mathrm{u}(t))\,\mathrm{d}t + K(\mathrm{x}(T)), \qquad \mathrm{x}(\tau) = x.$$

Definition (Value function or cost-to-go)

Consider the OC problem. The value function $\mathcal{V} : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ at state x and time τ is defined as the optimal cost-to-go over time horizon $[\tau,T]$ with initial state $\mathrm{x}(\tau) = x$, that is,

$$\mathcal{V}(x,\tau) = \inf_{\mathrm{u}:[\tau,T]\to\mathbb{U}} J_{[\tau,T]}(x,\mathrm{u}),$$

with $J_{[\tau,T]}$ as defined above.

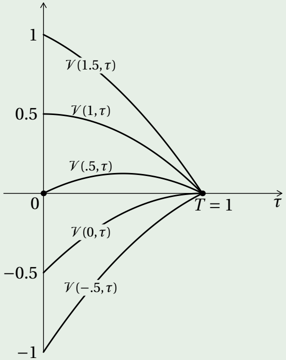
Example (Integrator with linear cost)

$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \qquad \mathtt{x}(0) = x_0, \qquad \mathbb{U} = [-1, 1], \qquad J_{[0,1]}(x_0, \mathtt{u}) = \int_0^1 \mathtt{x}(t)\,\mathrm{d}t.$$

$$\mathtt{u}_*(t) = -1,$$

$$\mathtt{x}_*(\tau) = x \implies \mathtt{x}_*(t) = x - (t - \tau)$$

$$\begin{aligned}\mathcal{V}(x,\tau) &= \int_\tau^1 x - (t-\tau)\ \mathrm{d}t \\ &= \left[xt - \tfrac{1}{2}(t-\tau)^2\right]_\tau^1 \\ &= x(1-\tau) - \tfrac{1}{2}(1-\tau)^2\end{aligned}$$



$$J_{[\tau,T]}(x,\mathrm{u}) = \int_{\tau}^{\tau+\epsilon} L(\mathrm{x}(t),\mathrm{u}(t))\,\mathrm{d}t \quad + \quad J_{[\tau+\epsilon,T]}(\mathrm{x}(\tau+\epsilon),\mathrm{u})$$

$$\mathcal{V}(x,\tau) = \min_{u:[\tau,T]\to\mathbb{U}} \left(\int_{\tau}^{\tau+\epsilon} L(\mathrm{x}(t),\mathrm{u}(t))\,\mathrm{d}t + J_{[\tau+\epsilon,T]}(\mathrm{x}(\tau+\epsilon),\mathrm{u}) \right), \qquad \mathrm{x}(\tau) = x$$

$$\mathcal{V}(x,\tau) = \min_{u:[\tau,\tau+\epsilon]\to\mathbb{U}} \left(\int_{\tau}^{\tau+\epsilon} L(\mathrm{x}(t),\mathrm{u}(t))\,\mathrm{d}t + \mathcal{V}(\mathrm{x}(\tau+\epsilon),\tau+\epsilon) \right)$$

$$0 = \min_{u:[\tau,\tau+\epsilon]\to\mathbb{U}} \frac{\int_{\tau}^{\tau+\epsilon} L(\mathrm{x}(t),\mathrm{u}(t))\,\mathrm{d}t + \mathcal{V}(\mathrm{x}(\tau+\epsilon),\tau+\epsilon) - \mathcal{V}(x,\tau)}{\epsilon}$$

$$0 = \min_{u\in\mathbb{U}} \left(L(\mathrm{x}(\tau),u) + \frac{\mathrm{d}\mathcal{V}(\mathrm{x}(\tau),\tau)}{\mathrm{d}\tau} \right)$$

$$0 = \min_{u \in \mathbb{U}} \left(L(\mathrm{x}(\tau), u) + \frac{\mathrm{d}\mathcal{V}(\mathrm{x}(\tau), \tau)}{\mathrm{d}\tau} \right)$$

$$\frac{\mathrm{d}\mathcal{V}(\mathrm{x}(\tau), \tau)}{\mathrm{d}\tau} = \frac{\partial \mathcal{V}(\mathrm{x}(\tau), \tau)}{\partial x^{\mathrm{T}}} f(\mathrm{x}(\tau), \mathrm{u}(\tau)) + \frac{\partial \mathcal{V}(\mathrm{x}(\tau), \tau)}{\partial \tau}$$

$$0 = \min_{u \in \mathbb{U}} \left(L(x, u) + \frac{\partial \mathcal{V}(x, \tau)}{\partial x^{\mathrm{T}}} f(x, u) + \frac{\partial \mathcal{V}(x, \tau)}{\partial \tau} \right)$$

$$0 = \frac{\partial \mathcal{V}(x, \tau)}{\partial \tau} + \min_{u \in \mathbb{U}} \left(\frac{\partial \mathcal{V}(x, \tau)}{\partial x^{\mathrm{T}}} f(x, u) + L(x, u) \right)$$

Theorem (Hamilton-Jacobi-Bellman (HJB))

Consider the OC problem. Suppose $V : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is C^1 and satisfies the HJB equation(s)

$$\frac{\partial V(x,t)}{\partial t} + \min_{u \in \mathbb{U}} \left(\frac{\partial V(x,t)}{\partial x^{\mathrm{T}}} f(x,u) + L(x,u) \right) = 0, \qquad V(x,T) = K(x)$$

for all $x \in \mathbb{R}^n$ and all $t \in [0, T]$. Then

1. $J_{[\tau,T]}(x, \mathtt{u}) \geq V(x, \tau)$ for every input $\mathtt{u}$.
2. If there is a function $\mathtt{u}_* : [0, T] \to \mathbb{U}$ for which the solution $\mathtt{x}_*$ of $\dot{\mathtt{x}}(t) = f(\mathtt{x}(t), \mathtt{u}_*(t))$ with $\mathtt{x}(0) = x_0$ is well defined, and $\mathtt{u}_*(t)$ minimizes $\frac{\partial V(\mathtt{x}_*(t),t)}{\partial x^{\mathrm{T}}} f(\mathtt{x}_*(t), u) + L(\mathtt{x}_*(t), u)$ over all $u \in \mathbb{U}$ at almost each $t \in [0, T]$, then $\mathtt{u}_*$ is a solution to the optimal control problem, and the optimal cost is $J_{[0,T]}(x_0, \mathtt{u}_*) = V(x_0, 0)$.
3. Suppose for each $x \in \mathbb{R}^n, t \in [0, T]$ the HJB equation has a solution u. Denote one such solution as $\mathtt{u}(x, t)$. If for every $x \in \mathbb{R}^n, \tau \in [0, T]$ the $\mathtt{x}_*(t)$ (with $\mathtt{x}_*(\tau) = x$) is well defined for all $t \in [\tau, T]$, then $V(x, \tau)$ is the value function and $\mathtt{u}_*(t) := \mathtt{u}(\mathtt{x}_*(t), t)$ is an optimal control.

Example (Integrator with quadratic cost)

$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \quad \mathtt{x}(0) = x_0, \quad \mathbb{U} = \mathbb{R}, \qquad J(x_0, \mathtt{u}) = \mathtt{x}^2(T) + \int_0^T \mathtt{u}^2(t)\, \mathrm{d}t$$

$$0 = \frac{\partial V(x,t)}{\partial t} + \min_{u \in \mathbb{R}} \left(\frac{\partial V(x,t)}{\partial x} u + u^2 \right), \qquad V(x,T) = x^2$$

$$\mathtt{u}(x,t) = -\frac{1}{2} \frac{\partial V(x,t)}{\partial x},$$

$$0 = \frac{\partial V(x,t)}{\partial t} - \frac{1}{4} \left(\frac{\partial V(x,t)}{\partial x} \right)^2, \qquad V(x,T) = x^2$$

$V(x,t) = x^2 P(t)$ seems a wild guess? It gives:

$$0 = x^2 \dot{P}(t) - \frac{1}{4}(2xP(t))^2, \qquad x^2 P(T) = x^2$$

Example

$$0 = x^2 \dot{P}(t) - \frac{1}{4}(2xP(t))^2, \qquad x^2 P(T) = x^2$$

$$\dot{P}(t) = P^2(t), \qquad P(T) = 1.$$

$$P(t) = \frac{1}{1+T-t} \qquad V(x,t) = x^2 \frac{1}{1+T-t}$$

$$\begin{aligned} \mathtt{u}_*(t) &= \mathtt{u}(\mathtt{x}_*(t), t) = -\frac{1}{2}\frac{\partial V(\mathtt{x}(t), t)}{\partial x} \\ &= -\frac{2\mathtt{x}(t)P(t)}{2} = -\frac{\mathtt{x}(t)}{1+T-t} \end{aligned}$$

$$\dot{\mathtt{x}}_*(t) = \mathtt{u}_*(t) = -\frac{\mathtt{x}_*(t)}{1+T-t} \qquad \leftarrow \text{closed-loop system}$$

Closed-loop DE has well-defined solution, so V is the value function, and $\mathtt{u}_*$ is optimal!!!

Notice: $\mathtt{u}_*$ is given as a state feedback

Example

$$\dot{\mathrm{x}}(t) = \mathrm{u}(t), \quad \mathrm{x}(0) = x_0, \quad \mathbb{U} = [-1, 1], \qquad J_{[0,T]}(x_0, \mathrm{u}) = -\alpha\,\mathrm{x}(T) + \int_0^T \mathrm{x}(t)\,\mathrm{d}t$$

$$0 = \frac{\partial V(x,t)}{\partial t} + \min_{u\in[-1,1]} \left(\frac{\partial V(x,t)}{\partial x} u + x \right), \qquad V(x,T) = -\alpha x.$$

guess $\quad V(x,t) = xP(t) + Q(t)$

$$0 = x\dot{P}(t) + \dot{Q}(t) + \min_{u\in[-1,1]} P(t)u + x, \qquad xP(T) + Q(T) = -\alpha x$$

$$\dot{P}(t) = -1, \quad \dot{Q}(t) = -\min_{u\in[-1,1]} P(t)u, \qquad P(T) = -\alpha, \quad Q(T) = 0$$

Example

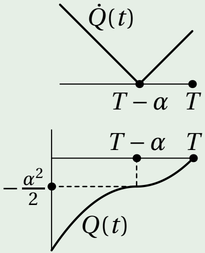
$$\dot{P}(t) = -1, \quad \dot{Q}(t) = -\min_{u\in[-1,1]} P(t)u, \qquad P(T) = -\alpha, \quad Q(T) = 0$$

$$P(t) = T - \alpha - t$$

$$\mathtt{u}_*(t) = \begin{cases} -1 & \text{if } t < T-\alpha \\ +1 & \text{if } t > T-\alpha \end{cases}$$

$$\dot{Q}(t) = \begin{cases} +(T-\alpha-t) & \text{if } t < T-\alpha \\ -(T-\alpha-t) & \text{if } t > T-\alpha \end{cases}$$

$$Q(t) = \begin{cases} -\frac{1}{2}(T-\alpha-t)^2 - \frac{\alpha^2}{2} & \text{if } t < T-\alpha \\ +\frac{1}{2}(T-\alpha-t)^2 - \frac{\alpha^2}{2} & \text{if } t > T-\alpha \end{cases}$$



Example (... continued (end))

$V(x,t) = xP(t) + Q(t)$ is continuously differentiable. The candidate optimal input is:

$$\mathtt{u}_*(t) = \begin{cases} -1 & \text{if } t < T - \alpha \\ +1 & \text{if } t > T - \alpha \end{cases}$$

Then solution $\mathtt{x}(t)$ of $\dot{\mathtt{x}}(t) = \mathtt{u}(t)$ well defined for all $t \in [0, T]$. Hence this $\mathtt{u}_*$ is the optimal input, the above $V(x,t)$ is the value function and $V(x_0, 0) = x_0 P(0) + Q(0)$ is the optimal cost.

What does Pontryagin say? $H(x, p, u) = pu + x$, and so $\dot{\mathtt{p}}_*(t) = -1, \mathtt{p}_*(T) = -\alpha$. Clearly this means that $\mathtt{p}_*(t) = T - \alpha - t$. Hence, as before, we get the same switching $\mathtt{u}_*(t)$.

But, of course, the fundamental difference is that the Minimum Principle assumes the existence of an optimal control, whereas satisfaction of the HJB equations proves that the control is optimal.

Lecture 10

- Recap Hamilton-Jacobi-Bellman (HJB)
- Example (quartics)
- § 3.5 Connection between HJB & MP
- § 3.6 Infinite horizon
- § 4.1 Intro to LQ
- § 4.2 LQ – Minimum Principle

$$\dot{\mathtt{x}}(t) = f(\mathtt{x}(t), \mathtt{u}(t)), \qquad \mathtt{x}(0) = x_0$$

$$J_{[0,T]}(x_0, \mathtt{u}) := \int_0^T L(\mathtt{x}(t), \mathtt{u}(t))\, \mathrm{d}t + K(\mathtt{x}(T)) \qquad \mathtt{x}(0) = x_0$$

$$J_{[\tau,T]}(x, \mathtt{u}) := \int_\tau^T L(\mathtt{x}(t), \mathtt{u}(t))\, \mathrm{d}t + K(\mathtt{x}(T)), \qquad \mathtt{x}(\tau) = x, \quad \tau \in [0, T]$$

$$V(x, \tau) = \inf_{\mathtt{u}:[\tau,T]\to\mathbb{U}} J_{[\tau,T]}(x, \mathtt{u}).$$

Theorem (Hamilton-Jacobi-Bellman (HJB))

Consider OC problem. Suppose a C^1 function $V : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ satisfies

$$\frac{\partial V(x,t)}{\partial t} + \min_{u \in \mathbb{U}} \left(\frac{\partial V(x,t)}{\partial x^{\mathrm{T}}} f(x,u) + L(x,u) \right) = 0, \qquad V(x,T) = K(x)$$

for all $x \in \mathbb{R}^n$ and all $t \in [0, T]$. Then

(1) $J_{[\tau,T]}(x, \mathtt{u}) \geq V(x, \tau)$ for every input $\mathtt{u}$.

(3) Suppose for each $x \in \mathbb{R}^n$, $t \in [0, T]$ the HJB eqn has a solution u. Denote one such solution as $\mathtt{u}(x,t)$. If for every $\tau \in [0,T]$, $x \in \mathbb{R}^n$ the input $\mathtt{u}(x,t)$ is admissible — meaning the solution $\mathtt{x}_*$ of

$$\dot{\mathtt{x}}(t) = f(\mathtt{x}(t), \mathtt{u}(\mathtt{x}(t), t)), \qquad \mathtt{x}(\tau) = x$$

is well defined for all $t \in [\tau, T]$ — then $V(x, \tau)$ is the value function and $\mathtt{u}_*(t) := \mathtt{u}(\mathtt{x}_*(t), t)$ is an optimal control.

Suppose $\mathtt{u}$ is admissible:

$$
\begin{aligned}
J_{[\tau,T]}(x,\mathtt{u}) &= \int_\tau^T L(\mathtt{x}(t),\mathtt{u}(t))\,\mathrm{d}t + K(\mathtt{x}(T)) \\
&= \int_\tau^T \frac{\partial V(\mathtt{x},t)}{\partial x^{\mathrm{T}}} f(\mathtt{x},\mathtt{u}) + L(\mathtt{x},\mathtt{u}) - \frac{\partial V(\mathtt{x},t)}{\partial x^{\mathrm{T}}} f(\mathtt{x},\mathtt{u})\,\mathrm{d}t + K(\mathtt{x}(T)) \\
&\geq \int_\tau^T \min_{u\in\mathbb{U}}\left(\frac{\partial V(\mathtt{x},t)}{\partial x^{\mathrm{T}}} f(\mathtt{x},u) + L(\mathtt{x},u)\right) - \frac{\partial V(\mathtt{x},t)}{\partial x^{\mathrm{T}}} f(\mathtt{x},\mathtt{u})\,\mathrm{d}t + K(\mathtt{x}(T)) \\
&= \int_\tau^T -\frac{\partial V(\mathtt{x},t)}{\partial t} - \frac{\partial V(\mathtt{x},t)}{\partial x^{\mathrm{T}}} f(\mathtt{x},\mathtt{u})\,\mathrm{d}t + K(\mathtt{x}(T)) \\
&= -\int_\tau^T \frac{\mathrm{d}V(\mathtt{x}(t),t)}{\mathrm{d}t}\,\mathrm{d}t + V(\mathtt{x}(T),T) = V(x,\tau).
\end{aligned}
$$

For $\mathtt{u}(x,t)$ the inequality " $\geq$" is an equality " $=$".
So, if $\mathtt{u}(x,t)$ is admissible then....

Example (Quartic control)

$$\dot{\mathrm{x}}(t) = \mathrm{u}(t),\ \mathrm{x}(0) = x_0,\ \mathbb{U} = \mathbb{R}, \qquad J_{[0,T]}(x_0, \mathrm{u}) = \int_0^T \mathrm{x}^4(t) + \mathrm{u}^4(t)\,\mathrm{d}t$$

$$0 = \frac{\partial V(x,t)}{\partial t} + \min_{u\in\mathbb{R}} \left(\frac{\partial V(x,t)}{\partial x} u + x^4 + u^4 \right), \qquad V(x,T) = 0$$

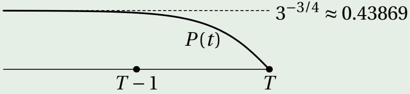
$$V(x,t) = x^4 P(t) \text{ guess:}$$

$$0 = x^4 \dot{P}(t) + \min_{u\in\mathbb{R}} \left(4x^3 P(t) u + x^4 + u^4 \right), \qquad x^4 P(T) = 0$$

$$u = -\sqrt[3]{P(t)}\,x$$

$$0 = x^4 \dot{P}(t) - 4x^4 P^{4/3}(t) + x^4 + x^4 P^{4/3}(t), \qquad x^4 P(T) = 0$$

$$-\dot{P}(t) = -3P^{4/3}(t) + 1, \qquad P(T) = 0$$

$P(t) =$ 

$$\mathrm{u}_*(t) = -\sqrt[3]{P(t)}\,\mathrm{x}_*(t) \qquad \dot{\mathrm{x}}_*(t) = -\sqrt[3]{P(t)}\,\mathrm{x}_*(t)$$

3.5 Connection with Minimum Principle

$$\frac{\partial V(x,t)}{\partial t} + \min_{u\in\mathbb{U}}\left(\frac{\partial V(x,t)}{\partial x^{\mathrm{T}}} f(x,u) + L(x,u)\right) = 0, \qquad V(x,T) = K(x)$$

$$H(x,p,u) = p^{\mathrm{T}} f(x,u) + L(x,u)$$

$$\frac{\partial V(x,t)}{\partial t} + \min_{u\in\mathbb{U}} H\left(x, \frac{\partial V(x,t)}{\partial x}, u\right) = 0, \qquad V(x,T) = K(x)$$

We also have

$$\mathtt{p}_*(T) = \frac{\partial K(\mathtt{x}_*(T))}{\partial x} = \frac{\partial V(\mathtt{x}_*(T),T)}{\partial x}$$

Theorem (Connection between costate & value function)

Assume $f(x,u)$, $L(x,u)$, $K(x)$ are all C^1. Let $\mathbb{U}=\mathbb{R}^m$ and suppose there is a C^2 function $V:\mathbb{R}^n\times[0,T]\to\mathbb{R}$ that satisfies the HJB equation

$$\frac{\partial V(x,t)}{\partial t}+\min_{u\in\mathbb{U}} H\Big(x,\frac{\partial V(x,t)}{\partial x},u\Big)=0, \qquad V(x,T)=K(x)$$

for all $x\in\mathbb{R}^n$ and all $t\in(0,T)$. Denote, for each x,t, one possible minimizer as $\mathtt{u}_*(x,t)$, and assume that all conditions of HJB thm are satisfied. In particular that $V(x,t)$ is the value function and that the differential equation $\dot{\mathtt{x}}_*(t)=f(\mathtt{x}_*(t),\mathtt{u}_*(\mathtt{x}_*(t),t))$ has a well defined solution $\mathtt{x}_*(t)$ for all $t\in[\tau,T]$ for every given $\mathtt{x}(\tau)\in\mathbb{R}^n$. Then $\mathtt{p}_*(t)$ defined as

$$\mathtt{p}_*(t)=\frac{\partial V(\mathtt{x}_*(t),t)}{\partial x}$$

is the solution of the Hamiltonian costate equation:

$$\dot{\mathtt{p}}_*(t)=-\frac{\partial H(\mathtt{x}_*(t),\mathtt{p}_*(t),\mathtt{u}_*(\mathtt{x}_*(t),t))}{\partial x}, \qquad \mathtt{p}_*(T)=\frac{\partial K(\mathtt{x}_*(T))}{\partial x}.$$

$$\mathtt{p}_*(t) = \frac{\partial V(\mathtt{x}_*(t), t)}{\partial x}$$

The costate at $t = 0$ captures the sensitivity of the optimal cost w.r.t. (changes in) the initial state x_0:

$$\mathtt{p}_*(0) = \frac{\partial V(\mathtt{x}_*(0), 0)}{\partial x}$$

Keep in mind:

- HJB is “sufficient” for optimality, while MP is necessary
- HJB is a partial DE, while MP is an ordinary DE
- HJB requires “high” degree of smoothness, MP does not.

3.6 Infinite Horizon and Lyapunov Functions

$$\dot{\mathrm{x}}(t) = f(\mathrm{x}(t), \mathrm{u}(t)), \quad \mathrm{x}(0) = x_0$$

$$J_{[0,\infty)}(x_0, \mathrm{u}) = \int_0^{\infty} L(\mathrm{x}(t), \mathrm{u}(t))\, \mathrm{d}t$$

$$\mathcal{V}(x, \tau) = \inf_{\mathrm{u}:[\tau,\infty)\to\mathbb{U}} J_{[\tau,\infty)}(x, \mathrm{u}),$$

$$\mathcal{V}(x) = \inf_{\mathrm{u}:[0,\infty)\to\mathbb{U}} J_{[0,\infty)}(x, \mathrm{u})$$

seems to imply: $\quad 0 = \min_{u\in\mathbb{U}} \left(\frac{\partial V(x)}{\partial x^{\mathrm{T}}} f(x,u) + L(x,u) \right)$????

Makes a connection with “closed-loop” Lyapunov function (cost-to-go):

Example (Design of optimal stabilizing inputs and Lyapunov function)

$$\dot{\mathrm{x}}(t) = \mathrm{u}(t), \quad J_{[0,\infty)}(x, \mathrm{u}) = \int_0^\infty \mathrm{x}^4(t) + \mathrm{u}^4(t)\, \mathrm{d}t, \qquad \mathbb{U} = \mathbb{R},$$

$$0 = \min_{u\in\mathbb{R}} \left(\frac{\partial V(x)}{\partial x} u + x^4 + u^4\right) \qquad u = -\left(\tfrac{1}{4}\frac{\partial V(x)}{\partial x}\right)^{1/3}$$

$$0 = \left(\tfrac{1}{4} - 1\right)\left(\tfrac{1}{4}\right)^{1/3}\left(\frac{\partial V(x)}{\partial x}\right)^{4/3} + x^4$$

$$\frac{\partial V(x)}{\partial x} = \pm 4(3^{-3/4})x^3 \qquad \text{choose: } V(x) = 3^{-3/4}x^4 \geq 0$$

$$\mathrm{u}_*(t) = -\left(\tfrac{1}{4}\frac{\partial V(\mathrm{x}(t))}{\partial x}\right)^{1/3} = -3^{-1/4}\mathrm{x}(t).$$

under this control: $\dot{V}(x) = \frac{\partial V(x)}{\partial x^{\mathrm{T}}} f(x, u_*) = -L(x, u_*) = -(x^4 + u_*^4) < 0$

$$J_{[0,\infty)}(x_0, \mathrm{u}) = \int_0^\infty L(\mathrm{x}, \mathrm{u})\,\mathrm{d}t \geq \int_0^\infty -\frac{\partial V(\mathrm{x})}{\partial x^{\mathrm{T}}} f(\mathrm{x}, \mathrm{u})\,\mathrm{d}t$$

$$= \int_0^\infty -\dot{V}(\mathrm{x})\,\mathrm{d}t = V(x_0) - \underbrace{V(\mathrm{x}(\infty))}_{0} = V(x_0),$$

and equality holds if $\mathrm{u} = \mathrm{u}_*$. So u_* is the control that minimizes J over all STABILIZING controls!

Part V

Chapter 4 — Linear Quadratic Control

Overview

4.1 Linear Systems with Quadratic Cost

Definition

Linear Quadratic (LQ) optimal control problem:

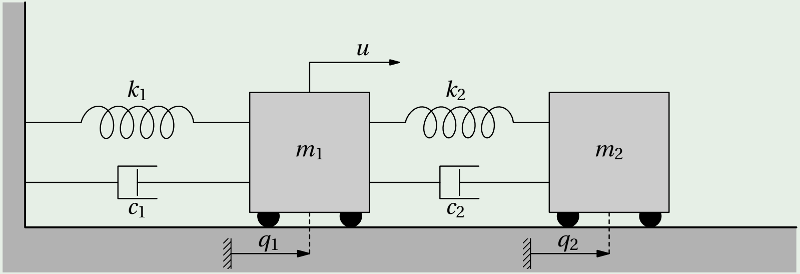
$$\dot{\mathtt{x}}(t) = A\mathtt{x}(t) + B\mathtt{u}(t), \qquad \mathtt{x}(0) = x_0 \in \mathbb{R}^n, \qquad \mathbb{U} = \mathbb{R}^m$$

$$J_{[0,T]}(x_0, \mathtt{u}) = \mathtt{x}^{\mathrm{T}}(T)S\mathtt{x}(T) + \int_0^T \mathtt{x}^{\mathrm{T}}(t)Q\mathtt{x}(t) + u^{\mathrm{T}}(t)R\mathtt{u}(t)\,\mathrm{d}t$$

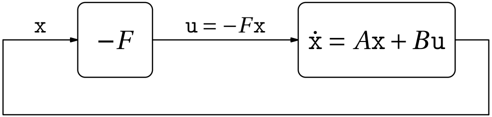
$$S = S^{\mathrm{T}} \geq 0$$
$$Q = Q^{\mathrm{T}} \geq 0$$
$$R = R^{\mathrm{T}} > 0$$

We solve it completely

Example (Two connected cars (§ 4.6))



Minimize $\int_0^{\infty} \mathrm{q}_2^2(t) + R\mathrm{u}^2(t)\,\mathrm{d}t$.....



4.2 Finite horizon LQ: Minimum Principle

$$J_{[0,T]}(x_0, \mathtt{u}) = \mathtt{x}^{\mathrm{T}}(T)S\mathtt{x}(T) + \int_0^T \mathtt{x}^{\mathrm{T}}(t)Q\mathtt{x}(t) + u^{\mathrm{T}}(t)R\mathtt{u}(t)\,\mathrm{d}t$$

$$H(x, 2p, u) = 2p^{\mathrm{T}}(Ax + Bu) + x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru$$

$$\begin{aligned} \dot{\mathtt{x}}(t) &= A\mathtt{x}(t) + B\mathtt{u}(t), & \mathtt{x}(0) &= x_0, \\ 2\dot{\mathtt{p}}(t) &= -A^{\mathrm{T}}2\mathtt{p}(t) - 2Q\mathtt{x}(t), & 2\mathtt{p}(T) &= 2S\mathtt{x}(T) \end{aligned}$$

$$\frac{\partial H(x, 2p, u)}{\partial u} = 2B^{\mathrm{T}}p + 2Ru \quad \Longrightarrow \quad u = -R^{-1}B^{\mathrm{T}}p$$

Hence we can eliminate $\mathtt{u}$ from the Hamiltonian equations:

$$\begin{aligned} \dot{\mathtt{x}}_*(t) &= A\mathtt{x}_*(t) - BR^{-1}B^{\mathrm{T}}\mathtt{p}_*(t), & \mathtt{x}_*(0) &= x_0, \\ \dot{\mathtt{p}}_*(t) &= -A^{\mathrm{T}}\mathtt{p}_*(t) - Q\mathtt{x}_*(t), & \mathtt{p}_*(T) &= S\mathtt{x}_*(T) \end{aligned}$$

$$\mathtt{u}_*(t) = -R^{-1}B^{\mathrm{T}}\mathtt{p}_*(t)$$

$$\dot{\mathtt{x}}_*(t) = A\mathtt{x}_*(t) - BR^{-1}B^{\mathrm{T}}\mathtt{p}_*(t), \qquad \mathtt{x}_*(0) = x_0,$$
$$\dot{\mathtt{p}}_*(t) = -A^{\mathrm{T}}\mathtt{p}_*(t) - Q\mathtt{x}_*(t), \qquad \mathtt{p}_*(T) = S\mathtt{x}_*(T)$$

$$\begin{bmatrix}\dot{\mathtt{x}}_*(t)\\ \dot{\mathtt{p}}_*(t)\end{bmatrix} = \underbrace{\begin{bmatrix}A & -BR^{-1}B^{\mathrm{T}}\\ -Q & -A^{\mathrm{T}}\end{bmatrix}}_{\mathscr{H}}\begin{bmatrix}\mathtt{x}_*(t)\\ \mathtt{p}_*(t)\end{bmatrix}, \qquad \begin{bmatrix}\mathtt{x}_*({\color{red}0})\\ \mathtt{p}_*({\color{red}T})\end{bmatrix} = \begin{bmatrix}x_0\\ S\mathtt{x}_*(T)\end{bmatrix}$$

$$\begin{bmatrix}\mathtt{x}_*(t)\\ \mathtt{p}_*(t)\end{bmatrix} = \mathrm{e}^{\mathscr{H}t}\begin{bmatrix}x_0\\ \mathtt{p}_*(0)\end{bmatrix}, \qquad \mathtt{p}_*(T) = S\mathtt{x}_*(T).$$

Lemma (Optimal cost)

For every solution $(\mathtt{x}_*, \mathtt{p}_*)$ of the Hamiltonian equations with $\mathtt{u}_*(t) := -R^{-1}B^{\mathrm{T}}\mathtt{p}_*(t)$, the cost equals

$$J_{[0,T]}(x_0, \mathtt{u}_*) = \mathtt{p}_*^{\mathrm{T}}(0)x_0.$$

It does not say that a solution $(\mathtt{x}_*, \mathtt{p}_*)$ exists! (but for LQ it does exist, as we will soon see)

Example

$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \qquad J_{[0,T]}(x_0, \mathtt{u}) = \int_0^T \mathtt{x}^2(t) + \mathtt{u}^2(t)\,\mathrm{d}t$$

$$\mathscr{H} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\mathrm{e}^{\mathscr{H}t} = \frac{1}{2}\begin{bmatrix} \mathrm{e}^t + \mathrm{e}^{-t} & -\mathrm{e}^t + \mathrm{e}^{-t} \\ -\mathrm{e}^t + \mathrm{e}^{-t} & \mathrm{e}^t + \mathrm{e}^{-t} \end{bmatrix}$$

$$\begin{bmatrix} \mathtt{x}_*(t) \\ \mathtt{p}_*(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathrm{e}^t + \mathrm{e}^{-t} & -\mathrm{e}^t + \mathrm{e}^{-t} \\ -\mathrm{e}^t + \mathrm{e}^{-t} & \mathrm{e}^t + \mathrm{e}^{-t} \end{bmatrix}\begin{bmatrix} x_0 \\ \mathtt{p}_*(0) \end{bmatrix}$$

$$\mathtt{p}_*(T) = S\mathtt{x}_*(T) = 0 \qquad \Longleftrightarrow \qquad \mathtt{p}_*(0) = \frac{\mathrm{e}^T - \mathrm{e}^{-T}}{\mathrm{e}^T + \mathrm{e}^{-T}}x_0$$

$$J_{[0,T]}(x_0, \mathtt{u}_*) = \mathtt{p}_*(0)x_0 = \frac{\mathrm{e}^T - \mathrm{e}^{-T}}{\mathrm{e}^T + \mathrm{e}^{-T}}x_0^2, \qquad \mathtt{u}_*(t) = -R^{-1}B^{\mathrm{T}}\mathtt{p}_*(t) = -\mathtt{p}_*(t)$$

$$\begin{bmatrix} \mathtt{x}_*(t) \\ \mathtt{p}_*(t) \end{bmatrix} = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} \begin{bmatrix} x_0 \\ p(0) \end{bmatrix}$$

$$\begin{aligned} 0 &= S\mathtt{x}_*(T) - \mathtt{p}_*(T) \\ &= \begin{bmatrix} S & -I \end{bmatrix} \begin{bmatrix} \mathtt{x}_*(T) \\ \mathtt{p}_*(T) \end{bmatrix} \\ &= \begin{bmatrix} S & -I \end{bmatrix} \begin{bmatrix} \Sigma_{11}(T) & \Sigma_{12}(T) \\ \Sigma_{21}(T) & \Sigma_{22}(T) \end{bmatrix} \begin{bmatrix} x_0 \\ \mathtt{p}_*(0) \end{bmatrix} \\ &= \big(S\Sigma_{11}(T) - \Sigma_{21}(T)\big)x_0 + \big(S\Sigma_{12}(T) - \Sigma_{22}(T)\big)\mathtt{p}_*(0). \end{aligned}$$

$$\iff \mathtt{p}_*(0) = Mx_0, \quad M = -\big(S\Sigma_{12}(T) - \Sigma_{22}(T)\big)^{-1}\big(S\Sigma_{11}(T) - \Sigma_{21}(T)\big).$$

Theorem (Existence and uniqueness of solution)

Suppose $Q \geq 0, S \geq 0, R > 0$. Then the Hamiltonian equations with mixed boundary conditions have a unique solution $(\mathtt{x}_*, \mathtt{p}_*)$ on $[0, T]$. Concretely, the above matrix M is well defined, and

$$\begin{bmatrix} \mathtt{x}_*(t) \\ \mathtt{p}_*(t) \end{bmatrix} = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} \begin{bmatrix} I \\ M \end{bmatrix} x_0 \qquad \forall t \in [0, T]$$

- The LQ problem satisfies the convexity assumptions of Thm. 2.8.1.
  Hence, $\mathtt{u}_* := -R^{-1}B^{\mathrm{T}}\mathtt{p}_*(t)$ is the optimal cost, and $\mathtt{p}_*^{\mathrm{T}}(0)x_0$ is the optimal cost.
- Exercise 4.3: if $R = I$ then

$$J_{[0,T]}(x_0,\mathtt{u}) - J_{[0,T]}(x_0,\mathtt{u}_*) = \int_0^T \mathtt{z}^{\mathrm{T}}(t)Q\mathtt{z}(t) + \mathtt{v}^{\mathrm{T}}(t)\mathtt{v}(t)\,\mathrm{d}t \geq 0.$$

  Here, $\dot{\mathtt{z}}(t) = A\mathtt{z}(t) + B\mathtt{v}(t),\ \mathtt{z}(0) = 0$ and $\mathtt{v} = \mathtt{u} - \mathtt{u}_*$.
- Optimality also follows later from HJB

Example

$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \qquad J_{[0,T]}(x_0, \mathtt{u}) = \mathtt{x}^2(T) + \int_0^T R\mathtt{u}^2(t)\, \mathrm{d}t$$

$$\mathcal{H} = \begin{bmatrix} 0 & -1/R \\ 0 & 0 \end{bmatrix} \qquad \mathrm{e}^{\mathcal{H}t} = \begin{bmatrix} 1 & -t/R \\ 0 & 1 \end{bmatrix}$$

$$0 = S\mathtt{x}_*(T) - \mathtt{p}_*(T) = \begin{bmatrix} S & -1 \end{bmatrix} \mathrm{e}^{\mathcal{H}T} \begin{bmatrix} x_0 \\ \mathtt{p}_*(0) \end{bmatrix} = x_0 - (T/R+1)\mathtt{p}_*(0)$$

$$\mathtt{p}_*(0) = \frac{x_0}{T/R+1}$$

$$J_{[0,T]}(x_0, \mathtt{u}_*) = \mathtt{p}_*(0)x_0 = \frac{x_0^2}{T/R+1}$$

$$\mathtt{u}_*(t) = -\frac{1}{R}\mathtt{p}_*(t) = -\frac{1}{R}\mathtt{p}_*(0) = -\frac{x_0}{T+R}$$

Lecture 11

- Exam formulas!
- Recap LQ & HJB
- § 4.3 Finite horizon LQ with dynamic programming
- § 4.4 Riccati Differential Equations (RDE's)

Exam formulas

Euler-Lagrange eqn: $$\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) F(t, \mathtt{x}(t), \dot{\mathtt{x}}(t)) = 0$$

Beltrami identity: $$F - \dot{\mathtt{x}}^{\mathrm{T}}\left(\frac{\partial F}{\partial \dot{x}}\right) = C$$

Standard Hamiltonian eqn: $$\dot{\mathtt{x}}(t) = \frac{\partial H(\mathtt{x}(t), \mathtt{p}(t), \mathtt{u}(t))}{\partial p}, \qquad \mathtt{x}(0) = x_0$$
$$\dot{\mathtt{p}}(t) = -\frac{\partial H(\mathtt{x}(t), \mathtt{p}(t), \mathtt{u}(t))}{\partial x}, \quad \mathtt{p}(T) = \frac{\partial K(\mathtt{x}(T))}{\partial x}$$

HJB eqn: $$\frac{\partial V(x,t)}{\partial t} + \min_{u \in \mathbb{U}}\left[\frac{\partial V(x,t)}{\partial x^{\mathrm{T}}} f(x,u) + L(x,u)\right] = 0, \qquad V(x,T) = K(x)$$

LQ Riccati DE: $$\dot{P}(t) = -P(t)A - A^{\mathrm{T}}P(t) + P(t)BR^{-1}B^{\mathrm{T}}P(t) - Q, \qquad P(T) = S$$

Definition (LQ optimal control)

$$\dot{\mathtt{x}}(t) = A\mathtt{x}(t) + B\mathtt{u}(t), \qquad \mathtt{x}(0) = x_0 \in \mathbb{R}^n, \qquad \mathbb{U} = \mathbb{R}^m$$

$$J_{[\tau,T]}(x,\mathtt{u}) = \mathtt{x}^{\mathrm{T}}(T)S\mathtt{x}(T) + \int_{\tau}^{T} \mathtt{x}^{\mathrm{T}}(t)Q\mathtt{x}(t) + \mathtt{u}^{\mathrm{T}}(t)R\mathtt{u}(t)\,\mathrm{d}t, \qquad \mathtt{x}(\tau) = x$$

$$S = S^{\mathrm{T}} \geq 0$$
$$Q = Q^{\mathrm{T}} \geq 0$$
$$R = R^{\mathrm{T}} > 0$$

And recall the HJB equation:

$$\frac{\partial V(x,t)}{\partial t} + \min_{u\in\mathbb{U}}\left(\frac{\partial V(x,t)}{\partial x^{\mathrm{T}}} f(x,u) + L(x,u)\right) = 0, \qquad V(x,T) = K(x)$$

4.3 Finite horizon LQ: Dynamic Programming

$$\dot{\mathtt{x}}(t) = A\mathtt{x}(t) + B\mathtt{u}(t), \qquad \mathtt{x}(0) = x_0, \qquad \mathbb{U} = \mathbb{R}^m$$

$$J_{[0,T]}(x_0, \mathtt{u}) = \mathtt{x}^{\mathrm{T}}(T)S\mathtt{x}(T) + \int_0^T \mathtt{x}^{\mathrm{T}}(t)Q\mathtt{x}(t) + \mathtt{u}^{\mathrm{T}}(t)R\mathtt{u}(t)\,\mathrm{d}t$$

$$0 = \frac{\partial V(x,t)}{\partial t} + \min_{u\in\mathbb{R}^m}\left(\frac{\partial V(x,t)}{\partial x^{\mathrm{T}}}(Ax+Bu) + x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru\right), \qquad V(x,T) = x^{\mathrm{T}}Sx$$

$$0 = \frac{\partial V(x,t)}{\partial t} + \min_{u \in \mathbb{R}^m} \left(\frac{\partial V(x,t)}{\partial x^{\mathrm{T}}} (Ax + Bu) + x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u \right), \qquad V(x,T) = x^{\mathrm{T}} S x$$

motivated by MP let's try quadratic: $V(x,t) = x^{\mathrm{T}} P(t) x, \quad P = P^{\mathrm{T}}$

$$0 = x^{\mathrm{T}} \dot{P}(t) x + \min_{u \in \mathbb{R}^m} \left(2x^{\mathrm{T}} P(t) (Ax + Bu) + x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u \right), \qquad x^{\mathrm{T}} P(T) x = x^{\mathrm{T}} S x$$

$$u = -R^{-1} B^{\mathrm{T}} P(t) x$$

$$0 = x^{\mathrm{T}} \left(\dot{P}(t) + 2P(t)A + Q - P(t) B R^{-1} B^{\mathrm{T}} P(t) \right) x, \qquad x^{\mathrm{T}} P(T) x = x^{\mathrm{T}} S x$$

$$0 = x^{\mathrm{T}} \left(\dot{P}(t) + P(t)A + A^{\mathrm{T}} P(t) + Q - P(t) B R^{-1} B^{\mathrm{T}} P(t) \right) x, \qquad x^{\mathrm{T}} P(T) x = x^{\mathrm{T}} S x$$

4.4 Riccati Differential Equations (RDE's)

$$x^{\mathrm{T}}\left(\dot{P}(t)+P(t)A+A^{\mathrm{T}}P(t)+Q-P(t)BR^{-1}B^{\mathrm{T}}P(t)\right)x=0$$

$$x^{\mathrm{T}}P(T)x=x^{\mathrm{T}}Sx$$

Definition (Riccati Differential Equations (RDE))

The differential equation

$$\dot{P}(t)=-P(t)A-A^{\mathrm{T}}P(t)+P(t)BR^{-1}B^{\mathrm{T}}P(t)-Q, \qquad P(T)=S, \qquad t\in[0,T]$$

is known as a Riccati Differential Equation (RDE).

A quadratic DE with final condition. Notice that $P(t)\in\mathbb{R}^{n\times n}$.

Lemma (Solution of the finite horizon LQ problem)

Let Q, S, R be symmetric and suppose $R > 0$. If the RDE has a continuously differentiable solution $P : [0, T] \to \mathbb{R}^{n\times n}$, then the LQ problem has a solution for every $x_0 \in \mathbb{R}^n$. In particular, the linear state feedback

$$\mathtt{u}_*(t) = -R^{-1}B^{\mathrm{T}}P(t)\mathtt{x}(t)$$

is the optimal input, and the optimal cost is quadratic in the state:

$$J_{[0,T]}(x_0, \mathtt{u}_*) = x_0^{\mathrm{T}}P(0)x_0,$$

and also its value function is quadratic: $V(x, t) := x^{\mathrm{T}}P(t)x$.

Proof.

$\mathtt{u}_*$ is "admissible" because the closed-loop, $\dot{\mathtt{x}}(t) = (A - BR^{-1}B^{\mathrm{T}}P(t))\mathtt{x}(t)$, is linear. □

Lemma assumes symmetry of S, Q but not positive semi-definiteness.
Also, existence of RDE solution is assumed, not guaranteed.

Example (Example ... continued)

$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \qquad J(x_0, \mathtt{u}) = \mathtt{x}^2(T) + \int_0^T R\mathtt{u}^2(t)\,\mathrm{d}t \quad (R > 0)$$

$$\dot{P}(t) = P^2(t)/R, \qquad P(T) = 1$$

$$P(t) = \frac{R}{R+T-t} = \frac{1}{1+(T-t)/R} \quad \text{is well defined for } t \in [0, T]$$

$$J(x_0, \mathtt{u}_*) = x_0^2 P(0) = \frac{x_0^2}{1+T/R}$$

$$\mathtt{u}_*(t) = -R^{-1}B^{\mathrm{T}}P(t)\mathtt{x}(t) = -\frac{P(t)\mathtt{x}(t)}{R} = -\frac{\mathtt{x}(t)}{R+T-t}$$

$P(t)$ exists for $t \in [0, T]$, so above state feedback is optimal.

Is existence a coincidence? Not really:

So far existence of $P(t)$ was assumed. But for standard LQ it is guaranteed:

Theorem (Existence of solution of RDE's)

If $S = S^{\mathrm{T}} \geq 0, Q = Q^{\mathrm{T}} \geq 0, R = R^{\mathrm{T}} > 0$ then the RDE has a unique continuously differentiable solution $P(t)$ for $t \in [0, T]$, and $P(t)$ is symmetric and positive semi-definite at every $t \in [0, T]$.

Consequently, the LQ problem has a unique solution

$$\mathtt{u}_*(t) = -R^{-1} B^{\mathrm{T}} P(t) \mathtt{x}(t)$$

with value function

$$V(x, t) = x^{\mathrm{T}} P(t) x$$

and optimal cost

$$J_{[0,T]}(x_0, \mathtt{u}_*) = x_0^{\mathrm{T}} P(0) x_0.$$

Example (Nonstandard LQ: negative Q)

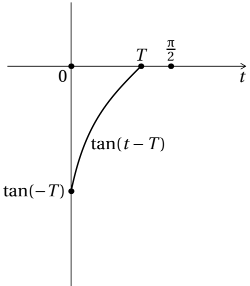
$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \quad J_{[0,T]}(x_0, \mathtt{u}) = \int_0^T -x^2(t) + u^2(t)\,\mathrm{d}t \qquad \Longrightarrow \; A = 0, B = 1, Q = -1 < 0, R = 1, S = 0$$

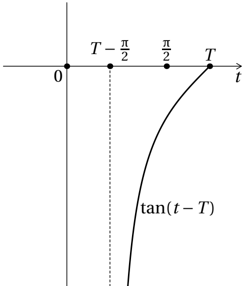
$$\dot{P}(t) = -P(t)A - A^{\mathrm{T}}P(t) + P(t)BR^{-1}B^{\mathrm{T}}P(t) - Q \qquad P(T) = S$$

$$\dot{P}(t) = P^2(t) + 1, \qquad P(T) = 0$$

$$P(t) = \tan(t - T) \qquad V(x,t) = x^2 \tan(t - T)$$

$$\mathtt{u}_*(t) = -R^{-1}B^{\mathrm{T}}P(t)\mathtt{x}(t) = -\tan(t - T)\mathtt{x}(t)$$





Connection between Hamiltonians and RDE's

$$2\mathtt{p}_*(t) = \frac{\partial V(\mathtt{x}_*(t), t)}{\partial x}$$

$$V(x,t) = x^{\mathrm{T}} P(t) x$$

$$\Longrightarrow \qquad \mathtt{p}_*(t) = P(t)\mathtt{x}_*(t)$$

recall:
$$\begin{bmatrix} \mathtt{x}_*(t) \\ \mathtt{p}_*(t) \end{bmatrix} = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} \begin{bmatrix} I \\ M \end{bmatrix} x_0$$

$$\begin{aligned} \mathtt{p}_*(t) &= (\Sigma_{21}(t) + \Sigma_{22}(t)M)x_0 \\ &= (\Sigma_{21}(t) + \Sigma_{22}(t)M)(\Sigma_{11}(t) + \Sigma_{12}(t)M)^{-1}\ \mathtt{x}_*(t) \end{aligned}$$

$$P(t) = \big(\Sigma_{21}(t) + \Sigma_{22}(t)M\big)\big(\Sigma_{11}(t) + \Sigma_{12}(t)M\big)^{-1}$$

Lecture 12

- Recap Hamiltonians & RDE
- § 4.5 Infinite horizon LQ: the Algebraic Riccati Equation (ARE)
- LQ with stability
- Connection between Hamiltonians ans ARE's
- § 4.6 Application (classic LQ)

Definition

LQ optimal control problem:

$$\dot{\mathrm{x}}(t) = A\mathrm{x}(t) + B\mathrm{u}(t), \qquad \mathrm{x}(0) = x_0 \in \mathbb{R}^n, \qquad \mathbb{U} = \mathbb{R}^m$$

$$J_{[0,T]}(x_0, \mathrm{u}) = \mathrm{x}^{\mathrm{T}}(T) S \mathrm{x}(T) + \int_0^T \mathrm{x}^{\mathrm{T}}(t) Q \mathrm{x}(t) + \mathrm{u}^{\mathrm{T}}(t) R \mathrm{u}(t)\, \mathrm{d}t$$

$$S = S^{\mathrm{T}} \geq 0, \qquad Q = Q^{\mathrm{T}} \geq 0, \qquad R = R^{\mathrm{T}} > 0$$

Theorem (Minimum Principle & Hamiltonian matrix)

Let $S = S^{\mathrm{T}} \geq 0, Q = Q^{\mathrm{T}} \geq 0, R = R^{\mathrm{T}} > 0$. The mixed boundary value problem

$$\begin{bmatrix} \dot{\mathrm{x}}(t) \\ \dot{\mathrm{p}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix}}_{\mathscr{H}} \begin{bmatrix} \mathrm{x}(t) \\ \mathrm{p}(t) \end{bmatrix}, \qquad \begin{bmatrix} \mathrm{x}(0) \\ \mathrm{p}(T) \end{bmatrix} = \begin{bmatrix} x_0 \\ S\mathrm{x}(T) \end{bmatrix}$$

has unique solution $(\mathrm{x}_*, \mathrm{p}_*)$, and $\mathrm{u}_*(t) = -R^{-1}B^{\mathrm{T}}\mathrm{p}_*(t)$ is an OC, and the optimal cost is $\mathrm{p}^{\mathrm{T}}(0)x_0$.

Theorem (Existence of solution of RDE's)

If $S = S^{\mathrm{T}} \geq 0, Q = Q^{\mathrm{T}} \geq 0, R = R^{\mathrm{T}} > 0$ then the

$$\dot{P}(t) = -P(t)A - A^{\mathrm{T}}P(t) + P(t)BR^{-1}B^{\mathrm{T}}P(t) - Q, \qquad P(T) = S$$

has a unique continuously differentiable solution $P(t)$ on $[0, T]$, and $P(t)$ is symmetric and positive semi-definite at every $t \in [0, T]$.

Consequently, the LQ problem has as solution

$$\mathtt{u}_*(t) = -R^{-1}B^{\mathrm{T}}P(t)\mathtt{x}(t)$$

with value function

$$V(x, t) = x^{\mathrm{T}}P(t)x$$

and optimal cost

$$J_{[0,T]}(x_0, \mathtt{u}_*) = x_0^{\mathrm{T}}P(0)x_0.$$

4.5 Infinite Horizon LQ and Algebraic Riccati Equations (ARE's)

Classic infinite-horizon version:

Definition (Infinite-horizon LQ)

Given $\dot{\mathtt{x}}(t) = A\mathtt{x}(t) + B\mathtt{u}(t)$ and $\mathtt{x}(0) = x_0$ minimize

$$J_{[0,\infty)}(x_0, \mathtt{u}) := \int_0^{\infty} \mathtt{x}^{\mathrm{T}}(t) Q \mathtt{x}(t) + \mathtt{u}^{\mathrm{T}}(t) R \mathtt{u}(t) \, \mathrm{d}t$$

over all $\mathtt{u} : [0, \infty) \to \mathbb{R}^m$.

Example

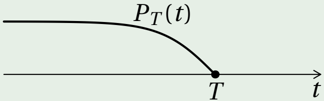
$$\dot{\mathrm{x}}(t) = \mathrm{u}(t), \qquad \mathrm{x}(0) = x_0,$$

$$J_{[0,T]}(x_0, \mathrm{u}) = \int_0^T \mathrm{x}^2(t) + \mathrm{u}^2(t)\,\mathrm{d}t$$

$$\dot{P}(t) = -P(t)A - A^{\mathrm{T}}P(t) + P(t)BR^{-1}B^{\mathrm{T}}P(t) - Q, \qquad P(T) = S$$

$$\dot{P}_T(t) = P_T^2(t) - 1, \qquad P_T(T) = 0.$$

$$P_T(t) = \tanh(T-t) = \frac{\mathrm{e}^{T-t} - \mathrm{e}^{-(T-t)}}{\mathrm{e}^{T-t} + \mathrm{e}^{-(T-t)}}.$$



Suggests that as $T \to \infty$:

$$\boxed{0 = A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q}$$

Theorem (Infinite horizon LQ)

Consider $\dot{\mathtt{x}}(t) = A\mathtt{x}(t) + B\mathtt{u}(t), \mathtt{x}(0) = x_0$, and suppose $Q \geq 0, R > 0$, and that for every x_0 an input exists that renders the cost $J_{[0,\infty)}(x_0, \mathtt{u})$ finite.

Then the solution $P_T(t)$ of RDE converges to a matrix independent of t as $T \to \infty$. That is, a constant matrix P exists such that

$$\lim_{T\to\infty} P_T(t) = P \qquad \forall t > 0.$$

This P is positive semi-definite and satisfies the Algebraic Riccati Equation (ARE):

$$0 = A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q.$$

Example

$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \qquad \mathtt{x}(0) = x_0$$

$$J_{[0,\infty)}(x_0, \mathtt{u}) = \int_0^\infty \mathtt{x}^2(t) + \mathtt{u}^2(t)\,\mathrm{d}t.$$

notice that $\mathtt{u}(t) := -\mathtt{x}(t)$ ensures cost is finite, so theorem applies:

$$0 = A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q$$

$$0 = -P^2 + 1$$

$$P = \pm 1, \qquad P = +1$$

$$\mathtt{u}_*(t) = -R^{-1}B^{\mathrm{T}}P\mathtt{x}(t) = -\mathtt{x}(t)$$

$$J(x_0, \mathtt{u}_*) = Px_0^2 = x_0^2.$$

Nice: $\mathtt{u}_*(t) = -\mathtt{x}(t)$ is a stabilizing input!

Officially we do not yet know that it solves the infinite horizon LQ

"Modern" version includes stability:

Definition (Infinite horizon LQ problem with stability)

Suppose $Q \geq 0, R > 0$ and consider $\dot{\mathtt{x}}(t) = A\mathtt{x}(t) + B\mathtt{u}(t), \mathtt{x}(0) = x_0$. The (infinite horizon) LQ problem with stability is to minimize

$$J_{[0,\infty)}(x_0, \mathtt{u}) = \int_0^\infty \mathtt{x}^{\mathrm{T}}(t) Q \mathtt{x}(t) + \mathtt{u}^{\mathrm{T}}(t) R \mathtt{u}(t) \, \mathrm{d}t$$

over all stabilizing inputs $\mathtt{u}$, i.e., inputs that achieve $\lim_{t\to\infty} \mathtt{x}(t) = 0$

Example

$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \quad \mathtt{x}(0) = x_0,$$

$$J_{[0,\infty)}(x_0, u) = \int_0^\infty \mathtt{x}^2(t) + \mathtt{u}^2(t)\,\mathrm{d}t$$

$$\mathtt{x}^2 + \mathtt{u}^2 = -2\mathtt{x}\mathtt{u} \quad + (\mathtt{x}+\mathtt{u})^2$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}(-\mathtt{x}^2) + (\mathtt{x}+\mathtt{u})^2$$

$$J_{[0,\infty)}(x_0, \mathtt{u}) = x_0^2 + \int_0^\infty (\mathtt{x}(t)+\mathtt{u}(t))^2\,\mathrm{d}t$$

$$\mathtt{u}_* = -\mathtt{x}$$

yes: it stabilizes!

$$J_{[0,\infty)}(x_0, \mathtt{u}) = x_0^2$$

Earlier we conjectured that $\mathtt{u} := -\mathtt{x}$ is optimal. Now we know it is optimal or, at least, optimal with respect to all stabilizing inputs.

Theorem (Solution of the LQ problem with stability)

There is at most one matrix $P \in \mathbb{R}^{n\times n}$ that satisfies the ARE

$$A^{\mathrm{T}}P + PA + Q - PBR^{-1}B^{\mathrm{T}}P = 0$$

with the property that $A - BR^{-1}B^{\mathrm{T}}P$ is asymptotically stable.
In that case P is symmetric, and the linear state feedback

$$\mathtt{u}_*(t) = -R^{-1}B^{\mathrm{T}}P\mathtt{x}(t)$$

is the solution of the LQ problem with stability, and $J(\mathtt{u}_*) = x_0^{\mathrm{T}}Px_0$.

Then P is said to be a stabilizing solution of the ARE.

For “stabilizable & detectable” systems there is a perfect result:

Lemma (Three ways to solve the LQ problem-with-stability)

Consider the LQ problem with stability (in particular $Q \geq 0, R > 0$).

If (A, B) is stabilizable and (Q, A) detectable then there is a unique stabilizing solution P of the ARE. Consequently, $\mathtt{u}_*(t) := -R^{-1}B^{\mathrm{T}}P\mathtt{x}(t)$ solves the LQ problem with stability.

Moreover, this P can be determined in the following 3 equivalent ways:

1. P equals $\lim_{T\to\infty} P_T(t)$ where $P_T(t)$ is the solution of RDE,
2. P is the unique symmetric, positive semi-definite solution of the ARE,
3. P is the unique stabilizing solution of the ARE.

Example

Consider again the integrator system and cost

$$\dot{\mathtt{x}}(t) = \mathtt{u}(t), \qquad \mathtt{x}(0) = x_0, \qquad J_{[0,\infty)}(x_0, \mathtt{u}) = \int_0^\infty \mathtt{x}^2(t) + \mathtt{u}^2(t)\,\mathrm{d}t$$

It is “stabilizable & detectable”, so these three are the same:

$$\text{RDE:} \quad \lim_{T\to\infty} P_T(t) = 1 \qquad \Longrightarrow \quad P = 1$$

$$\text{ARE, positive semi definite:} \quad -P^2 + 1 = 0, \quad P \geq 0 \qquad \Longrightarrow \quad P = 1$$

$$\text{ARE, stabilizing solution:} \quad -P^2 + 1 = 0, \quad A - BR^{-1}B^{\mathrm{T}}P = -P < 0 \qquad \Longrightarrow \quad P = 1$$

Connection between Hamiltonians and ARE's

$$PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q = 0$$

$$-Q - A^{\mathrm{T}}P = P(A - BR^{-1}B^{\mathrm{T}}P)$$

$$\underbrace{\begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix}}_{\mathscr{H}} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} (A - BR^{-1}B^{\mathrm{T}}P)$$

$$\begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} (A - BR^{-1}B^{\mathrm{T}}P)$$

Theorem

Assume $Q \geq 0, R > 0$. If (A, B) is stabilizable and (Q, A) detectable, then

1. $\mathscr{H}$ has no imaginary eigenvalues,
2. matrices $V \in \mathbb{R}^{(2n)\times n}$ of rank n exist such that

$$\begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Lambda \qquad \text{for some asymptotically stable } \Lambda \in \mathbb{R}^{n\times n},$$

3. for any such $V \in \mathbb{R}^{(2n)\times n}$, if we partition V as $V = \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right]$ with $V_1, V_2 \in \mathbb{R}^{n\times n}$, then V_1 is invertible,
4. the ARE has a unique stabilizing solution P. In fact $P := V_2 V_1^{-1}$ is the unique answer, and it is symmetric.

Example

$$\begin{bmatrix} \dot{\mathtt{x}}_1(t) \\ \dot{\mathtt{x}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathtt{x}_1(t) \\ \mathtt{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathtt{u}(t)$$

$$J_{[0,\infty)}(x_0, \mathtt{u}) = \int_0^\infty \mathtt{x}_1^2(t) + \mathtt{x}_2^2(t) + \mathtt{u}^2(t)\,\mathrm{d}t$$

$$\mathscr{H} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

Example (... continued)

$$\mathscr{H} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

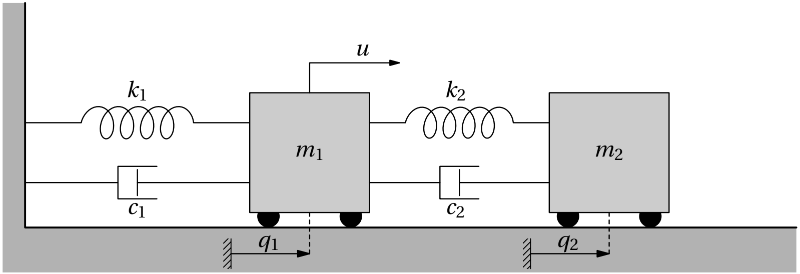
$$\lambda_{1,2} = -\tfrac{1}{2}\sqrt{3} \pm \tfrac{1}{2}\mathrm{i}, \qquad \lambda_{3,4} = +\tfrac{1}{2}\sqrt{3} \pm \tfrac{1}{2}\mathrm{i}$$

$$V = \begin{bmatrix} -\lambda_1 & -\lambda_2 \\ -\lambda_1^2 & -\lambda_2^2 \\ 1 & 1 \\ \lambda_1^3 & \lambda_2^3 \end{bmatrix}$$

$$P = V_2 V_1^{-1} = \begin{bmatrix} 1 & 1 \\ \lambda_1^3 & \lambda_2^3 \end{bmatrix} \begin{bmatrix} -\lambda_1 & -\lambda_2 \\ -\lambda_1^2 & -\lambda_2^2 \end{bmatrix}^{-1} = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}$$

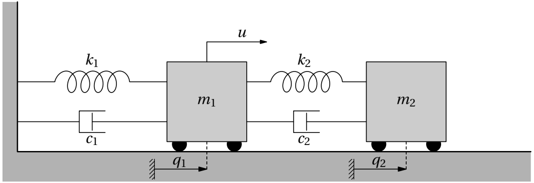
$$\mathtt{u}_*(t) = -R^{-1}B^{\mathrm{T}}P\mathtt{x}(t) = -\mathtt{x}_1(t) - \sqrt{3}\mathtt{x}_2(t)$$

4.6 Application: LQ Controller Design for Connected Cars



Task: control the second car with a force u that acts on the first car

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\mathtt{q}}_1(t) \\ \ddot{\mathtt{q}}_2(t) \end{bmatrix} + \begin{bmatrix} r_1 + r_2 & -r_2 \\ -r_2 & r_2 \end{bmatrix} \begin{bmatrix} \dot{\mathtt{q}}_1(t) \\ \dot{\mathtt{q}}_2(t) \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} \mathtt{q}_1(t) \\ \mathtt{q}_2(t) \end{bmatrix} = \begin{bmatrix} \mathtt{u}(t) \\ 0 \end{bmatrix}$$



For $m_1 = m_2 = 1, k_1 = k_2 = 1$ and $r_1 = r_2 = 0.1$, and $\mathtt{x} := (\mathtt{q}_1, \mathtt{q}_2, \dot{\mathtt{q}}_1, \dot{\mathtt{q}}_2)$:

$$\dot{\mathtt{x}}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -0.2 & 0.1 \\ 1 & -1 & 0.1 & -0.1 \end{bmatrix} \mathtt{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mathtt{u}(t).$$

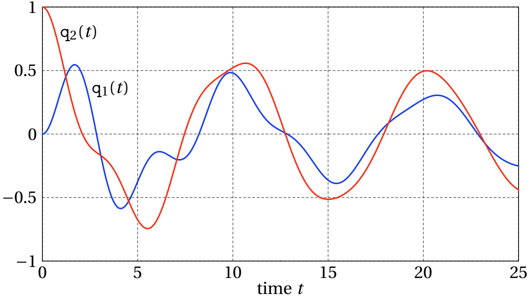
$$\int_0^\infty \mathtt{q}_2^2(t) + R\mathtt{u}^2(t)\,\mathrm{d}t.$$

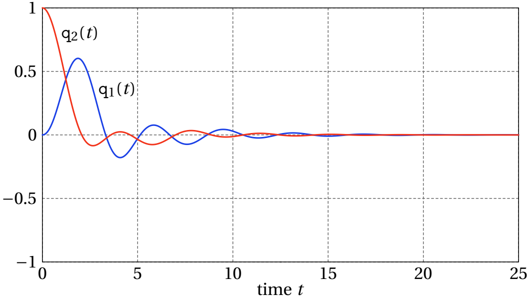
$$J = \int_0^\infty \mathtt{q}_2^2(t) + R\mathtt{u}^2(t)\,\mathrm{d}t, \qquad R = 0.2:$$

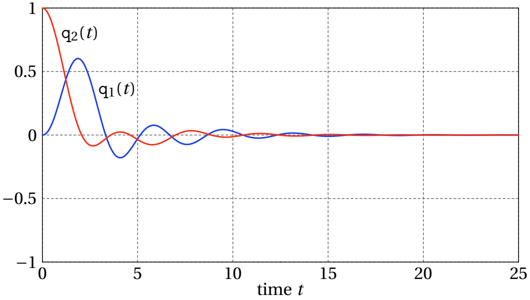
$$P = \begin{bmatrix} 0.4126 & 0.2286 & 0.2126 & 0.5381 \\ 0.2286 & 0.9375 & 0.0773 & 0.5624 \\ 0.2126 & 0.0773 & 0.2830 & 0.4430 \\ 0.5381 & 0.5624 & 0.4430 & 1.1607 \end{bmatrix}$$

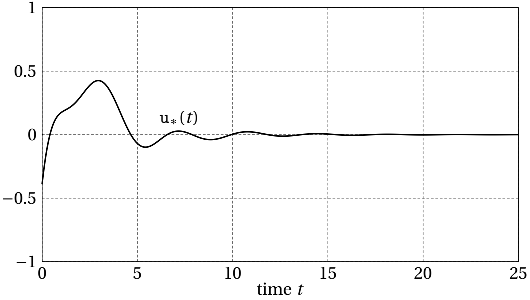
and then the optimal state feedback control is

$$\begin{aligned} \mathtt{u}_*(t) &= -R^{-1}B^{\mathrm{T}}P\mathtt{x}(t) \\ &= -\begin{bmatrix} 1.0628 & 0.3867 & 1.4151 & 2.2150 \end{bmatrix}\mathtt{x}(t) \end{aligned}$$


1
0.5
0
−0.5
−1
0
5
10
15
20
25
time t
$q_2(t)$
$q_1(t)$


1
0.5
0
−0.5
−1
0
5
10
15
20
25
time t
$q_2(t)$
$q_1(t)$


1
0.5
0
−0.5
−1
0
5
10
15
20
25
time t
$q_2(t)$
$q_1(t)$


1
0.5
0
−0.5
−1
0
5
10
15
20
25
time t
$u_*(t)$

You can, of course, compute P in MATLAB:

```
A=0;                          % whatever
B=1;                          % whatever
Q=1;                          % whatever (well)
R=1;                          % whatever (well, >0)

[F,P,CLeig]=lqr(A,B,Q,R);     % F = R^{-1} B^T P. So u_*(t) = -F x(t)
CLeig                         % the eigenvalues of A-BF

J=@(x0) x0'*P*x0;             % the optimal cost (infinite horizon)
```

The End