Optimal signal reconstruction

Quantification and graphical representation of optimal signal reconstructions

Master Thesis
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by

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Abstract

This report will use, in order to measure the performance of a (mathematical) system, the L^2 norm for systems. For a BIBO-stable (Bounded Input Bounded Output) and Linear Continuous Time Invariant (LCTI) system usually a transfer function is defined. Using this transfer function it is possible to calculate the L^2 norm of the system.

In the process of sampling and reconstruction of a signal two systems are used: a sampler and a hold. Most of the time these systems are not LCTI but only linear and h-shift invariant or equivalently Linear Discrete Time Invariant (LDTI). For this class of systems a way of calculating the L^2 system norm is presented. This calculation is based on the Frequency Power Response (FPR) of a system which is introduced in this report as well. This FPR is for an LDTI system what the frequency response, e.g. $|G(\mathrm{i}\omega)|^2$ is for an LCTI system.

It has already been shown that the optimal combination of sampler and hold for a given sampling period h is always LCTI. This means that the L^2 norm of the system can be calculated in a classical way. This report shows how to calculate the L^2 norm of the optimal combination of sampler and hold. Also a graphical interpretation is given for this optimal combination.

Because of the FPR, the L^2 norm can now be calculated not only for LCTI systems but for LDTI systems as well. And it is shown how to determine the optimal hold for a given sampler and sampling period h. Additionally the L^2 norm of the system can be calculated and graphically represented: how good (or how bad) is a certain hold in combination with the given sampler.

Preface

This report is part of my graduation project for the Master of Science program in Systems & Control at the University of Twente. The set up for the project came from my supervisor Dr. Ir. G. Meinsma who has done a lot of research in this area. I have had a great time working with him and I would like to thank him for his good advice, the way he tutored me and all the work and effort he put in reading parts of my report and answering my countless questions.
Furthermore I would like to thank all my colleagues with whom I attended lectures and spend numerous hours of studying and playing cards.
Last but certainly not least I would like to thank my friends and family for supporting me all these years.

Almar Snippe
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List of Symbols

$F(s)$	transfer function of the combination $\mathcal{HS}$, 4
N_k	k^{th} Nyquist band, 11
$\bar{u}[j]$	sampled input, 2
$\breve{f}[k](\tau)$	lifted representation of a signal f, 5
$\delta(t)$	Dirac delta function, 2
i	the imaginary unit $\sqrt{-1}$, 5
$\kappa(t,s)$	kernel of the combination $\mathcal{HS}$, 4
λ_i	eigenvalues of a matrix, 9
$\bar{u}_\omega[j]$	sampled harmonic input, 21
$\mathbb{1}_{[a,b]}(t)$	stepfunction, 2
$\mathcal{G}$	signal generator, 4
$\mathcal{H}$	hold, 3
$\mathcal{S}$	sampler, 2
ω_{nyq}	Nyquist frequency, 3
ω	frequency in radians per time unit, 5
ω_k	the k^t aliased frequency, 5
$\phi(t)$	hold function, 3
$\psi(t)$	sampling function, 2
e	error signal, 2
$g(t)$	impulse response, 4
$g(t,s)$	kernel of a general system $\mathcal{G}$, 4
h	sampling period, 2
n_u	dimension of the input u, 6
n_y	dimension of the output y, 6
$u(t)$	input signal, 2
$u_\omega(t)$	harmonic input, 18
$y(t)$	output signal, 3
$y_\omega(t)$	sampled-and-reconstructed harmonic input, 21
LCTI	Linear Continuous Time Invariant, 4
LDTI	Linear Discrete Time Invariant, 2
MIMO	Multi Input Multi Output, 6
SISO	Single Input Single Output, 6

1 Introduction

1.1 Motivation

In signal processing, sampling and reconstruction of signals is an important subject. Sampling is nothing more than the discretization of an analog signal, the device performing this transformation is called a *sampler*. Sampling for example can be done by measuring a signal at fixed moments in time. Reconstruction is exactly the opposite of sampling; it turns a number of samples into an analog signal. The device performing this transformation is called a *hold*. Reconstruction for example can be done by linearly interpolating between two samples (connecting the dots).
A reason of sampling is compressing signals or adjusting signals for storage at for example an CD. When choosing the right combination of sampler and hold, the sampled-and-reconstructed signal will look similar to the original signal. The goal is of course to minimize the difference between these two signals. In that case the sampling and reconstruction process is optimal. In order to measure the performance of a certain sampling and reconstruction process, a norm is assigned to the process. These norms are well defined for most processes, but their calculation is sometimes rather complex.
The goal of this report is to show some ways of calculating a norm of a sampling and reconstruction process and how to choose the sampler and/or hold in order to achieve optimal signal reconstruction.

1.2 History of Sampling

In 2000 Michael Unser wrote an article [6] about the development of sampling starting with Shannon because he, together with Nyquist, can be seen as the godfather of sampling. Shannon published an article in 1949 where he stated that a signal containing no frequency higher than a certain bound, is *completely* determined by giving its samples at a series of points spaced h time units apart. Furthermore he stated that in that case the signal can be reconstructed uniquely and error-free for which he presented a formula (based on the samples). Nowadays this is still referred to as Shannon's Theorem but he himself has not claimed the theorem as his own because he said the idea was already common knowledge in the communication art. At the present Shannon's Theorem is still alive and well. The whole research area was founded by his theorem and through the years many people have devoted their research (and life) to this topic. All kinds of different subjects are researched: varying time vs. constant time between samples, undersampling or reconstruction using weighted samples and splines for example. All of this has started with Shannon's Theorem from 1949.

2 Background Information

2.1 Introduction

The idea behind sampling is to reduce the data quantity of an (analog) signal or to be able to record the signal in a way that allows one to reconstruct the signal afterwards.

Example 2.1.1. In order to record music on a CD, the music is sampled and these samples are recorded on the CD. When playing the CD, the CD-player reconstructs an analog signal based on the samples recorded on the CD. □

The device that turns the analog signal u (in Example 2.1.1 the music) in discrete samples $\bar{u}$ is called a *sampler* and it is denoted by the symbol $\mathcal{S}$. The device that reconstructs an analog signal y (in Example 2.1.1 the sound leaving the speakers) from the samples $\bar{u}$ is called a *hold* and is denoted by the symbol $\mathcal{H}$. An illustration of this set-up is shown in Figure 1. Here the device $\mathcal{G}$ is called the *signal generator*. In the context of Example 2.1.1 this generator can be seen as the instruments producing the music.
Of special interest is the *error signal* e which is the difference between the original signal u and the sampled-and-reconstructed signal y. The smaller this signal e, the more the reconstructed signal looks like to the original signal.

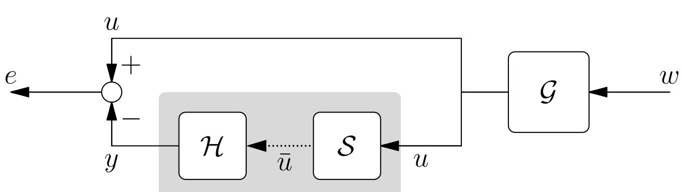


Figure 1: A system with sampler $\mathcal{S}$, hold $\mathcal{H}$, generator $\mathcal{G}$, generator signal w, input signal u, sampled input signal $\bar{u}$, output signal$y$ and error signal e

In general a continuous time signal is represented by an ordinary letter and round brackets, e.g. $u(t)$. Whereas the representation of a sampled (discrete) signal is a barred letter and square brackets, e.g. $\bar{u}[j]$.

2.1.1 Sampler

The reason for sampling and reconstruction of a signal is stated above. This subsection will focus on some properties of samplers. A sampler turns an analog input signal $u(t)$ into a discrete signal $\bar{u}[j]$, see Figure 2.
For a sampler the time between two consecutive samples is called the *sampling period*. This sampling period can be uniform (constant) or it can vary over time. For some applications it is desirable to use a varying sampling time whereas in this report the sampling period will be uniform

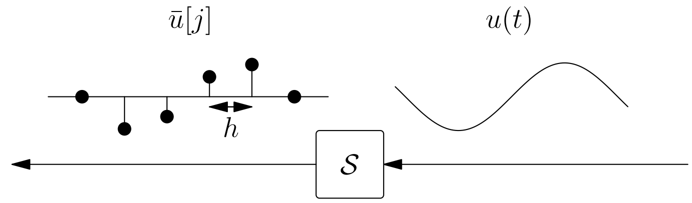


Figure 2: Example of a sampler that takes samples according to the value of the input function at multiples of the sampling period h (the ideal sampler)

and it is denoted by

$$h.$$

Besides a uniform sampling period, the samplers $\mathcal{S}$ in this report are assumed to be *Linear Discrete Time Invariant* (LDTI), see Definition A.1.8. This means that it is linear and that a shift of the analog input by a multiple k of the sampling period h results in a shift of the sampled (discrete) output by k samples

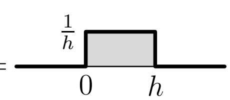
$$\mathcal{S}\sigma^{kh} = \bar{\sigma}^k \mathcal{S}.$$

It can be shown [3] that essentially every LDTI sampler can be written as a convolution

$$\bar{u} = \mathcal{S}u : \quad \bar{u}\,[j] = \int_{-\infty}^{\infty} \psi(jh - s)u(s)\mathrm{d}s, \qquad j \in \mathbb{Z} \quad (2.1)$$

for some function $\psi(t)$. This function $\psi(t)$ is called the *sampling function* and it defines the sampler, see Definition A.2.2. The most conventional sampler is the ideal sampler which can be obtained by taking $\psi(t)$ as the Dirac delta function $\delta(t)$. This results in samples that are just the values of the input at multiples of the sampling period, see Figure 2. Of course the class of samplers (2.1) is much richer than the ideal sampler. For instance, taking

$$\psi(t) = \tfrac{1}{h}\mathbb{1}_{[0,h]}(t) =$$



leads to samples that are averages of the input over one sampling period.
Clearly, sampling throws away an enormous amount of information since, based purely on the samples one cannot determine a unique analog input signal. Therefore, in order to reconstruct the original analog input signal to a certain extend, one must assume certain properties of the input signal. For example, if the (ideal) samples $\bar{u}[j] := u(jh)$ are all zero, the input signal might have been the zero signal $u(t) = 0$. But it might also have been the signal

$$u(t) = \sin\left(\frac{\pi}{h}t\right)$$

which has its zeros in multiples of h. This shows that it is not clear what the input signal has been considering only

the information of the samples. For example, information about the frequency of the signal can be used in order to reconstruct the signal to a certain extend.
The famous result of Shannon (see Theorem A.2.4) shows that if the maximal frequency of the analog input signal $u(t)$ is bounded by the Nyquist frequency ω_{nyq}, defined as

$$\omega_{nyq} := \frac{\pi}{h}$$

(see Definition A.2.1), then the signal can be constructed uniquely and error-free. Note that this is an assumption on the input signal. The ideal sampler

$$\psi(t) = \delta(t).$$

in combination with the correct hold, which will be mentioned shortly in Subsection 2.1.2, will achieve this error-free reconstruction.

2.1.2 Hold

In this subsection some properties of holds will be reviewed. A hold turns a discrete signal $\bar{u}[j]$ into an analog output signal $y(t)$, see Figure 3.

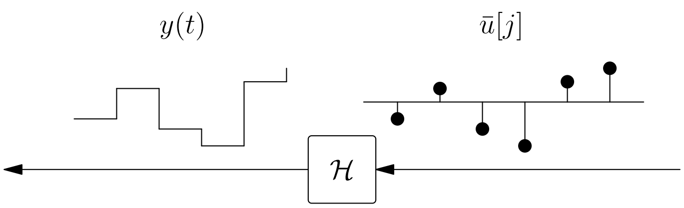


Figure 3: Example of a hold that holds the output constant over one sampling period (zero-order hold)

The holds $\mathcal{H}$ in this report are assumed to be LDTI devices as are the samplers. In this case this means that it is linear and that a shift of the discrete input by k samples results in a shift of the reconstructed (analog) output by a multiple k of the sampling period h

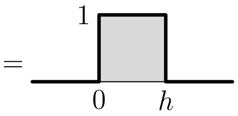
$$\mathcal{H}\bar{\sigma}^k = \sigma^{kh}\mathcal{H}.$$

Similar to the sampler it can be shown [3] that essentially every LDTI hold can be written as a convolution

$$y = \mathcal{H}\bar{u}: \quad y(t) = \sum_{j\in\mathbb{Z}} \phi(t - jh)\bar{u}[j], \quad t \in \mathbb{R} \tag{2.2}$$

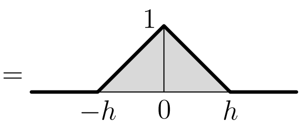
for some function $\phi(t)$. This function $\phi(t)$ is called the *hold function* and it defines the hold, see Definition A.2.3. Just like for the sampler, the class of holds (2.2) is a rich class containing numerous holds. For example the zero-order hold which keeps the analog output signal constant over each sampling period can be obtained by using the hold function

$$\phi(t) = \mathbb{1}_{[0,h]}(t) =$$



A schematic representation of this hold is shown in Figure 3. Another example of a hold is the one that linearly interpolates between consecutive samples. This hold is defined by the hold function

$$\phi(t) = \left(1 - \tfrac{|t|}{h}\right)\mathbb{1}_{[-h,h]}(t) =$$



and is illustrated in Figure 4.

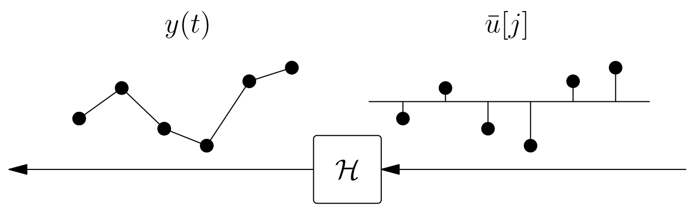


Figure 4: Example of a hold that linearly interpolates between two consecutive samples (first-order hold)

From these two examples one can see that the quality of the analog output can vary a lot using different holds. The zero-order hold only uses the information from one sample, whereas the first-order hold also uses the information from the neighboring samples. As mentioned in Subsection 2.1.1 Shannon's theorem (Theorem A.2.4) also provides the hold function that will reconstruct a signal error-free and uniquely if the maximum frequency of the signal is smaller than ω_{nyq}. Shannon's reconstruction formula reads

$$f(t) = \sum_{k\in\mathbb{Z}} \text{sinc}\,(t - kh)\, f(kh).$$

Hence the hold function defining this hold (referred to as the *sinc-hold*) is the sinc:

$$\text{sinc}\,(t) := \frac{\sin(\pi t)}{\pi t}.$$

So by Shannon's theorem using the ideal sampler and the sinc-hold leads to an error-free reconstruction of the analog input signal if the maximal frequency of the input signal is smaller than ω_{nyq}. And thus as explained previously Shannon's sampler and hold are defined by the functions

$$\psi(t) = \delta(t) \tag{2.3}$$
$$\phi(t) = \text{sinc}\,(t)\,. \tag{2.4}$$

2.1.3 Sampler and Hold combination

The combination of sampler and hold sometimes has the special property of being *Linear Continuous Time Invariant* (LCTI), see Definition A.1.7, and this has some useful consequences.
In general both sampler and hold are LDTI which means that they have certain shift-properties (see Subsections 2.1.1 and 2.1.2). In addition if a device is LCTI, the properties are somewhat extended. Sometimes the combination of sampler and hold $\mathcal{HS}$ is LCTI whereas both sampler and hold individually are not. If the combination $\mathcal{HS}$ is LCTI, then by definition it is linear and a shift of the analog input by any real number τ results in a shift of the analog output by τ

$$(\mathcal{HS})\sigma^\tau = \sigma^\tau(\mathcal{HS}).$$

Since systems that are not LCTI have no classic transfer function (see Subsection 2.3) the individual transfer functions of $\mathcal{H}$ and $\mathcal{S}$ mostly do not exist (only in the special case that both $\mathcal{H}$ and $\mathcal{S}$ are LCTI and stable, see Subsection 2.3). Sometimes the combination $\mathcal{HS}$ does have a transfer function. In order to avoid notational confusion, if $\mathcal{HS}$ has a transfer function, its notation is

$$F(s) := (HS)(s).$$

In general the combination $\mathcal{HS}$ is assumed to be LDTI and its mapping can be constructed by substituting the expression for $\bar{u}[j]$ (2.1) in the expression for $y(t)$ (2.2). This results in a mapping from the input u to the output $y = \mathcal{HS}u$

$$y(t) = \sum_{j\in\mathbb{Z}} \phi(t - jh) \int_{-\infty}^{\infty} \psi(jh - s)u(s)\,\mathrm{d}s.$$

Since the summation is independent of the variable s it can be taken inside the integral leading to a product of the hold- and sampler function

$$y(t) = \int_{-\infty}^{\infty} \sum_{j\in\mathbb{Z}} \phi(t - jh)\psi(jh - s)u(s)\,\mathrm{d}s.$$

Thus $\mathcal{HS}$ is an integral operator of the form

$$y(t) = \int_{-\infty}^{\infty} \kappa(t, s)u(s)\,\mathrm{d}s$$

where $\kappa(t, s)$ is called the *kernel* of $\mathcal{HS}$ and it equals

$$\kappa(t, s) := \sum_{j\in\mathbb{Z}} \phi(t - jh)\psi(jh - s). \tag{2.5}$$

Note that this kernel h-shift invariant, i.e. for all $l \in \mathbb{Z}$

$$\begin{aligned}
\kappa(t + lh, s + lh) &= \sum_{k\in\mathbb{Z}} \phi(t + lh - jh)\psi(jh - s - lh) \\
&= \sum_{k\in\mathbb{Z}} \phi(t + (l - j)h)\psi((j - l)h - s) \\
&= \sum_{k\in\mathbb{Z}} \phi(t - jh)\psi(jh - s) \\
&= \kappa(t, s).
\end{aligned}$$

2.1.4 Signal Generator

The last device from the setting in Figure 1 is the signal generator $\mathcal{G}$. This device is assumed to be LCTI (see Definition A.1.7) and is assumed to have a strictly proper and stable transfer function $G(s)$ (see Subsection 2.3). It can be shown [3] that every LCTI generator $\mathcal{G}$ can be written as a convolution

$$u = \mathcal{G}w : \quad u(t) = \int_{-\infty}^{\infty} g(t - \tau)w(\tau)\,\mathrm{d}\tau, \quad k \in \mathbb{Z}$$

for some function $g(t)$. This function $g(t)$ is called the *impulse response* and it defines the generator.
In this report the symbol $\mathcal{G}$ will also be used as the symbol for a general (not explicitly specified) system. From the context it will be clear when $\mathcal{G}$ refers to a signal generator and when to a general system.
It can be shown [3] that a *general* LDTI system $y = \mathcal{G}u$ is of the form

$$y(t) = \int_{-\infty}^{\infty} g(t, s)u(s)\,\mathrm{d}s$$

where $g(t, s)$ is called the *kernel* and it is h-shift invariant:

$$g(t + h, s + h) = g(t, s).$$

2.2 Signals

This subsection will focus on the representation of signals and some of their properties. Furthermore a way of representing a continuous-time signal as a kind of discrete signal (Lifting) is mentioned.

2.2.1 Laplace transform

In signal processing the most straight forward way to represent a signal f is the *time domain representation*, i.e.

$$f(t), \quad \forall t \in \mathbb{R}.$$

However for some applications it is convenient to know the *Laplace transform* of a signal. The Laplace transform $F(s)$ of a signal $f(t)$ is defined as

$$F(s) := \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t \tag{2.6}$$

for those $s \in \mathbb{C}$ for which this integral exists. This transformation is called the *two-sided* Laplace transform because the signal is integrated from $-\infty$ to ∞. The *one-sided* Laplace transform is defined as well

$$F_1(s) := \int_0^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t$$

for those $s \in \mathbb{C}$ for which this integral exists. Clearly this one-sided Laplace transform throws away a lot of information about the signal if the signal is non-causal (see Definition A.1.9). A signal $f(t)$ is said to be *causal* if

$$f(t) = 0 \quad \forall t < 0.$$

This report will consider non-causal signals as well, therefore the two-sided Laplace transform will be used. So from now on the two-sided Laplace transform (2.6) will be referred to as the *Laplace transform*.

2.2.2 Fourier transform

It can be shown [1] that for an absolutely integrable signal $f(t)$, i.e.

$$\int_{-\infty}^{\infty} |f(t)| \,\mathrm{d}t < \infty$$

the Laplace transform (2.6) exists for all $s \in \mathbb{C}$ with $\mathrm{Re}\,(s) = 0$. This means that the Laplace transform exists on the entire imaginary axis. Write the complex number s as $\sigma + \mathrm{i}\omega$ with σ and ω real numbers and i being the imaginary unit $\sqrt{-1}$. Now the Laplace transform looks like

$$\begin{aligned} F(\sigma + \mathrm{i}\omega) &= \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\sigma t - \mathrm{i}\omega t} \,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\sigma t}\mathrm{e}^{-\mathrm{i}\omega t} \,\mathrm{d}t \end{aligned}$$

for $\sigma = 0$ which leads to

$$F(\mathrm{i}\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t} \,\mathrm{d}t. \tag{2.7}$$

where ω is the frequency in radians per time unit. This transform exists for all $\omega \in \mathbb{R}$ and is a special case of the Laplace transform, it is called the *Fourier transform*. The Fourier transform has an inverse given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\mathrm{i}\omega)\mathrm{e}^{\mathrm{i}\omega t} \,\mathrm{d}\omega. \tag{2.8}$$

In this report, the Fourier transform $F(\mathrm{i}\omega)$ of a signal $f(t)$ is always denoted by a capital. Using Equations (2.7) and (2.8) one can switch between the time domain representation and the frequency domain representation of a signal.
If a signal $f(t)$ is square integrable, i.e. the energy of the signal E_f is finite

$$E_f := \int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t < \infty$$

then the two representations have a special property captured in Parseval's theorem (see Theorem A.3.6). This theorem shows that integrating the signal over all time equals integrating the signal over all frequencies, except for a contstant

$$\int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega. \tag{2.9}$$

This means that the energy of the signal equals the energy of its Fourier transform, except for the constant 2π.

2.2.3 Lifting

Lifting is a technique to represent a continuous time signal $f(t)$ with $t \in \mathbb{R}$ on a smaller, finite time interval $[0, h)$. In fact the signal is cut into an infinite number of intervals, each of length h, see Definition A.4.1. However this means that the *lifted signal* $\breve{f}$ now is a function of two variables, i.e. k and τ, and it is defined as

$$\breve{f}[k](\tau) = f(kh + \tau) \quad k \in \mathbb{Z}, \tau \in [0, h).$$

In other words, with lifting, a signal f on $\mathbb{R}$ is considered as a sequence of functions on the interval $[0, h)$. A positive aspect of this process is that there is no loss of information. The process of lifting is illustrated in Figure 5. The idea

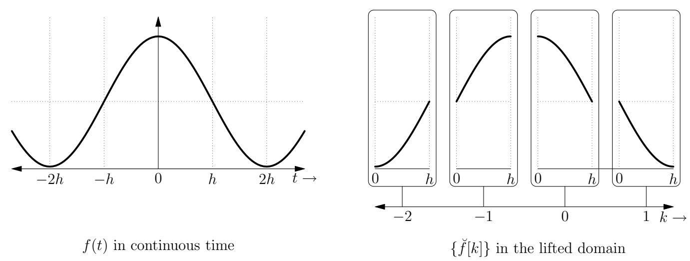


Figure 5: Lifting the analog signal $f(t) = 1 + \cos(\frac{\pi}{2h}t)$

behind this representation is to allow only time shifts that are multiples of h. This implies that if a continuous-time system $y = \mathcal{G}u$ (see Subsection 2.3) is h-periodic, then in lifted representation $\breve{u} = \breve{\mathcal{G}}\breve{y}$ is shift invariant (i.e. a shift in k) [3].
It turns out [3] that the Fourier transform of a lifted signal $\breve{f}$ exists if the Fourier transform of f itself exists, and it is given by

$$\mathfrak{F}\{\breve{f}\} = \breve{f}(\mathrm{e}^{\mathrm{i}\omega h}; \tau) := \sum_{k \in \mathbb{Z}} \breve{f}\,[k]\,(\tau)\mathrm{e}^{-\mathrm{i}\omega kh}$$

where the frequency ω is 2π-periodic, see Definition A.4.2. A very useful result [3] is a theorem that shows that there exists a bijection from the lifted fourier transform $\breve{f}(\mathrm{e}^{\mathrm{i}\omega h}; \tau)$ and the classical Fourier transform $F(\mathrm{i}\omega)$, see Theorem A.4.3. The projection in one direction is given by

$$\breve{f}(\mathrm{e}^{\mathrm{i}\omega h}; \tau) = \frac{1}{h} \sum_{k \in \mathbb{Z}} F(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\omega_k \tau} \tag{2.10}$$

for all $\tau \in [0, h)$, where $\omega_k = \omega + 2\omega_{nyq}k$ is the k^{th} aliased frequency (see Definition A.2.1). And its inverse is given by

$$F(\mathrm{i}\omega_k) = \int_0^h \breve{f}(\mathrm{e}^{\mathrm{i}\omega h}; \tau)\mathrm{e}^{-\mathrm{i}\omega_k \tau} \mathrm{d}\tau.$$

This allows to switch between the classical representation of a signal and its lifted representation using both (the ordinary and the lifted) Fourier transforms.

2.3 Systems

This subsection reviews mathematical systems, their properties and why it is convenient to use systems in signal reconstruction.
In general a mathematical *input-output system* is a device that receives an input signal u and produces a output signal y based on this input signal. Figure 6 shows a graphical interpretation of an input-output system. Examples of such systems are samplers, holds and signal generators. In general the input- and output signals of a system are multidimensional. This means that the system is multi input multi output (MIMO). In order to understand the results derived in this report, using MIMO systems is simply unnecessarily complicated. Therefore this report will focus on systems that are single input single output (SISO) only, but all results can be extended to MIMO systems by use of matrix operations.

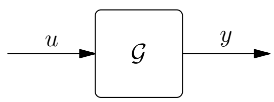


Figure 6: A system $\mathcal{G}$ with input u, output y

If a system $\mathcal{G}$ is LCTI (see Definition A.1.7), then there exists a *convolution* (see Definition A.1.10) that obtains an output y based on the input u

$$y(t) = (g * u)(t) = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)\,\mathrm{d}\tau. \tag{2.11}$$

The function $g(t)$ describes the system.
A system $\mathcal{G}$ is said to be BIBO-*stable* (see Definition A.1.2) if the output is bounded for every bounded input. $\mathcal{G}$ is BIBO-stable iff the infinite integral of $|g(t)|$ is finite:

$$\int_{-\infty}^{\infty} |g(t)|\,\mathrm{d}t < \infty$$

and then the (two sided) Laplace transform of the function $g(t)$ exists as well on the imaginary axis. If additionally the in- and output have Laplace transforms, the system can be written as

$$Y(s) = G(s)U(s)$$

often notated as simply

$$y = G(s)u.$$

Note that this transfer function can only exist if the system is LCTI, if the system is only LDTI it has no classic transfer function. In general the in- and output signals can be multidimensional which results in a transfer matrix. The dimensions of the in- and output signals are denoted by n_u and n_y respectively. In this report n_u and n_y are both one.

If the transfer function $G(s)$ is *rational* and *proper* the transfer function can be written in the form

$$G(s) = C(sI - A)^{-1}B + D \tag{2.12}$$

with real matrices A, B, C and D. A rational transfer function is said to be *proper* if the degree of the numerator does not exceed the degree of the denominator. If additionally the transfer function is *strictly* proper (the degree of the numerator is smaller than the degree of the denominator), then D equals zero.
Furthermore, if the system (2.12) is considered *causal* and proper, then the impulse response $g(t)$ of (2.12) is

$$g(t) = C\mathrm{e}^{At}B \cdot \mathbb{1}(t) + D\delta(t). \tag{2.13}$$

This corresponds to a state space representation

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{2.14}$$

with a new variable x: the internal *state* of the system. Figure 7 shows a block-diagram of a proper state space representation.

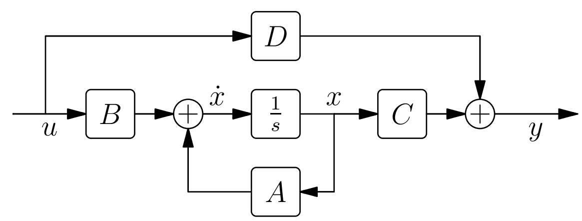


Figure 7: The state space respresentation of system (2.14) with input u, output y, real matrices A, B, C and D and where $\frac{1}{s}$ denotes a pure integrator

A common notation for the state space representation of a transfer function is

$$G(s) = C(sI-A)^{-1}B+D \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \Leftrightarrow \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du. \end{cases}$$

Equations (2.11) and (2.13) combined provide the solution for the output y

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} g(t-\tau)u(\tau)\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} \Big(C\mathrm{e}^{A(t-\tau)}B \cdot \mathbb{1}(t-\tau) + D\delta(t-\tau)\Big)\, u(\tau)\,\mathrm{d}\tau \\ &= \int_{-\infty}^{t} C\mathrm{e}^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau + Du(t). \end{aligned}$$

So, to conclude this subsection, if an LCTI system $\mathcal{G}$ is stable it has a transfer function $G(s)$ for $\mathrm{Re}\,(s) = 0$. If additionally the system is causal and the transfer function is rational and proper, then the system has a state space representation of the form (2.14).
This report will focus on strictly proper systems, i.e. matrix D is the zero matrix.

2.4 Norms

In order to decide which of two sampler-and-hold combinations is the best one, some kind of measure will be used to compare multiple options. Such a measure is called a *norm*, see Definition A.3.1. This subsection will show some examples and applications of norms. Of special interest are the norms suitable for signals and systems. Some physical interpretations are mentioned as well.
In general, a norm (denoted by $||\cdot||$) is measure on a vector space that assigns a non-negative number to every element of the space. This number is the size of the element, measured by this specific norm. One vector space can have multiple norms with different (physical) interpretations.

Example 2.4.1. In the vector space $\mathbb{R}^3$ every element $x \in \mathbb{R}^3$ is of the form

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{2.15}$$

The Euclidean norm on $\mathbb{R}^3$

$$||x||_2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

represents the distance of the element x to the origin. Whereas the norm

$$||x||_\infty = \max\{x_1, x_2, x_3\}$$

represents the maximum distance in one direction (x_1-, x_2- or x_3-axis) of the element x to the origin. Both $||\cdot||_2$ and $||\cdot||_\infty$ are norms on the space $\mathbb{R}^3$ but they have different interpretations. □

Example 2.4.1 shows that there exist several norms for one vector space. The norm that is most convenient for signals is studied in Subsection 2.4.1. For a system it is not straight forward how to compute its norm, Subsection 2.4.2 will show the solution to this problem.

2.4.1 Signal Norms

For signals it is convenient to work with a norm that represents the energy of the signal. Before introducing this norm, first the vector space on which the signals live needs to be introduced. In this report this is the space $L^2(\mathbb{R})$ see Definition A.3.3. The space $L^2[a, b]$ consists of all (Lebesque-integrable) functions $f(t)$ with finite energy on the interval $[a, b]$:

$$\int_a^b |f(t)|^2 \,\mathrm{d}t < \infty.$$

For all elements $f(t)$ in the space $L^2[a, b]$ the norm

$$||f||_{L^2} := \sqrt{\int_a^b |f(t)|^2 \,\mathrm{d}t}$$

represents the square root of the signal's energy. Note that $L^2(\mathbb{R})$ is besides a normed space an inner product space as well (see Definition A.3.2). The inner product between two elements in $L^2(\mathbb{R})$ is defined as

$$\langle f, g\rangle := \int_{-\infty}^{\infty} f(t)\overline{g(t)} \,\mathrm{d}t.$$

By definition of the inner product, two elements are orthogonal if their inner product equals zero. Furthermore, the relation between the inner product and the norm of a signal is the following: the norm of a signal is the square root of the inner product of the signal with itself

$$||f||_{L^2} = \sqrt{\langle f, f\rangle}.$$

2.4.2 LCTI System Norms

As mentioned in the introduction of this subsection, it is slightly more complicated to calculate norms of a system. However, if a system $\mathcal{G}$ is stable and LCTI the norm of the system can be defined in a similar way as the signal norms. Recall that if $\mathcal{G}$ is stable and LCTI the system has a transfer function $G(s)$ mapping the input on the output. Define y_δ as the response of the system to the Dirac delta function

$$y_\delta := \mathcal{G}\delta.$$

If a system is stable and LCTI, the L^2 system norm is the L^2 signal norm of the system's response to the Dirac delta function

$$\begin{aligned} ||\mathcal{G}||_{L^2} := ||y_\delta||_{L^2} &= \sqrt{\int_{-\infty}^{\infty} |\mathcal{G}\delta(t)|^2 \,\mathrm{d}t} \\ &= \sqrt{\int_{-\infty}^{\infty} |(g * \delta)(t)|^2 \,\mathrm{d}t} \\ &= \sqrt{\int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} g(t-\tau)\delta(\tau)\,\mathrm{d}\tau\right|^2 \mathrm{d}t} \\ &= \sqrt{\int_{-\infty}^{\infty} |g(t)|^2 \,\mathrm{d}t} \\ &= ||g||_{L^2}. \end{aligned} \tag{2.16}$$

Note that $||\mathcal{G}||_{L^2}$ is a system norm whereas $||y_\delta||_{L^2}$ and $||g||_{L^2}$ are signal norms. Furthermore, the L^2-norm for systems has an interpretation in terms of stochastic signals. If the input signal is *white noise* (see Definition A.1.11), then the squared norm of the system $||\mathcal{G}||_{L^2}^2$ is exactly the variance or *power* of the output.
Equation (2.16) is the definition of the L^2-norm for systems but it is not straightforward how to calculate this norm. The following equation shows how to compute the

system norm in the frequency domain

$$\begin{aligned} ||\mathcal{G}||_{L^2} = ||y_\delta||_{L^2} &= \sqrt{\int_{-\infty}^{\infty} |\mathcal{G}\delta(t)|^2 \, \mathrm{d}t} \\ &= \sqrt{\int_{-\infty}^{\infty} |(g * \delta)(t)|^2 \, \mathrm{d}t} \\ &= \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\mathrm{i}\omega) \cdot 1|^2 \, \mathrm{d}\omega} \\ &= \sqrt{\frac{1}{\pi} \int_{0}^{\infty} |G(\mathrm{i}\omega)|^2 \, \mathrm{d}\omega} \end{aligned} \tag{2.17}$$

where $G(\mathrm{i}\omega)$ is the transfer function $G(s)$ evaluated in the purely imaginary points $\mathrm{i}\omega$. The function $G(\mathrm{i}\omega)$ is the Fourier transform of the impulse response $g(t)$. Note that $|G(\mathrm{i}\omega)|$ is an even function and that the derivations above only hold for LCTI systems.
In order to calculate $|G(\mathrm{i}\omega)|^2$ the conjugate of a real transfer matric $G^{\sim}(s)$, defined as

$$G^{\sim}(s) := [G(-s)]^T \tag{2.18}$$

will be used. Similar to the scalar case, the squared absolute value of a multidimensional real transfer function is the function itself times its conjugate

$$|G(\mathrm{i}\omega)|^2 = G^{\sim}(\mathrm{i}\omega)G(\mathrm{i}\omega).$$

This leads to the following expression for the L^2 system norm of an LCTI system $\mathcal{G}$:

$$||\mathcal{G}||_{L^2} = \sqrt{\frac{1}{\pi} \int_{0}^{\infty} G^{\sim}(\mathrm{i}\omega)G(\mathrm{i}\omega) \, \mathrm{d}\omega}. \tag{2.19}$$

This is the expression for the L^2 system norm for an LCTI system that will be used in further sections of this report.

2.4.3 LDTI System Norms

The L^2 system norm of an LDTI system $\mathcal{G}$ is defined as

$$||\mathcal{G}||_{L^2} := \sqrt{\frac{1}{h} \int_{0}^{h} ||\mathcal{G}\delta(\cdot - t)||_{L^2}^2 \, \mathrm{d}t} \tag{2.20}$$

which can be seen as the integral over the response of the system to a series of Dirac delta functions. There does not exist a nice expression for this norm that is easy to work with yet.

2.5 Calculation of the L^2 system norm

An expression for the L^2 norm for a system was derived in Subsection 2.4.2 and given by (2.19), still this expression is not solvable in a clear way. This subsection will provide an explicit solution for the L^2 system norm and it contains a few examples to show how this norm is calculated.

2.5.1 Classical Calculation

If the matrix A of the transfer function

$$G(s) = C(sI - A)^{-1}B$$

is stable (see Definition A.3.8) the L^2 system norm of $\mathcal{G}$ (2.19) can be calculated in a classic way [5]. A Matrix A is stable if all its eigenvalues λ_i lie in the open left half of the complex plane $\mathbb{C}$:

$$\mathrm{Re}\,(\lambda_i) < 0 \quad \forall i.$$

If so, the system has a unique solution P of the Lyapunov equation [5]:

$$A^T P + PA = -C^T C. \tag{2.21}$$

It is a classic result that if A is stable, then the calculation of the L^2 system norm of $\mathcal{G}$ can be reduced to

$$||\mathcal{G}||_{L^2} = \sqrt{B^T P B}. \tag{2.22}$$

Example 2.5.1. Consider the system $\mathcal{G}$ with transfer function

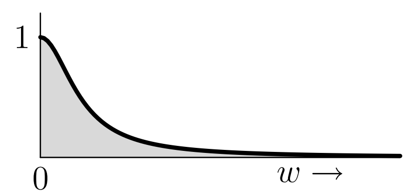
$$G(s) = \frac{1}{1+s}.$$

This corresponds to a state space representation

$$\begin{aligned} \dot{x} &= -1x + 1u \\ y &= 1x \end{aligned}$$

and the squared magnitude of $G(\mathrm{i}\omega)$ looks like

$$\left|\frac{1}{1+\mathrm{i}\omega}\right|^2 =$$



Note that all matrices are only scalars and that therefore A has only one eigenvalue, i.e. -1 and thus A is stable. The Lyapunov equation reduces to a simple, scalar equation

$$-P - P = -1$$

which has the solution $P = \frac{1}{2}$. So the L^2 system norm (2.22) equals

$$\begin{aligned} ||\mathcal{G}||_{L^2} &= \sqrt{1 \cdot \frac{1}{2} \cdot 1} \\ &= \sqrt{\frac{1}{2}}. \end{aligned} \tag{2.23}$$

□

2.5.2 Alternative Calculation

From Subsection 2.3 it is known that a real, rational and strictly proper transfer function $G(s)$ can be written in the form

$$G(s) = C(sI - A)^{-1}B \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right].$$

In combination with (2.18) this gives an expression for the conjugate $G^{\sim}(s)$ of the transfer matrix

$$\begin{aligned} G^{\sim}(s) &= \left[C(-sI - A)^{-1}B\right]^T \\ &= B^T\left[(-sI - A)^T\right]^{-1} C^T \\ &= -B^T(sI + A^T)^{-1}C^T. \end{aligned}$$

In Equation (2.19) the transfer function $G(s)$ and its conjugate $G^{\sim}(s)$ form a coupled system $G^{\sim}G$ which means that the output y of G is the input for $G^{\sim}$. Define K as this coupled system

$$K(s) := G^{\sim}(s)G(s).$$

K also has a state space representation which will be shown next [5]. Say, G has input u, output y and state x whereas $G^{\sim}$ has input y, output z and state q, then the coupled system can be written as

$$\begin{aligned} y &= G(s)u \\ z &= G^{\sim}(s)y \\ z &= G^{\sim}(s)G(s)u \\ z &= K(s)u. \end{aligned}$$

Since both transfer functions G and $G^{\sim}$ are real, rational and strictly proper, they both have a state space realization

$$\begin{aligned} y = G(s)u &\Leftrightarrow \left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{array}\right. \\ z = G^{\sim}(s)y &\Leftrightarrow \left\{ \begin{array}{l} \dot{q}(t) = -A^Tq(t) - C^Ty(t) \\ z(t) = B^Tq(t). \end{array}\right. \end{aligned}$$

Combining these two state space realizations and merging them into one vector notation leads to

$$\left[\begin{array}{c} \dot{x} \\ \dot{q} \\ \hline z \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ -C^TC & -A^T & 0 \\ \hline 0 & B^T & 0 \end{array}\right] \left[\begin{array}{c} x \\ q \\ \hline u \end{array}\right]$$

which is the state space representation of $K(s)$. Define the real matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ as

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right] := \left[\begin{array}{cc|c} A & 0 & B \\ -C^TC & -A^T & 0 \\ \hline 0 & B^T & 0 \end{array}\right]. \tag{2.24}$$

This means that $K(s)$ can be written as

$$K(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}.$$

Now, the L^2 system norm (2.19) reduces to the integral over $K(s)$ for which a state space representation exists. Note that $\tilde{C}\tilde{B} = 0$. It can be shown [5] that if $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are real matrices and if $\tilde{C}\tilde{B} = 0$, then the semi-infinite integral of $K(s)$ can be determined explicitly:

$$\int_0^{\infty} K(\mathrm{i}\omega)\,\mathrm{d}\omega = \mathrm{i}\tilde{C}\log\left(\mathrm{i}\tilde{A}\right)\tilde{B}. \tag{2.25}$$

This equation only holds as long as $\tilde{A}$ has no eigenvalues λ_i on the imaginary axis:

$$\mathrm{Re}\,(\lambda_i) \neq 0 \quad \forall i.$$

Note that this does not mean that $\tilde{A}$ has to be stable (see Definition A.3.8); $\tilde{A}$ can have eigenvalues in the entire complex plane as long as they do not lie on the imaginary axis.

Equation (2.25) uses the principal logarithm (see Definition A.3.7) of a matrix. MATLAB® has a command that generates the principal logarithm for any square matrix of which the real eigenvalues are strictly positive.

Equation (2.25) provides that the L^2 system norm of a system $\mathcal{G}$ can be written as

$$||\mathcal{G}||_{L^2} = \sqrt{\frac{\mathrm{i}}{\pi}\left[\tilde{C}\log\left(\mathrm{i}\tilde{A}\right)\tilde{B}\right]} \tag{2.26}$$

provided that $\tilde{A}$ has no eigenvalues on the imaginary axes.

Example 2.5.2. Consider the same system $\mathcal{G}$ as in Example 2.5.1

$$G(s) = \frac{1}{1+s} \stackrel{s}{=} \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right].$$

This example will show how to calculate the L^2 system norm of this system in the way explained in Subsection 2.5.2

In order to calculate the L^2 system norm, first $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are computed:

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \\ \tilde{B} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \tilde{C} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \end{aligned}$$

Note that the eigenvalues of $\tilde{A}$ are -1 and 1, so Equation (2.26) can be applied to this system. The principal logarithm of $\mathrm{i}\tilde{A}$ is computed using MATLAB®:

$$\log\left(\mathrm{i}\tilde{A}\right) = \begin{bmatrix} -\frac{\pi}{2}\mathrm{i} & 0 \\ -\frac{\pi}{2}\mathrm{i} & \frac{\pi}{2}\mathrm{i} \end{bmatrix}.$$

Now Equation (2.26) can be exploited to determine the L^2

norm of the system $G(s) = \frac{1}{1+s}$

$$
\begin{aligned}
||\mathcal{G}||_{L^2} &= \sqrt{\frac{\mathrm{i}}{\pi}\left(\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{\pi}{2}\mathrm{i} & 0 \\ -\frac{\pi}{2}\mathrm{i} & \frac{\pi}{2}\mathrm{i} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)} \\
&= \sqrt{-\frac{\mathrm{i}}{\pi}\cdot\frac{\pi\mathrm{i}}{2}} \\
&= \sqrt{\frac{1}{2}}. \qquad (2.27)
\end{aligned}
$$

Of course, the L^2 norm of this system can also be determined analytically in order to verify the validation of Equation (2.26). To do this, Equation (2.17) will be exploited together with the conjugate

$$G^{\sim}(s) = \frac{1}{1-s}$$

of the transfer function $G(s)$. The squared magnitude of the transfer function reads (see Equation (2.18))

$$
\begin{aligned}
|G(\mathrm{i}\omega)|^2 &= \frac{1}{1+\mathrm{i}\omega}\cdot\frac{1}{1-\mathrm{i}\omega} \\
&= \frac{1}{1+\omega^2}.
\end{aligned}
$$

Now the L^2 norm of the system $\mathcal{G}$ can be calculated analytically

$$
\begin{aligned}
||\mathcal{G}||_{L^2} &= \sqrt{\frac{1}{\pi}\int_0^\infty |G(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega} \\
&= \sqrt{\frac{1}{\pi}\int_0^\infty \frac{1}{1+\omega^2}\,\mathrm{d}\omega} \\
&= \sqrt{\frac{1}{\pi}\lim_{\omega\to\infty}\arctan(\omega)} \\
&= \sqrt{\frac{1}{\pi}\cdot\frac{\pi}{2}} \\
&= \sqrt{\frac{1}{2}}.
\end{aligned}
$$

Note that this norm is exactly the same as the one calculated using the principal logarithm (2.27) and the one calculated using the classical expression (2.23). □

In Example 2.5.2 the transfer function is SISO therefore it is rather easy to calculate the L^2 system norm analytically. Whereas calculating the norm using the principal logarithm is a more complex calculation. In general, if the transfer function is MIMO it is a lot more complicated to calculate the norm analytically. Therefore it is very convenient to work with Equation (2.22) or (2.26) in order to calculate the L^2 norm of a system.

3 Truncated System Norm

3.1 Introduction

The signal- and system norms on the vector space L^2 are introduced in Subsections 2.4.1 and 2.4.2 respectively. This subsection will focus on the concept of *frequency truncated system norms*. Figure 1 on page 2 shows the set up for a sample-and-reconstruction problem, this system will be referred to as the *error system*. The signal w is the input signal for $\mathcal{G}$ that will generate the signal to be sampled and reconstructed u. The mapping from w to e reads

$$e = (\mathcal{I} - \mathcal{HS})\mathcal{G}w.$$

The goal is, given a (fixed) sampling period h, to minimize the error e in some sense. For instance that the mapping from w to e is minimized according to the L^2 system norm:

$$\min_{\mathcal{H},\mathcal{S}} ||(\mathcal{I} - \mathcal{HS})\mathcal{G}||_{L^2} . \tag{3.1}$$

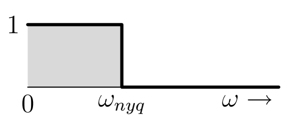
Here the norm is minimized over all stable and LDTI samplers and holds. The interpretations of equation (3.1) reads that the smaller the norm the more the sampled-and-reconstructed signal y looks like the original signal u. It can be shown [5] that if $\mathcal{G}$ is LCTI, then the combination of sampler and hold that minimizes the norm (3.1) is in fact LCTI and stable as well. This means that the combination $\mathcal{F} = \mathcal{HS}$ has a transfer function (see Subsection 2.1.3):

$$F(s).$$

3.2 Monotonically decreasing response

If additionally the system $\mathcal{G}$ has a monotonically decreasing magnitude $|G(\mathrm{i}\omega)|$ for positive ω, then the combination of sampler and hold is the ideal low pass filter:

$$F(\mathrm{i}\omega) = \mathbb{1}_{[-\omega_{nyq},\omega_{nyq}]}(\omega) =$$



This low-pass filter can be achieved using a low pass filter in combination with the ideal sampler and sinc-hold from Shannon's Theorem, see Equations (2.3) and (2.4) on page 3.

The system $(\mathcal{I} - \mathcal{HS})$ with the minimizing sampler and hold is in fact an ideal high-pass filter feeding through all frequencies higher than the Nyquist frequency ω_{nyq}.

So for a fixed sampling period h and an LTCI system G with monotonically decreasing magnitude $|G(\mathrm{i}\omega)|^2$, the best one can do is filter out the first *Nyquist band* N_1 consisting of the frequencies $[0, \omega_{nyq})$, from the frequency response (see Definition A.2.1).

Now the question arrises what the L^2 norm of the system $(\mathcal{I} - \mathcal{HS})\mathcal{G}$ with optimal sampler-and-hold combination is. This is in fact the object that was minimized in the first place, see Equation (3.1). The norm of the error system can be calculated in the same way as in Subsection 2.5. Now the optimal sampler-and-hold combination causes the magnitude $|(I - F(\mathrm{i}\omega))G(\mathrm{i}\omega)|^2$ of the whole system to be of the form

$$|(I - F(\mathrm{i}\omega))G(\mathrm{i}\omega)|^2 = \begin{cases} 0 & 0 < \omega \leq \omega_{nyq} \\ |G(\mathrm{i}\omega)|^2 & \omega > \omega_{nyq} \end{cases}$$

since $(I - F)(\mathrm{i}\omega)$ is a high-pass filter. This leads to the following expression of the system norm

$$\begin{aligned} ||(\mathcal{I} - \mathcal{HS})\mathcal{G}||_{L^2} &= \sqrt{\frac{1}{\pi}\int_0^\infty |(I - F(\mathrm{i}\omega))G(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega} \\ &= \sqrt{\frac{1}{\pi}\int_{\omega_{nyq}}^\infty |G(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega}. \end{aligned} \tag{3.2}$$

Equation (3.2) is called the *truncated* L^2 *system norm* of the system $\mathcal{G}$ and is denoted by

$$||\mathcal{G}||_{\omega_{nyq}} := \sqrt{\frac{1}{\pi}\int_{\omega_{nyq}}^\infty |G(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega}. \tag{3.3}$$

Note that Equation (3.3) is in fact not really a norm (see Definition A.3.1) since $||\mathcal{G}||_{\omega_{nyq}} = 0$ does not necessarily imply $G(\mathrm{i}\omega) = 0$ for all $\omega \in [0, \infty]$.

3.2.1 Unstable matrix A

The *truncated* L^2 system norm (3.3) is almost the same as the L^2 system norm (2.17) on page 8 except that the integral of the *truncated* L^2 system norm (3.3) starts at the Nyquist frequency instead of at zero. It turns out that the *truncated* L^2 system norm (3.3) can be calculated in a similar way as the oridinary L^2 system norm (2.17) using the state space representation

$$K(s) = \tilde{C}(sI + \tilde{A})^{-1}\tilde{B}$$

as defined in Subsection 2.5. Now the truncated L^2 system of $\mathcal{G}$ can be expressed in terms of the real matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$

$$\begin{aligned} ||\mathcal{G}||_{\omega_{nyq}} &= \sqrt{\frac{1}{\pi}\int_{\omega_{nyq}}^\infty |G(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega} \\ &= \sqrt{\frac{1}{\pi}\int_{\omega_{nyq}}^\infty G^{\sim}(\mathrm{i}\omega)G(\mathrm{i}\omega) \,\mathrm{d}\omega} \\ &= \sqrt{\frac{1}{\pi}\int_{\omega_{nyq}}^\infty K(\mathrm{i}\omega) \,\mathrm{d}\omega}. \end{aligned}$$

This is the same derivation as used for Equation (2.17). It can be shown [5] that the semi-infinite integral of $K(\mathrm{i}\omega)$ also exists if the lower bound of the integral is larger than zero:

$$\int_{\omega_{nyq}}^\infty K(\mathrm{i}\omega) \,\mathrm{d}\omega = \mathrm{i}\tilde{C}\log\left(\omega_{nyq}I + \mathrm{i}\tilde{A}\right)\tilde{B}$$

provided that $\omega_{nyq} > \omega_{\max} := \max|\omega_k|$, where the maximum is taken over all pure imaginary eigenvalues $\mathrm{i}\omega_k$ of $\tilde{A}$. This leaves a concrete expression for the truncated system norm

$$||\mathcal{G}||_{\omega_{nyq}} = \sqrt{\frac{\mathrm{i}}{\pi}\left[\tilde{C}\log\left(\omega_{nyq}I + \mathrm{i}\tilde{A}\right)\tilde{B}\right]} \tag{3.4}$$

which can be used to calculate the truncated L^2 system norm explicitly. So if the sampler-and-hold combination is optimal, the L^2 norm of the error system reduces to

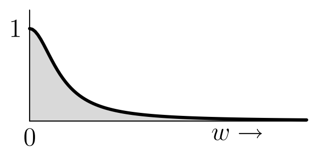
$$||(\mathcal{I}-\mathcal{HS})\mathcal{G}||_{L^2} = \sqrt{\frac{\mathrm{i}}{\pi}\left[\tilde{C}\log\left(\omega_{nyq}I + \mathrm{i}\tilde{A}\right)\tilde{B}\right]}. \tag{3.5}$$

Example 3.2.1. Consider the same system $\mathcal{G}$ as in Example 2.5.1 with the transfer function

$$G(s) = \frac{1}{1+s}$$

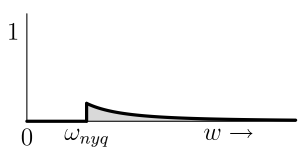
and sampling period $h = 1$. The squared magnitude of the transfer function looks like

$$|G(\mathrm{i}\omega)|^2 =$$



Since the magnitude of G is monotonically decreasing, the optimal hold-and-sampler combination will cut of the first Nyquist band:

$$|(I - F(\mathrm{i}\omega))\,G(\mathrm{i}\omega)|^2 =$$



The matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are the same as in Example 2.5.2 which leads to the calculation of the truncated L^2 system norm using Equation (3.5). In this case the Nyquist frequency $\omega_{nyq} = \frac{\pi}{h}$ equals π

$$\begin{aligned}
||\mathcal{G}||_{\omega_{nyq}} &= \sqrt{\frac{\mathrm{i}}{\pi}\left(\begin{bmatrix} 0 & 1 \end{bmatrix}\log\left(\begin{bmatrix} \pi - \mathrm{i} & 0 \\ -\mathrm{i} & \pi + \mathrm{i} \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)} \\
&= \sqrt{\frac{\mathrm{i}}{\pi}\cdot -0.3082\mathrm{i}} \\
&= 0.3132.
\end{aligned}$$

Again, the norm can be determined analytically. Example 2.5.2 already derived the anti-derivative of the integrant

$$\begin{aligned}
||\mathcal{G}||_{\omega_{nyq}} &= \sqrt{\frac{1}{\pi}\int_{\pi}^{\infty}\frac{1}{\omega^2+1}\,\mathrm{d}\omega} \\
&= \sqrt{\frac{1}{\pi}\left(\lim_{\omega\to\infty}\arctan(\omega) - \arctan(\pi)\right)} \\
&= \sqrt{\frac{1}{\pi}\left(\frac{\pi}{2} - 1.2626\right)} \\
&= 0.3132.
\end{aligned}$$

Note that this norm, which is calculated analytically, is exactly the same as the on calculated using the principle logarithm.

In order to get an indication how much energy of the original system $\mathcal{G}$ is preserved by sampling and reconstruction, the following formula is exploited

$$\begin{aligned}
\frac{||\mathcal{G}||^2_{L^2} - ||(\mathcal{I}-\mathcal{HS})\mathcal{G}||^2_{L^2}}{||\mathcal{G}||^2_{L^2}} \times 100\% &= \frac{0.5 - 0.0981}{0.5} \times 100\% \\
&= 80.4\%.
\end{aligned}$$

In this formula the numerator is the energy of the system itself minus the energy of the error system. So the numerator consists of thet total energy that is preserved by sampling and reconstruction. Dividing this by the energy of the system and multiplying by 100 gives the percentage of energy that is preserved.

So 80.4% of the system's energy is preserved by sampling and reconstruction if a sampling period $h = 1$ is used in combination with the optimal sampler and hold combination. □

3.2.2 Stable matrix A

Subsection 3.2.1 showed how to calculate the norm

$$||(\mathcal{I}-\mathcal{HS})\mathcal{G}||_{L^2} \tag{3.6}$$

of the error system for the optimal sampler-and-hold combination

$$F(\mathrm{i}\omega) = \mathbb{1}_{[-\omega_{nyq},\omega_{nyq}]}(\omega). \tag{3.7}$$

The system $\mathcal{G}$ is assumed to be LCTI and to have a monotonically decreasing magnitude $|G(\mathrm{i}\omega)|^2$, and the sampling period, h, is fixed.

In this case the L^2 norm of the optimal error system $(\mathcal{I}-\mathcal{HS})\mathcal{G}$ reduces to the truncated L^2 system norm of only $\mathcal{G}$ as shown in Subsection 3.2

$$\begin{aligned}
||(\mathcal{I}-\mathcal{HS})\mathcal{G}||_{L^2} &= ||\mathcal{G}||_{\omega_{nyq}} \\
&= \sqrt{\frac{\mathrm{i}}{\pi}\left[\tilde{C}\log\left(\omega_{nyq}I + \mathrm{i}\tilde{A}\right)\tilde{B}\right]}.
\end{aligned}$$

However, calculating $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ requires a lot of the computational capacity since the dimensions of $\tilde{A}$ are twice as large as of A itself. The computational burden can be reduced if the matrix A of the transfer function

$$G(s) = C(sI - A)^{-1}B$$

is stable. Subsection 2.5.1 showed that if A is stable the L^2 system norm of $\mathcal{G}$ reduces to

$$||\mathcal{G}||_{L^2} = \sqrt{B^T PB}.$$

It can be shown [5] that if $\mathcal{G}$ is stable, strictly proper and if $\mathcal{G}$ has the state space representation $G(s) = C(sI - A)^{-1}B$ with real matrices A, B and C and A is stable, then

$$\begin{aligned} ||G||_{\omega_{nyq}} &= \sqrt{-\frac{2}{\pi}\text{Im}\ \left(B^T P \log\left(\omega_{nyq} I + \mathrm{i}A\right) B\right)} \\ &= \sqrt{||G||^2_{L^2} - \frac{2}{\pi}\text{Im}\ \left(B^T P \log\left(\mathrm{i}\omega_{nyq} I - \mathrm{i}A\right) B\right)} \end{aligned}$$

where P is the unique solution of the Lyapunov equation (2.21) on page 8. This provides the possibility to calculate the L^2 norm of the error system (3.6) with the optimal sampler-and-hold combination (3.7), without the computational burden of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$.
So if the matrix A is stable and the sampler-and-hold combination is optimal, the L^2 norm of the error system reduces to

$$||(\mathcal{I} - \mathcal{HS})\mathcal{G}||_{L^2} = \sqrt{-\frac{2}{\pi}\text{Im}\ \left(B^T P \log\left(\omega_{nyq} I + \mathrm{i}A\right) B\right)}. \tag{3.8}$$

Example 3.2.2. This example will show how to calculate the truncated L^2 system norm using the Equation (3.8). Consider the same system $\mathcal{G}$ as in Example 2.5.1 with the transfer function

$$G(s) = \frac{1}{1+s}$$

and sampling period $h = 1$. The cut-off frequency ω_{nyq} is again π. Matrix P is again $\frac{1}{2}$ just as in Example 2.5.1 and thus the truncated L^2 system norm reduces to

$$\begin{aligned} ||\mathcal{G}||_{\omega_{nyq}} &= \sqrt{-\frac{2}{\pi}\text{Im}\left(1 \cdot \frac{1}{2} \cdot \log\left(\pi - \mathrm{i}\right) \cdot 1\right)} \\ &= \sqrt{-\frac{2}{\pi}\text{Im}\left(\frac{1}{2} \cdot (1.1930 - 0.3082\mathrm{i})\right)} \\ &= \sqrt{-\frac{2}{\pi}\left(-0.154\right)} \\ &= 0.3132. \end{aligned}$$

Note that this norm is the same as the one from Example 3.2.1. □

3.3 Folding

This subsection will discuss how the optimal sampler and hold combination $\mathcal{HS}$ will look when the squared magnitude $|G(\mathrm{i}\omega)|^2$ of the system $\mathcal{G}$ is *not* monotonically decreasing. Still the goal is to minimize

$$||(\mathcal{I} - \mathcal{HS})\mathcal{G}||_{L^2}$$

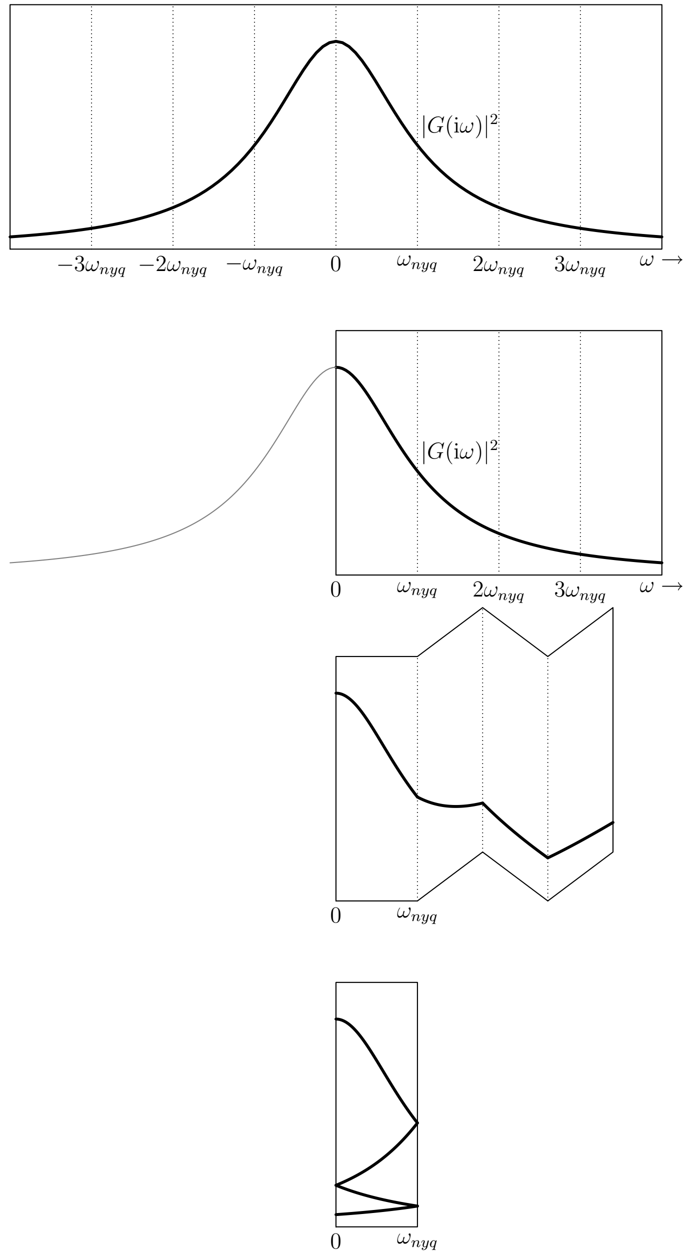


Figure 8: Folding the response of $|G(\mathrm{i}\omega)|^2$

over all stable and LDTI samplers and holds.
The assumption that $\mathcal{G}$ is strictly proper still holds, so eventually there will be a frequency from where on $|G(\mathrm{i}\omega)|^2$ will be monotonically decreasing. This frequency is denoted by

$$\omega^*.$$

It can be shown [4] that the optimal combination $\mathcal{HS}$ filters a finite number of frequency bands out of the response $|G(\mathrm{i}\omega)|^2$. Though the total length of these frequency bands equals the length of one Nyquist Band: ω_{nyq}. So the possibilities of reducing the L^2 system norm are limited by ω_{nyq} and thus by h. The next thing is to find the frequency bands that need to be filtered out, in order to

achieve a minimal L^2 norm of the error system. It turns out that to find these frequency bands, one must *fold* the response $|G(\mathrm{i}\omega)|^2$ like a harmonica. The response is folded in multiples of the Nyquist frequency and since $|G(\mathrm{i}\omega)|^2$ is an even function, this only needs to be done for positive frequencies. This process is illustrated in Figure 8.
Once the response has been folded, the maximum over the folded part is determined. In Figure 8 this is just the first Nyquist band N_1 because the response is monotonically decreasing, but in general the maximum will consist of several small frequency bands each corresponding to another Nyquist band N_k. This is the case in Figure 9 and Example 3.3.1.
In order to determine the maximum over the folded function, all Nyquist bands N_k will be projected on the interval $[0, \omega_{nyq})$

$$h_k(\zeta) = \begin{cases} |G(\mathrm{i}\,((k-1)\omega_{nyq} + \zeta))|^2 & k = 1,3,5,... \\ |G(\mathrm{i}\,(k\omega_{nyq} - \zeta))|^2 & k = 2,4,6,... \end{cases}$$

with $\zeta \in [0, \omega_{nyq})$ and $k \in \mathbb{Z}^+$ the Nyquist band index. Now for every ζ in the domain the maximum over all functions h_k will be determined numerically

$$\max_k h_k(\zeta).$$

Every folded frequency $\zeta_m \in [0, \omega_{nyq})$ has a maximum in one of the Nyquist bands, indicated by the k_m corresponding to this maximum. So the maximum corresponding to ζ_m, lies in the Nyquist band N_{k_m}. In order to determine the original frequency corresponding to the maxima, the k_m and ζ_m of every maximum are used to project the folded frequency back on the original frequency domain (shown in Figure 9)

$$\omega_m = \begin{cases} k_m\omega_{nyq} + \zeta_m & k_m = 1,3,5,... \\ k_m\omega_{nyq} + (\omega_{nyq} - \zeta_m) & k_m = 2,4,6,... \end{cases}$$

Folding does not imply that the frequencies with the largest peak of the response are filtered out, but it filters out the maximum of the folded response.
Important for folding is to know from where on the frequency response is monotonically decreasing. Here the transfer matrix $K(s)$ will be exploited once again, recall

$$|G(s)|^2 = G^{\sim}(s)G(s) = K(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}.$$

As stated in the beginning of this subsection, the frequency response will decrease monotonically after ω^*. This frequency is the largest frequency for which the derivative of $K(\mathrm{i}\omega)$

$$\frac{\mathrm{d}}{\mathrm{d}\mathrm{i}\omega}K(\mathrm{i}\omega) = -\tilde{C}(\mathrm{i}\omega I - \tilde{A})^{-2}\tilde{B}$$

equals zero. By transferring this expression back to a (MIMO) transfer matrix it is just a matter of equaling the (multiple) numerator(s) to zero. The largest frequency for

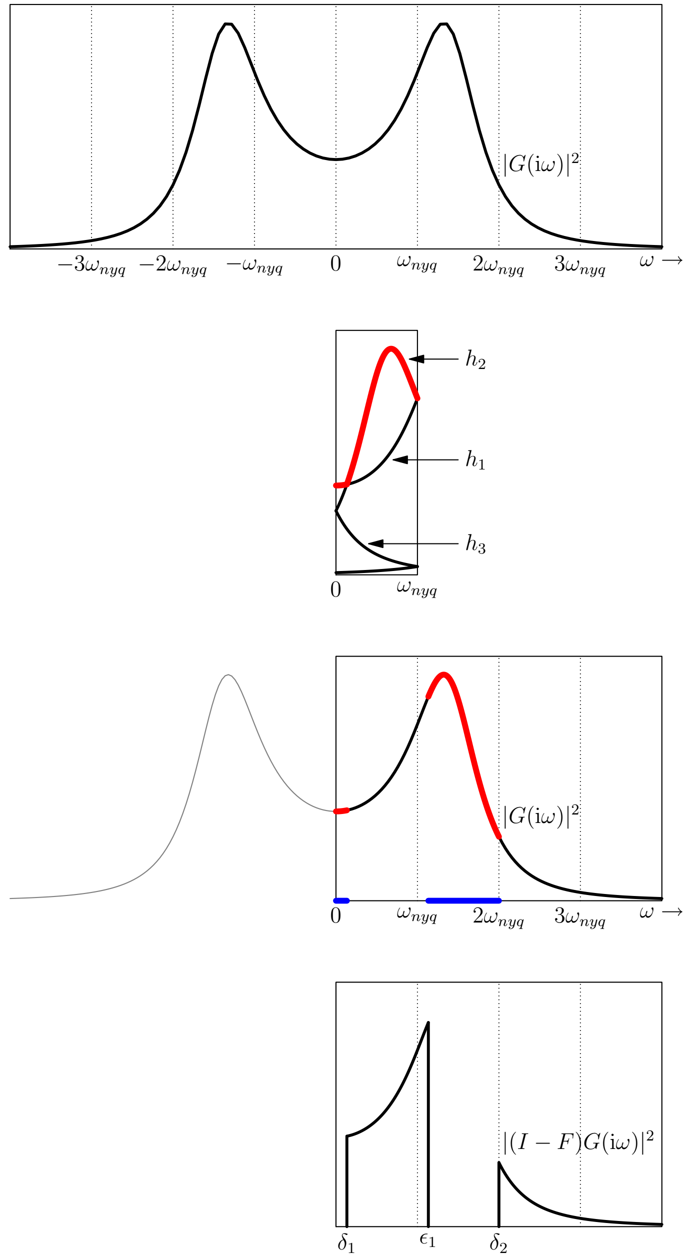


Figure 9: Unfolding the response of $|G(\mathrm{i}\omega)|^2$ after determining the maxima (red). In blue the frequencies that will be filtered in order to achieve a minimal L^2 norm of the error system. There is one unfiltered band $[\delta_1, \epsilon_1]$ and the tail $[\delta_n, \infty)$, so in this case n equals 1

which (one of) the numerator(s) equals zero, is ω^*. It is not difficult to determine in which Nyquist band ω^* lies. This Nyquist band is called

$$N_{k^*}.$$

Folding needs to been done up till Nyquist band N_{k^*+1} because this band can still tribute to the maximum due to folding.

In the case where $|G(\mathrm{i}\omega)|^2$ is monotonically decreasing, the calculation of the L^2 norm of the error system (3.1) consist of only one integral. Since the optimal combination of sampler-and-hold $\mathcal{HS}$ filters several frequency bands if the response is not monotonically decreasing, the calculation is somewhat more complicated. The number of frequency bands that are filtered out is finite, so the number frequency bands that are *unchanged* is finite as well (say n). Additionally the "tail" of the frequency response contributes to the norm as well

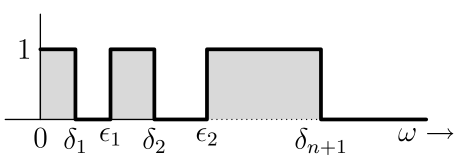
$$
\begin{aligned}
&||(\mathcal{I}-\mathcal{HS})\mathcal{G}||_{L^2} \\
&= \sqrt{\frac{1}{\pi}\sum_{k=1}^{n}\int_{\delta_k}^{\epsilon_k} |G(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega + \frac{1}{\pi}\int_{\delta_{n+1}}^{\infty} |G(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega} \\
&= \sqrt{\frac{1}{\pi}\sum_{k=1}^{n} -\mathrm{i}\tilde{C}\log\left(\Omega_k\right)\tilde{B} + ||\mathcal{G}||^2_{\delta_{n+1}}} \qquad (3.9)
\end{aligned}
$$

with

$$\Omega_k := \left(\epsilon_k I + \mathrm{i}\tilde{A}\right)\left(\delta_k I + \mathrm{i}\tilde{A}\right)^{-1}$$

and $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ as defined in (2.24). Furthermore ϵ_k and δ_k are respectively the under- and lower bound of the *unfiltered* frequency bands. The norm $||\mathcal{G}||_{\delta_{n+1}}$ is defined in the same way as the truncated L^2 system norm $||\mathcal{G}||_{\omega_{nyq}}$ (3.3), only with a lower bound δ_{n+1} instead of ω_{nyq}.
The optimal sampler-and-hold combination $\mathcal{HS}$ now looks like a series concatenated step functions

$$F(\mathrm{i}\omega) = \mathbb{1}_{[0,\delta_1]} + \sum_{k=1}^{n} \mathbb{1}_{[\epsilon_k,\delta_{k+1}]}$$

$F(\mathrm{i}\omega) =$ 

where the following holds for ϵ_k and δ_k

$$(\delta_1 - 0) + \sum_{k=1}^{n}(\delta_{k+1} - \epsilon_k) = \omega_{nyq}$$

since the optimal combination $\mathcal{HS}$ can only filter multiple frequency bands with a total length of ω_{nyq}.
Note that the above also holds for negative frequencies since the function $F(\mathrm{i}\omega)$ is an even function.

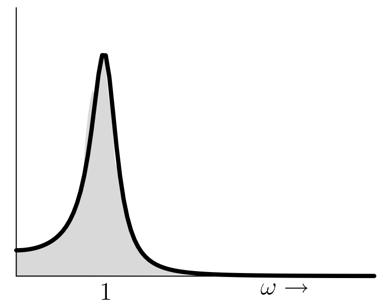
Example 3.3.1. Consider the ideal sampler, a sampling period $h = 4$ and the system

$$G(s) = \frac{1}{(s+0.2)^2+1}.$$

This corresponds to a transfer function $G(s) = C(sI - A)^{-1}B$ with

$$A = \begin{bmatrix} -0.4 & -1.04 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The squared magnitude of the transfer function looks like

$|G(\mathrm{i}\omega)|^2 =$ 

The matrix $\tilde{A}$ (as defined in Equation (2.24) on page 9) now looks like

$$\tilde{A} = \begin{bmatrix} -0.4 & -1.04 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & -1 \\ 0 & -1 & 1.04 & 0 \end{bmatrix}$$

which has the eigenvalues $-0.2 \pm \mathrm{i}$ and $0.2 \pm \mathrm{i}$. This means that $\tilde{A}$ has no pure imaginary eigenvalues, so the L^2 system norm of $\mathcal{G}$ can be calculated using Equation (2.26).

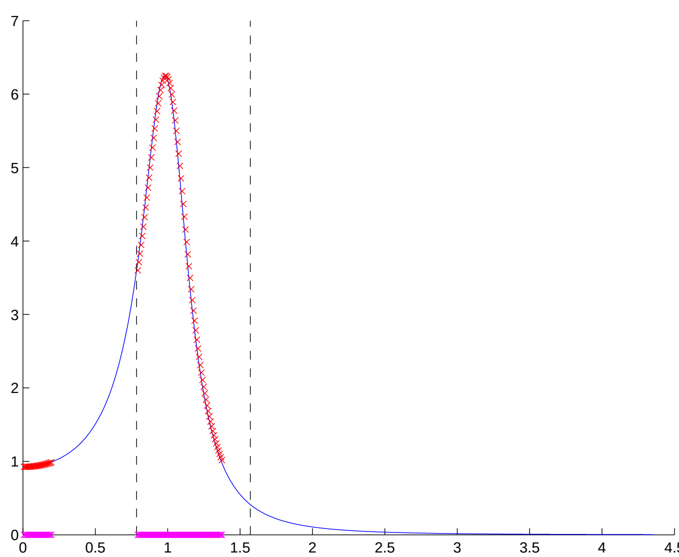


Figure 10: Frequency Response of $G = 1/((s+0.2)^2+1)$. In red the maxima found by folding and in pink the frequencies that will be filtered out by the optimal hold-and-sampler combination

Figure 10 shows the magnitude of the frequency response of the system $\mathcal{G}$. In this figure, the dotted lines are multiples of the Nyquist frequency $\omega_{nyq} = \frac{\pi}{4}$. These are the lines over which the function is folded, similar to Figure 8. After determining the maxima (in red) the function is folded back (similar to Figure 9) and the frequencies corresponding to the maxima are highlighted in pink. The L^2 norm of the error system corresponding to the optimal sampler-and-hold combination $\mathcal{HS}$ for this specific $\mathcal{G}$ can be calculated using Equation (3.9)

$$||(\mathcal{I} - \mathcal{HS})\mathcal{G}||_{L^2} = 0.6487.$$

In order to get an indication how much energy of the original system $\mathcal{G}$ is preserved by sampling and reconstruction, the following formula is exploited

$$\begin{aligned} &\frac{||\mathcal{G}||_{L^2}^2 - ||(\mathcal{I} - \mathcal{HS})\mathcal{G}||_{L^2}^2}{||\mathcal{G}||_{L^2}^2} \times 100\% \\ &= \frac{1.2019 - 0.4208}{1.2019} \times 100\% \\ &= 65.0.\% \end{aligned}$$

This means that 65.0% of the systems energy is preserved after sampling and reconstruction if a sampling period $h = 4$ is used in combination with the optimal sampler and hold combination. □

Intuitively one might think that reducing the sampling period h leads to better performance of the system. After all, reducing the sampling period leads to more samples (more data) and therefore it might be expected that the sampled-and-reconstructed signal y looks more similar to the input signal u. Though it turns out that this is not true. Example 3.3.2 shows that reducing the sampling period does not automatically lead to a smaller L^2 norm of the error system. Of course, eventually the error will go to zero but this is only for very small h. Example 3.3.2 shows a lower bound for the L^2 norm of the error system as well.

Example 3.3.2. [4] Consider the same system as in Example 3.3.1:

$$G(s) = \frac{1}{(s+0.2)^2 + 1}.$$

Now instead of taking the sampling period h fixed and determining the error for this one sample period , the sample period will vary and hence the norm of the system will be a function of h. In this example the norm of G is exploited in three different ways:

$$\begin{aligned} m(h) &:= ||\mathcal{G}||_{L^2}^2 \\ p(h) &:= ||(\mathcal{I} - \mathcal{HS})\mathcal{G}||_{L^2}^2 \\ q(h) &:= ||\mathcal{G}||_{L^2}^2 - \frac{||\mathcal{G}||_{L^\infty}^2}{h} \end{aligned}$$

where the infinity norm is defined as

$$||\mathcal{G}||_{L^\infty} := \operatorname*{ess\,sup}_{\omega \in \mathbb{R}} |G(\mathrm{i}\omega)| \,.$$

The function $q(h)$ is a lower bound for the L^2 norm of the error system $p(h)$ which can be seen in the proceedings of the example. For every sampling period h the norm is based on the corresponding optimal sampler and hold combination, in the same way as in Example 3.3.1.
In this example the fundamental limit is where $q(h)$ equals zero, i.e.

$$h_{\mathcal{G}} := \frac{||\mathcal{G}||_{L^2}^2}{||\mathcal{G}||_{L^\infty}^2} = \frac{2.5^2}{125/104} = 5.2.$$

Figure 11 shows the L^2-norm, the truncated norm and the difference between the L^2-norm and the scaled L^∞-norm for different sample periods. □

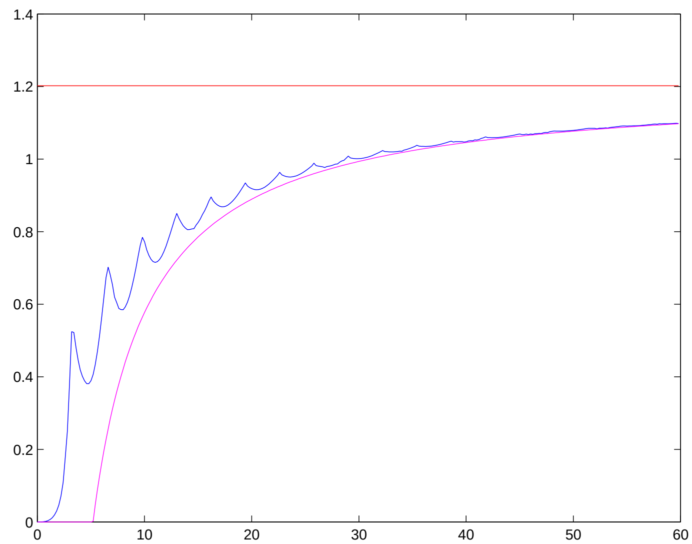


Figure 11: In red the L^2-norm of $\mathcal{G}$; $m(h)$, in blue the L^2 norm of the error system $(\mathcal{I} - \mathcal{HS})\mathcal{G}$; $p(h)$ and in pink the difference between the L^2 norm and the scaled L^∞ norm of $\mathcal{G}$; $q(h)$. On the horizontal axis the sampling period h and on the vertical axis the size of the norm. Here it is clear that $q(h)$ is indeed a lower bound for $||(\mathcal{I} - \mathcal{HS})\mathcal{G}||$

To conclude this section, as long as the combination sampler-and-hold $\mathcal{HS}$ is optimal, it is LCTI and has a transfer function $F(s)$ on the imaginary axis. This means that the L^2 norm of the error system can be calculated in a nice way for the minimizing combination $\mathcal{HS}$.
For a monotonically decreasing frequency response $|G(\mathrm{i}\omega)|^2$ of the generator $\mathcal{G}$ the minimizing combination $\mathcal{HS}$ filters the first Nyquist band N_1. In this case the norm can be calculated using either Equation (3.5) or Equation (3.8) depending on whether the matrix A of the transfer function $G(s)$ is stable or not.
If the response is not monotonically decreasing the optimal sampler-and-hold combination can be constructed by the concept of folding. This will cause $\mathcal{HS}$ to filter a

finite number of frequency bands from the frequency response. In this case the norm can be calculated using Equation (3.9).

4 Frequency Power Response

4.1 Introduction

This section will study another kind of problem as studied in Section 3. Instead of minimizing the L^2 norm

$$||(\mathcal{I} - \mathcal{HS})\mathcal{G}||_{L^2}$$

over the combination of sampler-and-hold $\mathcal{HS}$, the sampler is always the ideal sampler (thus fixed) and the norm will be minimized over all possible LDTI and stable holds $\mathcal{H}$. This means that (almost always) the combination $\mathcal{HS}$ is not LCTI, but only LDTI. Therefore the L^2 norm of the error system

$$||(\mathcal{I} - \mathcal{HS})\mathcal{G}||_{L^2} = \sqrt{\frac{1}{\pi}\int_0^\infty |(I - F(\mathrm{i}\omega))G(\mathrm{i}\omega)|^2 \,\mathrm{d}\omega}$$

is defined, but it can not be calculated in this way because the transfer function $F(s)$ of $\mathcal{HS}$ does not exist. In order to be able to minimize the error system, in this section another form of the L^2 system norm will be derived.

4.2 Frequency Power Response

This subsection will introduce the set up for a possible new expression of the L^2 system norm. This expression is based on the *Frequency Power Response* (FPR) of the system.

Definition 4.2.1. For an LDTI system $\mathcal{G}$ the *Frequency Power Response* (FPR) is defined as

$$P_{\mathcal{G}}(\omega) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\mathcal{G}u_\omega(t)|^2 \,\mathrm{d}t \tag{4.1}$$

where u_ω is defined as the *harmonic* input

$$u_\omega(t) := \mathrm{e}^{\mathrm{i}\omega t}. \tag{4.2}$$

□

The FPR is the LDTI alternative for what $|G(\mathrm{i}\omega)|^2$ is to an LCTI system $\mathcal{G}$. Note that for an LCTI system $\mathcal{G}$ the the FPR $P_{\mathcal{G}}$ equals $|G(\mathrm{i}\omega)|^2$. Before continuing with the FPR a small remark is in place.

Remark 4.2.2. Note that the response $\mathcal{G}u_\omega(t)$ has an h-periodic magnitude, i.e.

$$\begin{aligned}
|(\mathcal{G}u_\omega)(t+h)| &= |\mathcal{G}(\mathrm{e}^{\mathrm{i}\omega(t+h)})| \\
&= |\mathcal{G}(\mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega h})| \\
&= |\mathcal{G}(\mathrm{e}^{\mathrm{i}\omega t})\cdot \mathrm{e}^{\mathrm{i}\omega h}| \\
&= |\mathcal{G}(\mathrm{e}^{\mathrm{i}\omega t})|\cdot |\mathrm{e}^{\mathrm{i}\omega h}| \\
&= |(\mathcal{G}u_\omega)(t)|\cdot 1.
\end{aligned}$$

□

Remark 4.2.2 implies that the FPR of an LDTI system $\mathcal{G}$ reduces to

$$P_{\mathcal{G}}(\omega) = \frac{1}{h}\int_0^h |\mathcal{G}u_\omega(t)|^2 \,\mathrm{d}t. \tag{4.3}$$

Equation (4.3) is the formula of the FPR that will be used in further sections of this report.

Example 4.2.3. This example will show how the FPR will be used and that its interpretation is the same as $|G(\mathrm{i}\omega)|^2$ for an LCTI system.
Consider a sampling period $h = 1$ and the series of functions $\{f_k\}$ defined as

$$f_k(t,s) := \mathrm{e}^{-(s-kh)}\mathbb{1}_{[0,\infty)}(s-kh)\cdot\cos\left(\frac{2\pi}{h}t\right)$$

for $s, t \in \mathbb{R}$. Now define the kernel $g(t,s)$ of the system $\mathcal{G}$ as the concatenation of these functions

$$g(t,s) := f_k(t,s) \quad \text{if} \quad t \in [kh,(k+1)h).$$

Note that $g(t+lh, s+lh) = g(t,s)$. The mapping $y = \mathcal{G}u$ is given by

$$y(t) = \int_{-\infty}^{\infty} g(t,s)u(s)\,\mathrm{d}s.$$

Using Equation (4.3), the expression for the FPR of the LDTI system $\mathcal{G}$ can be derived

$$\begin{aligned}
P_{\mathcal{G}}(\omega) &= \frac{1}{h}\int_0^h \left|\int_{-\infty}^{\infty} g(t,s)\mathrm{e}^{-\mathrm{i}\omega s}\,\mathrm{d}s\right|^2 \mathrm{d}t \\
&= \frac{1}{h}\int_0^h \left|\int_{-\infty}^{\infty} f_0(t,s)\mathrm{e}^{-\mathrm{i}\omega s}\,\mathrm{d}s\right|^2 \mathrm{d}t \\
&= \frac{1}{h}\int_0^h \left|\int_{-\infty}^{\infty} \mathrm{e}^{-s}\mathbb{1}_{[0,\infty)}(s)\cdot\cos\left(\frac{2\pi}{h}t\right)\mathrm{e}^{-\mathrm{i}\omega s}\,\mathrm{d}s\right|^2 \mathrm{d}t \\
&= \frac{1}{h}\int_0^h \left|\cos\left(\frac{2\pi}{h}t\right)\int_0^{\infty} \mathrm{e}^{(-\mathrm{i}\omega-1)s}\,\mathrm{d}s\right|^2 \mathrm{d}t \\
&= \frac{1}{h}\int_0^h \left|\cos\left(\frac{2\pi}{h}t\right)\frac{1}{\mathrm{i}\omega+1}\right|^2 \mathrm{d}t \\
&= \frac{1}{h}\int_0^h \cos^2\left(\frac{2\pi}{h}t\right)\frac{1}{\omega^2+1}\,\mathrm{d}t \\
&= \frac{1}{2(\omega^2+1)}.
\end{aligned}$$

Figure 12 shows the plot of $P_{\mathcal{G}}(\omega)$. Note that this plot looks similar to the one of $|G(\mathrm{i}\omega)|^2$ in Example 3.2.1. □

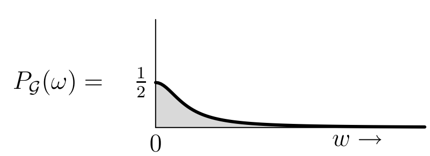


Figure 12: Plot of $P_{\mathcal{G}}(\omega)$

4.3 FPR Theorem

Before introducing the FPR Theorem first a Lemma is proven that will be used in the proof of the FPR Theorem. This lemma proves a property of the kernel $g(t,s)$ of an LDTI system $\mathcal{G}$. In fact it proves that integrating the kernel over a horizontal strip $(-\infty,\infty)\times[0,h]$ equals integrating the kernel over an vertical strip $[0,h]\times(-\infty,\infty)$.

Lemma 4.3.1. For an LDTI system $\mathcal{G}$ that is of the form

$$y=\mathcal{G}u \qquad y(t)=\int_{-\infty}^{\infty} g(t,s)u(s)\,\mathrm{d}s$$

with h-shift invariant kernel $g(t,s)$, the following holds

$$\int_0^h\int_{-\infty}^{\infty} g(t,s)\,\mathrm{d}s\,\mathrm{d}t=\int_0^h\int_{-\infty}^{\infty} g(s,t)\,\mathrm{d}s\,\mathrm{d}t.$$

Proof

Recall that the h-shift invariance of the kernel means that

$$g(t+kh,s+kh)=g(t,s)$$

for every $k\in\mathbb{Z}$. And thus

$$\int_0^h\int_0^h g(s+kh,t+kh)\,\mathrm{d}s\,\mathrm{d}t=\int_0^h\int_0^h g(s,t)\,\mathrm{d}s\,\mathrm{d}t.$$

Knowing this gives

$$\begin{aligned}
&\int_0^h\int_{-\infty}^{\infty} g(t,s)\,\mathrm{d}s\,\mathrm{d}t\\
&=\sum_{k\in\mathbb{Z}}\int_0^h\int_{kh}^{(k+1)h} g(t,s)\,\mathrm{d}s\,\mathrm{d}t\\
&=\sum_{k\in\mathbb{Z}}\int_0^h\int_0^h g(t,s+kh)\,\mathrm{d}s\,\mathrm{d}t\\
&=\sum_{k\in\mathbb{Z}}\int_0^h\int_0^h g(t-kh,s+kh-kh)\,\mathrm{d}s\,\mathrm{d}t\\
&=\sum_{k\in\mathbb{Z}}\int_0^h\int_0^h g(t-kh,s)\,\mathrm{d}s\,\mathrm{d}t\\
&=\sum_{k\in\mathbb{Z}}\int_{kh}^{(k+1)h}\int_0^h g(t,s)\,\mathrm{d}s\,\mathrm{d}t\\
&=\int_{-\infty}^{\infty}\int_0^h g(t,s)\,\mathrm{d}s\,\mathrm{d}t\\
&=\int_0^h\int_{-\infty}^{\infty} g(s,t)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

□

This lemma can be explained intuitively by the fact that $g(t,s)$ is h-shift invariant. The kernel can be seen as a large matrix with blocks of $h\times h$. This matrix is constant over all (sub)diagonals (because of the shift invariance). All rows of hight h contain all different existing blocks, as do all columns of width h. Therefore integrating over one row of hight h is the same as integrating over one column of width h. This is illustrated in Figure 13 where the vertical and horizontal strips are indicated in red an blue respectively. Both strips contain the same blocks.

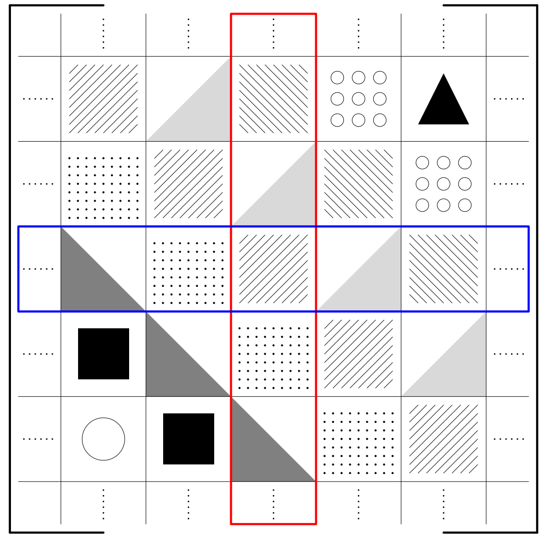


Figure 13: The kernel $\kappa(t,s)$ represented as a matrix. In blue a horizontal strip and in red a vertical strip

The FPR Theorem links the L^2 norm of an LDTI system $\mathcal{G}$ to the Frequency Power Response $P_{\mathcal{G}}$ of the system and is therefore very useful in order to calculate the L^2 system norm of an LDTI system. Recall that the L^2 norm for an LDTI system $\mathcal{G}$ was defined as

$$||\mathcal{G}||_{L^2}=\sqrt{\frac{1}{h}\int_0^h ||\mathcal{G}\delta(\cdot-t)||_{L^2}^2\ \mathrm{d}t}.$$

where $||\mathcal{G}\delta(\cdot-t)||_{L^2}$ is an L^2 signal norm whereas $||\mathcal{G}||_{L^2}$ is an L^2 system norm.

Theorem 4.3.2. [Frequency Power Response] *For an LDTI system $\mathcal{G}$ that is of the form*

$$y=\mathcal{G}u \qquad y(t)=\int_{-\infty}^{\infty} g(t,s)u(s)\,\mathrm{d}s$$

with h-shift invariant kernel $g(t,s)$, the following holds

$$||\mathcal{G}||_{L^2}^2=\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{\mathcal{G}}(\omega)\mathrm{d}\omega. \tag{4.4}$$

Proof

The system $\mathcal{G}$ is LDTI and it is given that

$$y(\tau)=\int_{-\infty}^{\infty} g(\tau,s)u(s)\,\mathrm{d}s$$

with $g(t,s)$ being the kernel of the system $\mathcal{G}$. This kernel is h-periodic:

$$g(t+lh, s+lh) = g(t,s)$$

for all $l \in \mathbb{Z}$. For a harmonic input u_ω the output $y_\omega := \mathcal{G}u_\omega$ has a squared magnitude

$$\begin{aligned}|y_\omega(t)|^2 &= \left|\int_{-\infty}^{\infty} g(t,s)\mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}s\right|^2 \\ &= \left|\int_{-\infty}^{\infty} g_t(s)\mathrm{e}^{-\mathrm{i}\omega s}\,\mathrm{d}s\right|^2 \\ &= |G_t(\mathrm{i}\omega)|^2 \qquad (4.5)\end{aligned}$$

where $g_t(s)$ is the function $g(t,s)$ where t is treated as a fixed variable (independent of s) and $G_t(\mathrm{i}\omega)$ is its Fourier transform.
By definition the L^2 norm for LDTI systems (see Subsection 2.4), $||\mathcal{G}||_{L^2}^2$ is given by

$$||\mathcal{G}||_{L^2}^2 = \frac{1}{h}\int_0^h ||\mathcal{G}\delta(\cdot - t)||_{L^2}^2\,\mathrm{d}t.$$

For $y = \mathcal{G}\delta(\cdot - t)$ the following expression exists

$$\begin{aligned}y(\tau) &= \int_{-\infty}^{\infty} g(\tau,s)\delta(s-t)\,\mathrm{d}s \\ &= g(\tau,t).\end{aligned}$$

This means that the L^2 system norm can be written as

$$\begin{aligned}||\mathcal{G}\delta(\cdot - t)||_{L^2}^2 &= ||y||_{L^2}^2 \\ &= \int_{-\infty}^{\infty} |y(\tau)|^2\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} |g(\tau,t)|^2\,\mathrm{d}\tau \\ &= ||g(\cdot,t)||_{L^2}^2\,.\end{aligned}$$

And thus reduces the expression for the L^2 system norm of $\mathcal{G}$ to

$$||\mathcal{G}||_{L^2}^2 = \frac{1}{h}\int_0^h ||g(\cdot,t)||_{L^2}^2\,\mathrm{d}t.$$

Substituting the definition of the L^2 norm for a (non-causal) signal $g(s,t)$ in this integral gives

$$||\mathcal{G}||_{L^2}^2 = \frac{1}{h}\int_0^h \int_{-\infty}^{\infty} |g(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t.$$

This can be written in the following form using Lemma 4.3.1

$$||\mathcal{G}||_{L^2}^2 = \frac{1}{h}\int_0^h \int_{-\infty}^{\infty} |g(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t. \qquad (4.6)$$

The right hand side from the equation in the theorem can be expressed in terms of the kernel as well

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{\mathcal{G}}(\omega)\,\mathrm{d}\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{h}\int_0^h |\mathcal{G}u_\omega(t)|^2\,\mathrm{d}t\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{h}\int_0^h |y_\omega(t)|^2\,\mathrm{d}t\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{h}\int_0^h |G_t(\mathrm{i}\omega)|^2\,\mathrm{d}t\,\mathrm{d}\omega\end{aligned}$$

where Equation (4.5) is used. By interchanging the order of integration, Parseval's Theorem (2.9) on page 5 can be applied on $g_t(s)$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{\mathcal{G}}(\omega)\,\mathrm{d}\omega &= \frac{1}{h}\int_0^h \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_t(\mathrm{i}\omega)|^2\,\mathrm{d}\omega\,\mathrm{d}t \\ &= \frac{1}{h}\int_0^h \int_{-\infty}^{\infty} |g_t(s)|^2\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

This means that the right hand side of the equation in the theorem can be written like

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{\mathcal{G}}(\omega)\,\mathrm{d}\omega = \frac{1}{h}\int_0^h \int_{-\infty}^{\infty} |g(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t. \qquad (4.7)$$

And thus the left hand side (4.6) of the theorem's equation equals the right hand side (4.7) □

5 Construction of Optimal Hold

5.1 Introduction

Just like in Section 4, this section will focus on the minimizing problem where the sampler is always the ideal sampler. This means that the L^2 norm of the error system

$$||(\mathcal{I} - \mathcal{HS})\mathcal{G}||_{L^2}$$

will be minimized over all LDTI and stable holds $\mathcal{H}$. Section 4 already showed that the L^2 system norm of an LDTI system can be expressed in terms of its kernel $\kappa(t, s)$ using the Frequency Power Response.

5.2 Harmonic input for $\mathcal{HS}$

In this subsection the input to the system $\mathcal{HS}$ is *harmonic*. This means that the input u in Figure 1 on page 2 is harmonic. So the input u of $\mathcal{S}$ is as defined in (4.2):

$$u_\omega(t) = \mathrm{e}^{\mathrm{i}\omega t}$$

for some $\omega \in \mathbb{R}$. Now the sampled-and-reconstructed signal y based on the harmonic input u_ω is defined as

$$y_\omega(t) := \mathcal{HS}\mathrm{e}^{\mathrm{i}\omega t}$$

for some $\omega \in \mathbb{R}$. The explicit form of y_ω will be derived in this subsection.

5.2.1 Sampler

The form of the sampled signal $\bar{u}$ will be determined for a harmonic input, applied to a general LDTI sampler $\mathcal{S}$:

$$\begin{aligned}\bar{u}_\omega[j] := \mathcal{S}u_\omega(t) &= \int_{-\infty}^{\infty} \psi(jh - s)u_\omega(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty} \psi(jh - s)\mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}s.\end{aligned}$$

The expression on the right hand side is almost the same as the Fourier transform (see Definition A.3.5). The next derivation shows that it is in fact a special form of the Fourier transform of $\psi(t)$.

$$\begin{aligned}\bar{u}_\omega[j] &= \int_{-\infty}^{\infty} \psi(jh - s)\mathrm{e}^{\mathrm{i}\omega s + \mathrm{i}\omega jh - \mathrm{i}\omega jh}\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty} \psi(jh - s)\mathrm{e}^{-\mathrm{i}\omega(jh-s) + \mathrm{i}\omega jh}\,\mathrm{d}s \\ &= \mathrm{e}^{\mathrm{i}\omega jh}\int_{-\infty}^{\infty} \psi(jh - s)\mathrm{e}^{-\mathrm{i}\omega(jh-s)}\,\mathrm{d}s \\ &= \mathrm{e}^{\mathrm{i}\omega jh}\Psi(\mathrm{i}\omega).\end{aligned}$$

5.2.2 Hold

The form of the sampled-and-reconstructed signal y_ω will be determined based on the sampled input $\bar{u}_\omega$

$$\begin{aligned}y_\omega(t) = \mathcal{H}\bar{u}_\omega[j] &= \sum_{j\in\mathbb{Z}} \phi(t - jh)\bar{u}_\omega[j] \\ &= \sum_{j\in\mathbb{Z}} \phi(t - jh)\mathrm{e}^{\mathrm{i}\omega jh}\Psi(\mathrm{i}\omega).\end{aligned}$$

By slightly modifying the equation on the right hand side, it can be seen as the Fourier transform of the lifted function $\breve{\phi}[j](\tau)$ where τ is the residual after dividing t by h ($t = mh + \tau$)

$$\begin{aligned}y_\omega(t) &= \sum_{j\in\mathbb{Z}} \phi(\tau + jh)\mathrm{e}^{-\mathrm{i}\omega jh}\Psi(\mathrm{i}\omega) \\ &= \sum_{j\in\mathbb{Z}} \breve{\phi}[j](\tau)\mathrm{e}^{-\mathrm{i}\omega jh}\Psi(\mathrm{i}\omega) \\ &= \breve{\phi}(\mathrm{e}^{\mathrm{i}\omega h}; \tau)\Psi(\mathrm{i}\omega).\end{aligned}$$

Using the Key Lifting Theorem (2.10) on page 5, this equation can be expressed in terms of the classical Fourier transform of $\psi(t)$

$$\breve{\phi}(\mathrm{e}^{\mathrm{i}\omega h}; \tau) = \frac{1}{h}\sum_{k\in\mathbb{Z}} \Phi(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\omega_k\tau}$$

for all $\tau \in [0, h)$. This means that if the input to the system $\mathcal{HS}$ is of the form u_ω, then the sampled-and-reconstructed output y_ω is of the form

$$y_\omega(t) = \left(\frac{1}{h}\sum_{k\in\mathbb{Z}} \Phi(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\omega_k\tau}\right)\Psi(\mathrm{i}\omega). \tag{5.1}$$

5.3 Calculation of the FPR

Although it has been shown that the L^2 norm of an LDTI system can be expressed using the FPR, it is not clear how to explicitly calculate the L^2 system norm when applying Theorem 4.3.2. This subsection will provide a solution for the calculation of the L^2 system norm of an LDTI system. In order to apply the FPR Theorem (4.4) to the problem of this report, the FPR of the system $(\mathcal{I} - \mathcal{HS})\mathcal{G}$ needs to be determined

$$\begin{aligned}P_{(\mathcal{I}-\mathcal{HS})\mathcal{G}}(\omega) &= \frac{1}{h}\int_0^h |(\mathcal{I} - \mathcal{HS})\mathcal{G}\mathrm{e}^{\mathrm{i}\omega t}|^2\,\mathrm{d}t \\ &= \frac{1}{h}\int_0^h |(\mathcal{I} - \mathcal{HS})\mathrm{e}^{\mathrm{i}\omega t}|^2 \cdot |G(\mathrm{i}\omega)|^2\,\mathrm{d}t \\ &= P_{\mathcal{I}-\mathcal{HS}}(\omega) \cdot |G(\mathrm{i}\omega)|^2.\end{aligned}$$

So in fact, only the FPR of the system $\mathcal{I} - \mathcal{HS}$ needs to be determined. In the end this will be multiplied with $|G(\mathrm{i}\omega)|^2$ in order to determine the FPR of the error system.

By definition, the FPR of $\mathcal{I}-\mathcal{HS}$ equals

$$P_{\mathcal{I}-\mathcal{HS}}(\omega) = \frac{1}{h}\int_0^h |(\mathcal{I}-\mathcal{HS})\mathrm{e}^{\mathrm{i}\omega t}|^2\,\mathrm{d}t = \frac{1}{h}\int_0^h |\mathrm{e}^{\mathrm{i}\omega t} - y_\omega(t)|^2\,\mathrm{d}t$$

where y_ω is as in Equation (5.1). Substituting this expression, dividing the whole equation by $\mathrm{e}^{\mathrm{i}\omega t}$ and using the definition of $\omega_k = \omega + 2\omega_{nyq}k$ leaves

$$\begin{aligned} &P_{\mathcal{I}-\mathcal{HS}}(\omega) \\ &= \frac{1}{h}\int_0^h |\mathrm{e}^{\mathrm{i}\omega t} - \left(\frac{1}{h}\sum_{k\in\mathbb{Z}}\Phi(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\omega_k t}\right)\Psi(\mathrm{i}\omega)|^2\,\mathrm{d}t \\ &= \frac{1}{h}\int_0^h |1 - \frac{1}{h}\sum_{k\in\mathbb{Z}}\Phi(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}2\pi kt/h}\Psi(\mathrm{i}\omega)|^2\,\mathrm{d}t. \end{aligned} \tag{5.2}$$

The derivation of the FPR of the system $\mathcal{I}-\mathcal{HS}$ can be extended which is stated in the following lemma.

Lemma 5.3.1. The FPR of $\mathcal{I}-\mathcal{HS}$ is of the form

$$\begin{aligned} &P_{\mathcal{I}-\mathcal{HS}}(\omega) \\ &= 1 - \frac{2}{h}\mathrm{Re}\left(\Phi(\mathrm{i}\omega)\Psi(\mathrm{i}\omega)\right) + \frac{1}{h^2}\left|\Psi(\mathrm{i}\omega)\right|^2\sum_{k\in\mathbb{Z}}\left|\Phi(\mathrm{i}\omega_k)\right|^2. \end{aligned} \tag{5.3}$$

Proof

To simplify the derivation of Equation (5.2) the following notations are introduced

$$\begin{aligned} B_k &:= \Phi(\mathrm{i}\omega_k)\Psi(\mathrm{i}\omega) \\ A_k &:= B_k\mathrm{e}^{\mathrm{i}2\pi kt/h}. \end{aligned}$$

This means the the FPR of $\mathcal{I}-\mathcal{HS}$ reduces to

$$P_{I-HS}(\omega) = \frac{1}{h}\int_0^h |1 - \frac{1}{h}\sum_{k\in\mathbb{Z}}A_k|^2\,\mathrm{d}t. \tag{5.4}$$

The summation of this integral can be written more explicitly, using the complex conjugate

$$\begin{aligned} &\left|1 - \frac{1}{h}\sum_{k\in\mathbb{Z}}A_k\right|^2 \\ &= \left(1 - \frac{1}{h}\sum_{k\in\mathbb{Z}}A_k\right)\overline{\left(1 - \frac{1}{h}\sum_{n\in\mathbb{Z}}A_n\right)} \\ &= 1 - \frac{1}{h}\sum_{k\in\mathbb{Z}}A_k - \frac{1}{h}\sum_{n\in\mathbb{Z}}\overline{A_n} + \frac{1}{h^2}\sum_{k\in\mathbb{Z}}A_k\sum_{n\in\mathbb{Z}}\overline{A_n} \\ &= 1 - \frac{2}{h}\mathrm{Re}\left(\sum_{k\in\mathbb{Z}}A_k\right) + \frac{1}{h^2}\sum_{k=n}A_k\overline{A_n} + \frac{1}{h^2}\sum_{k\neq n}A_k\overline{A_n}. \end{aligned}$$

Note that the complex conjugate of A_k is

$$\overline{A_k} = \overline{B_k}\mathrm{e}^{-\mathrm{i}2\pi kt/h}$$

which means that

$$A_k\overline{A_k} = B_k\overline{B_k}\mathrm{e}^{\mathrm{i}2\pi(k-n)t/h}.$$

Substituting this in the equation of the FPR (5.4) gives the somewhat more complicated expression

$$\begin{aligned} P_{\mathcal{I}-\mathcal{HS}}(\omega) = \frac{1}{h}\int_0^h 1 - \frac{2}{h}\mathrm{Re}\left(\sum_{k\in\mathbb{Z}}A_k\right) \\ + \frac{1}{h^2}\sum_{k=n}A_k\overline{A_n} + \frac{1}{h^2}\sum_{k\neq n}A_k\overline{A_n}\,\mathrm{d}t \end{aligned} \tag{5.5}$$

nevertheless, this expression can be simplified extensively. Consider the following integral for all $k \in \mathbb{Z}$.
For $k \neq 0$:

$$\begin{aligned} \int_0^h \mathrm{e}^{\mathrm{i}2\pi kt/h}\,\mathrm{d}t &= \left[\frac{h}{\mathrm{i}2\pi k}\mathrm{e}^{\mathrm{i}2\pi kt/h}\right]_0^h \\ &= \frac{h}{\mathrm{i}2\pi k}\left(\mathrm{e}^{\mathrm{i}2\pi k} - \mathrm{e}^0\right) \\ &= \frac{h}{\mathrm{i}2\pi k}\left(\cos(2\pi k) + \mathrm{i}\sin(2\pi k) - \mathrm{e}^0\right) \\ &= \frac{h}{\mathrm{i}2\pi k}\left(1 - \mathrm{e}^0\right) \\ &= 0. \end{aligned}$$

And for $k = 0$:

$$\begin{aligned} \int_0^h \mathrm{e}^{\mathrm{i}2\pi kt/h}\,\mathrm{d}t &= \int_0^h \mathrm{e}^0\,\mathrm{d}t \\ &= h. \end{aligned}$$

This means that the last term in the integral (5.5) equals zero. Since in this last term the $k-n$ of the exponential is always unequal to zero, therefore by the derivation above, this last term is always zero. In combination with the expression for B_k the FPR of $\mathcal{I}-\mathcal{HS}$ now reduces to

$$\begin{aligned} P_{\mathcal{I}-\mathcal{HS}}(\omega) &= \frac{1}{h}\int_0^h 1 - \frac{2}{h}\mathrm{Re}\left(\sum_{k\in\mathbb{Z}}A_k\right) + \frac{1}{h^2}\sum_{k=n}A_k\overline{A_n}\,\mathrm{d}t \\ &= 1 - \frac{1}{h}\int_0^h \frac{2}{h}\mathrm{Re}\left(\sum_{k\in\mathbb{Z}}A_k\right)\mathrm{d}t + \frac{1}{h^2}\sum_{k\in\mathbb{Z}}|B_k|^2 \end{aligned}$$

Now substituting the expression for A_k and adapting the integration into the summation leaves

$$\begin{aligned} &P_{\mathcal{I}-\mathcal{HS}}(\omega) \\ &= 1 - \frac{2}{h^2}\mathrm{Re}\left(\int_0^h \sum_{k\in\mathbb{Z}}B_k\mathrm{e}^{\mathrm{i}2\pi kt/h}\,\mathrm{d}t\right) + \frac{1}{h^2}\sum_{k\in\mathbb{Z}}|B_k|^2 \\ &= 1 - \frac{2}{h^2}\mathrm{Re}\left(\sum_{k\in\mathbb{Z}}\int_0^h B_k\mathrm{e}^{\mathrm{i}2\pi kt/h}\,\mathrm{d}t\right) + \frac{1}{h^2}\sum_{k\in\mathbb{Z}}|B_k|^2. \end{aligned}$$

The definitions for A_k and B_k as introduced in the beginning of this subsection can be substituted in order to get an expression in terms of the sampler- and hold function. In fact not the actual sampler- and hold functions are used, but their Fourier transforms. Note that ω_0 is simply ω and hence:

$$
\begin{aligned}
&P_{\mathcal{I}-\mathcal{HS}}(\omega)\\
&= 1 - \frac{2}{h^2}\mathrm{Re}\,(hB_o) + \frac{1}{h^2}\sum_{k\in\mathbb{Z}} |B_k|^2\\
&= 1 - \frac{2}{h}\mathrm{Re}\,(\Phi(\mathrm{i}\omega_0)\Psi(\mathrm{i}\omega)) + \frac{1}{h^2}\sum_{k\in\mathbb{Z}} |\Phi(\mathrm{i}\omega_k)|^2\,|\Psi(\mathrm{i}\omega)|^2\\
&= 1 - \frac{2}{h}\mathrm{Re}\,(\Phi(\mathrm{i}\omega)\Psi(\mathrm{i}\omega)) + \frac{1}{h^2}\,|\Psi(\mathrm{i}\omega)|^2\sum_{k\in\mathbb{Z}} |\Phi(\mathrm{i}\omega_k)|^2\,. \quad \square
\end{aligned}
$$

5.4 Find the optimal Hold

Recall that the goals is to minimize the L^2 norm of the error system over all stable and LDTI holds:

$$\min_{\mathcal{H}} ||(\mathcal{I}-\mathcal{HS})\mathcal{G}||_{L^2} \tag{5.6}$$

Theorem 4.3.2 provides an expression of the L^2 norm for LDTI systems in terms of the FPR (4.4) and Subsection 5.3 provides the following expression for the FPR of the error system

$$P_{(\mathcal{I}-\mathcal{HS})\mathcal{G}}(\omega) = P_{\mathcal{I}-\mathcal{HS}}(\omega)\cdot |G(\mathrm{i}\omega)|^2.$$

These two expressions in combination with Lemma 5.3.1 will be used in this subsection to find the optimal hold for a given sampler. In this case the sampler is assumed to be the ideal sampler

$$\psi(t) = \delta(t) \quad \Leftrightarrow \quad \Psi(\mathrm{i}\omega) = 1$$

since this one is used frequently in signal processing. Of course, the derivation can be performed for a general sampler, but this makes it unnecessarily complicated. The optimal hold will depend on the generator $\mathcal{G}$.

Theorem 5.4.1. *The hold $\mathcal{H}_*$ that minimizes the L^2 norm of the error system (5.6) for the ideal sampler $\psi(t) = \delta(t)$, is*

$$\Phi_*(\mathrm{i}\omega) = \frac{h\,|G(\mathrm{i}\omega)|^2}{\sum_{k\in\mathbb{Z}}|G(\mathrm{i}\omega_k)|^2}. \tag{5.7}$$

Proof

First the expression for the L^2 norm of the error system is rewritten

$$
\begin{aligned}
||(\mathcal{I}-\mathcal{HS})\mathcal{G}||^2_{L^2} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{(\mathcal{I}-\mathcal{HS})\mathcal{G}}(\omega)\,\mathrm{d}\omega\\
&= \frac{1}{2\pi}\int_{-\omega_{nyq}}^{\omega_{nyq}} \sum_{n\in\mathbb{Z}} P_{(\mathcal{I}-\mathcal{HS})\mathcal{G}}(\omega_n)\,\mathrm{d}\omega
\end{aligned}
$$

where ω_n is as always defined as

$$\omega_n := \omega + 2n\omega_{nyq}$$

for any given ω. This transformation of the integral is allowed because in this way the function is still integrated over all frequencies. For the sake of notational convenience the following notations are introduced

$$
\begin{aligned}
\Phi_n &:= \Phi(\mathrm{i}\omega_n)\\
G_n &:= G(\mathrm{i}\omega_n).
\end{aligned}
$$

Using Equation (5.3) the summation over the FPR of the error system can be written as

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} P_{(\mathcal{I}-\mathcal{HS})\mathcal{G}}(\omega_n)\\
&= \sum_{n\in\mathbb{Z}}\left(\left[1 - \frac{2}{h}\mathrm{Re}\,(\Phi_n) + \frac{1}{h^2}\sum_{k\in\mathbb{Z}}|\Phi_{n+k}|^2\right]|G_n|^2\right).
\end{aligned} \tag{5.8}
$$

Now a few remarks are in place. First of all, the optimal hold will be a real function, in fact it will be a *positive real* function. If this is not the case, there exists a positive real function that makes Equation (5.8) smaller. Therefore some parts can be simplified:

$$
\begin{aligned}
\mathrm{Re}\,(\Phi_n) &= \Phi_n\\
|\Phi_{n+k}|^2 &= \Phi^2_{n+k}.
\end{aligned}
$$

Secondly for any *fixed* n the following holds

$$\sum_{k\in\mathbb{Z}}\Phi^2_{n+k} = \sum_{k\in\mathbb{Z}}\Phi^2_k$$

because it is an infinite sum. This reduces the formula

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} P_{(\mathcal{I}-\mathcal{HS})\mathcal{G}}(\omega_n)\\
&= \sum_{n\in\mathbb{Z}}\left(\left[1 - \frac{2}{h}\Phi_n + \frac{1}{h^2}\sum_{k\in\mathbb{Z}}\Phi^2_k\right]|G_n|^2\right)\\
&= \sum_{n\in\mathbb{Z}}|G_n|^2 - \frac{2}{h}\sum_{n\in\mathbb{Z}}\Phi_n|G_n|^2 + \frac{1}{h^2}\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\Phi^2_k|G_n|^2.
\end{aligned} \tag{5.9}
$$

Now the optimal hold function for one specific frequency ω_m can be found by differentiating Equation (5.9) with respect to Φ_m and equalling this to zero

$$
\begin{aligned}
&\frac{\partial}{\partial\Phi_m}\sum_{n\in\mathbb{Z}} P_{(\mathcal{I}-\mathcal{HS})\mathcal{G}}(\omega_n) = 0\\
&\Leftrightarrow\\
&0 - \frac{2}{h}|G_m|^2 + \frac{2}{h^2}\Phi_m\sum_{n\in\mathbb{Z}}|G_n|^2 = 0\\
&\Leftrightarrow\\
&\Phi_m = \frac{h\,|G_m|^2}{\sum_{n\in\mathbb{Z}}|G_n|^2}.
\end{aligned}
$$

This means that the optimal hold $\mathcal{H}_*$ is

$$\Phi_*(\mathrm{i}\omega) = \frac{h\ |G(\mathrm{i}\omega)|^2}{\sum_{k\in\mathbb{Z}} |G(\mathrm{i}\omega_k)|^2}.$$

□

This is the same result as shown in [4] but the derivation is different.

5.5 Comparison with other Holds

The great advantage of the optimal hold (5.7) compared to standard holds like the zero- or first-order hold, is that it changes if the generator $\mathcal{G}$ changes. This will be demonstrated by a few examples. To compare the optimal hold, several frequently used holds are exploited in these examples. Recall the zero-order hold

$$\phi(t) = \mathbb{1}_{[0,h]}(t)$$

the first-order hold

$$\phi(t) = \left(1 - \frac{|t|}{h}\right) \mathbb{1}_{[-h,h]}(t)$$

and the sinc hold

$$\phi(t) = \mathrm{sinc}(t)$$

from Subsection 2.1.2.

Example 5.5.1. Consider the same system $\mathcal{G}$ as in Example 2.5.1 on page 8:

$$G(s) = \frac{1}{s+1}.$$

The L^2 system norm of $(\mathcal{I} - \mathcal{HS})\mathcal{G}$ will be calculated for different holds using Theorem 4.3.2. The sampler is assumed to be the ideal sampler $\psi(t) = \delta(t)$ and the sampling period h equals 1. The optimal hold is compared with the zero-order hold, the first-order hold and the sinc hold.

Figure 14 shows plots of $P_{(\mathcal{I}-\mathcal{HS})\mathcal{G}}(\omega)$ for the different holds. Table 1 shows the L^2 norm of the error system for the different holds.

Hold	$\|\|(\mathcal{I} - \mathcal{HS})\mathcal{G}\|\|^2_{L^2}$
Zero-order	0.3655
First-order	0.1572
Sinc	0.1942
Optimal	0.1549

Table 1: L^2 norm of the error system for different holds for the system $G(s) = \frac{1}{s+1}$

In order to get an indication how much energy of the original system $\mathcal{G}$ is preserved by sampling and reconstruction, the following formula is exploited

$$\frac{||\mathcal{G}||^2_{L^2} - ||(\mathcal{I} - \mathcal{HS})\mathcal{G}||^2_{L^2}}{||\mathcal{G}||^2_{L^2}} \times 100\% = \frac{0.5 - 0.1549}{0.5} \times 100\% = 69.0\%.$$

So 69.0% of the system's energy is preserved by sampling and reconstruction if a sampling period $h = 1$ in combination with the optimal hold is used. In Example 3.2.1 the preserved energy with optimal $\mathcal{HS}$ was 80.4% and now with this fixed sampler 69.0% is the best one can do. Another explanation of the formula above is that if the input signal is white noise (see Definition A.1.11) then 69.0% of its power is preserved by sampling and reconstruction with the optimal hold in combination with this fixed sampler and a sampling period $h = 1$. □

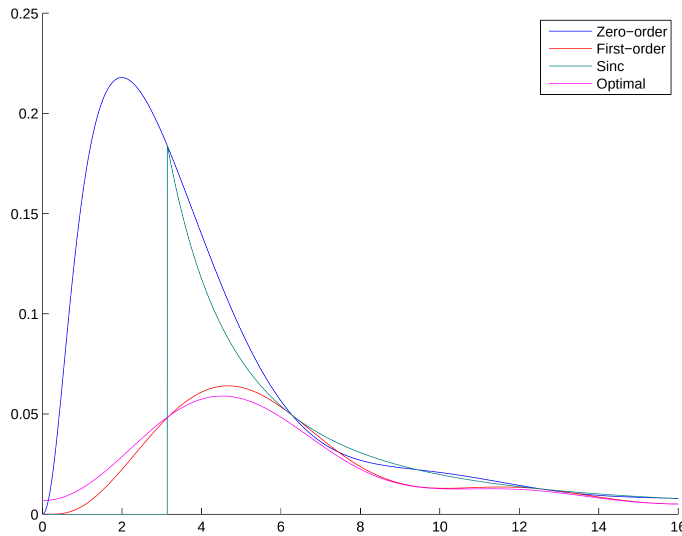


Figure 14: Plots of the FPR of $(\mathcal{I} - \mathcal{HS})\mathcal{G}$ for $G(s) = \frac{1}{s+1}$, the ideal sampler and different holds. In blue the zero-order hold, in red the first-order hold, in green the sinc hold and in pink the optimal hold

Example 5.5.2. Consider the same system $\mathcal{G}$ as in Example 3.3.1 on page 15:

$$G(s) = \frac{1}{(s+0.2)^2+1}.$$

Similar to Example 5.5.1, the L^2 system norm of $(\mathcal{I} - \mathcal{HS})\mathcal{G}$ will be calculated for different holds using Theorem 4.3.2. The optimal hold is compared with the same holds and again the sampler is assumed to be the ideal sampler. In this example is the sampling period $h = 4$, unlike Example 5.5.1.

Figure 15 shows plots of $P_{(\mathcal{I}-\mathcal{HS})\mathcal{G}}(\omega)$ for the different holds. Table 2 shows the L^2 norm of the error system for the different holds.

In order to get an indication how much energy of the original system $\mathcal{G}$ is preserved by sampling and reconstruction, the following formula is exploited

$$\frac{||\mathcal{G}||^2_{L^2} - ||(\mathcal{I} - \mathcal{HS})\mathcal{G}||^2_{L^2}}{||\mathcal{G}||^2_{L^2}} \times 100\% = \frac{1.2019 - 0.5492}{1.2019} \times 100\% = 54.3\%.$$

Hold	$\|\|(\mathcal{I}-\mathcal{HS})\mathcal{G}\|\|_{L^2}^2$
Zero-order	2.2918
First-order	1.0238
Sinc	1.6379
Optimal	0.5492

Table 2: L^2 norm of the error system for different holds for the system $G(s) = \frac{1}{(s+0.2)^2+1}$

So 54.3% of the system's energy is preserved by sampling and reconstruction if a sampling period $h = 4$ in combination with the optimal hold is used. In Example 3.3.1 the preserved energy with optimal $\mathcal{HS}$ was 65.0% and now with this fixed sampler 54.3% is the best one can do.
Another explanation of the formula above is that if the input signal is white noise (see Definition A.1.11) then 54.3% of its power is preserved by sampling and reconstruction with the optimal hold in combination with this fixed sampler and a sampling period $h = 4$.
In this example the L^2 norm of the error system for the sinc hold and the zero-order hold are larger than the L^2 norm of the system itself $\|\|\mathcal{G}\|\|_{L^2}^2 = 1.2019$. This means that it is better to take the zero hold $\phi(t) := 0, \forall t \in \mathbb{R}$ because then the L^2 norm of the error system is smaller. □

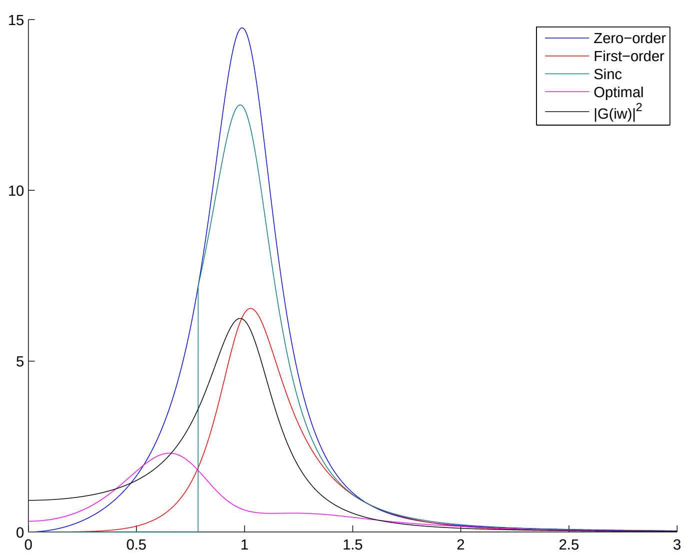


Figure 15: Plots of the FPR of $(\mathcal{I}-\mathcal{HS})\mathcal{G}$ for $G(s) = \frac{1}{(s+0.2)+1}$, the ideal sampler and different holds. In blue the zero-order hold, in red the first-order hold, in green the sinc hold, in pink the optimal hold and in black the response of the system itself $|G(\mathrm{i}\omega)|^2$

6 Concluding Remarks

In this report it has been achieved to calculate the L^2 norm of the error system for the optimal combination of sampler and hold for both monotonically decreasing and non-monotically decreasing frequency response $|G(\mathrm{i}\omega)|^2$. Furthermore the graphical interpretation of this optimal combination is presented.
Next the Frequency Power Response is introduced in order to calculate the L^2 norm of an LDTI system and it has been proven that the L^2 norm of an LDTI system can indeed be expressed in terms of this FPR. Using this expression of the L^2 norm of an LDTI system, the optimal hold is constructed for a given sampler and sampling period.
And last, this optimal hold is compared to several other holds. For different holds the L^2 norm of the error system is calculated and the graphical interpretation (using the FPR) of the error system for these holds is presented.

Appendices

A Definitions and Theorems

A.1 Classical system theory

Definition A.1.1. A system is an *input/output system* if

- the input u is not restricted by the system.
- the ouput y is completely determined by the input u.

Figure 16 is an illustration of an input/output system. □

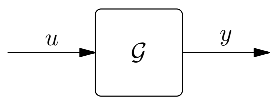


Figure 16: A system with input u, output y and mapping $\mathcal{G}$

Definition A.1.2. A system $\mathcal{G}$ is said to be BIBO-*stable* if its output y is bounded as long as its input u is bounded as well:

$$|u(t)| < M_1 \quad \Rightarrow \quad |y(t)| < M_2.$$

□

Definition A.1.3. A system has a *state space representation* if it can be written in the following form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{A.1}$$

where A, B and C are real matrices. Figure 17 shows the block representation of the state space representation of the system. □

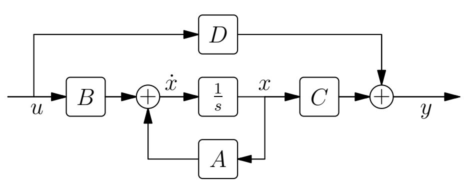


Figure 17: The state space respresentation of system (A.1) with input u, output y, real matrices A, B, C and D and where $\frac{1}{s}$ denotes a pure integrator

Definition A.1.4. Transforming system (A.1) to the Laplace domain gives a simple set of equations without derivatives (differentiating in the time domain reduces to multiplying by s in the Laplace domain):

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s). \end{aligned}$$

Substituting the solution of $X(s)$ of the first equation in the second equation (provided that the inverse of $(sI - A)$ exists) leads to a *transfer function* $G(s)$ that maps the input u on the output y:

$$\begin{aligned} Y(s) &= [C(sI - A)^{-1}B + D]U(s) \\ Y(s) &= G(s)U(s) \end{aligned}$$

which often has the following notation

$$y = G(s)u.$$

□

Definition A.1.5. For a real transfer matrix G the *conjugate* is defined as:

$$G^{\sim}(s) := [G(-s)]^T.$$

Furthermore this conjugate is used to computate the magnitude of the frequency response of the transfer function:

$$|G(\mathrm{i}\omega)|^2 = G^{\sim}(\mathrm{i}\omega)G(\mathrm{i}\omega)$$

□

Definition A.1.6. A system $\mathcal{G}$ is said to be *linear* if

$$\mathcal{G}(\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 \mathcal{G} y_1 + \lambda_2 \mathcal{G} y_2$$

for every two inputs y_1, y_2 and scalars λ_1, λ_2. □

Definition A.1.7. To define continuous time invariance, first the continuous time shift of a signal σ^τ is defined:

$$(\sigma^\tau f)(t) := f(\tau + t).$$

A system $\mathcal{G}$ is said to be *Linear Continuous Time Invariant* (LCTI) if it is linear (see Definition A.1.6) and if

$$\mathcal{G}(\sigma^\tau) = \sigma^\tau(\mathcal{G})$$

holds for every $\tau \in \mathbb{R}$. □

Definition A.1.8. To define discrete time invariance, the discrete time shift of a signal $\bar{\sigma}^k$ is defined:

$$(\bar{\sigma}^k f)(t) := f(kh + t)$$

for a fixed $h \in \mathbb{R}$.
A system $\mathcal{G}$ is said to be *Linear Discrete Time Invariant* (LDTI) if it is linear (see Definition A.1.6) and if

$$\mathcal{G}(\bar{\sigma}^k) = \bar{\sigma}^k(\mathcal{G})$$

holds for every $k \in \mathbb{Z}$. □

Definition A.1.9. A signal $f : \mathbb{R} \to \mathbb{C}$ is said to be causal if $f(t) = 0$ for all $t < 0$. □

Definition A.1.10. The *convolution* of two continuous signals $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ is the signal $(f * g) : \mathbb{R} \to \mathbb{C}$ defined as

$$(f * g)(t) := \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\mathrm{d}\tau.$$

□

Definition A.1.11. A signal $w(t)$ is said to be *white noise* if its expectation equals zero:

$$\mathbb{E}(w(t)) = 0$$

and its standard deviation equals σ:

$$\mathbb{E}(w(t)w(s)) = \sigma^2\delta(t-s).$$

□

A.2 Sampling theory

Definition A.2.1.

- The *Nyquist frequency* is defined as

$$\omega_{nyq} := \frac{\pi}{h}$$

where h is the sampling period of the sampler $\mathcal{S}$.

- The k^{th} *Nyquist band* is defined as the interval

$$N_k := [(k-1)\omega_{nyq}; k\omega_{nyq}) .$$

- The k^{th} *aliased frequency* of ω is defined as

$$\omega_k := \omega + 2\omega_{nyq}k$$

with $k \in \mathbb{Z}$. □

Definition A.2.2. A sampling device $\mathcal{S}$ is assumed to be an LDTI device transforming a function (or analog signal) $u(t) : \mathbb{R} \to \mathbb{C}^{n_u}$ into a function $\bar{u}[j] : \mathbb{Z} \to \mathbb{C}^{n_{\bar{u}}}$ where n_u and $n_{\bar{u}}$ are the dimensions of the input signal u and the sampled output signal $\bar{u}$ respectively. Assuming that

$$\mathcal{S}(u(\cdot - h)) = (\mathcal{S}u)[\cdot - 1]$$

or equivalently

$$\mathcal{S}\sigma^{kh} = \bar{\sigma}^k \mathcal{S}$$

which can be seen as Analog/Discrete shift invariance, a general model for such a device is

$$\bar{u} = \mathcal{S}u : \quad \bar{u}[j] = \int_{-\infty}^{\infty} \psi(jh - t)u(t)\mathrm{d}t, \qquad j \in \mathbb{Z}$$

for some $\psi(t)$, called the *sampling function* [3]. □

Definition A.2.3. A hold device $\mathcal{H}$ is assumed to be an LDTI device transforming a function (or discrete signal) $\bar{u}[j] : \mathbb{Z} \to \mathbb{C}^{n_{\bar{u}}}$ into a function $y(t) : \mathbb{R} \to \mathbb{C}^{n_y}$ where $n_{\bar{u}}$ and n_y are the dimensions of the sampled input signal $\bar{u}$ and the output signal y respectively. Assuming that

$$\mathcal{H}(\bar{u}[\cdot - 1]) = (\mathcal{H}\bar{u})(\cdot - h)$$

or equivalently

$$\mathcal{H}\bar{\sigma}^k = \sigma^{kh}\mathcal{H}$$

which can be seen as Discrete/Analog shift invariance, a general model for such a device is

$$y = \mathcal{H}\bar{u} : \quad y(t) = \sum_{j \in \mathbb{Z}} \phi(t - jh)\bar{u}[j], \quad t \in \mathbb{R}$$

for some $\phi(t)$, called the *sampling function* [3]. □

Theorem A.2.4. (Shannon) *If a function $f(t)$ contains no frequencies higher than ω_N (in radians per time unit), it is completely determined by giving its samples $f(kh)$, $k \in \mathbb{Z}$ at a series of points spaced h time units apart. The reconstruction formula is*

$$\begin{aligned} f(t) &= \sum_{k \in \mathbb{Z}} f(kh)\mathrm{sinc}\left(\frac{t}{h} - k\right) \\ &= \sum_{k \in \mathbb{Z}} f(kh)\frac{\sin(\pi(\frac{t}{h} - k))}{\pi(\frac{t}{h} - k)} \end{aligned} \qquad \text{(A.2)}$$

in which the equidistant samples $f(kh)$ can be seen as coefficients of a shifted sinc-function. [6] □

A.3 General mathematics

Definition A.3.1. A real-valued function

$$||\cdot|| : \mathcal{F} \to \mathbb{R}$$

is called a *norm* on a vector space $\mathcal{F}$ if it satisfies

1. $||f|| \geq 0$ for all $f \in \mathcal{F}$, and $||f|| = 0 \Leftrightarrow f = 0$
2. $||\lambda f|| = |\lambda| \cdot ||f||$ for all $\lambda \in \mathbb{R}$ and $f \in \mathcal{F}$
3. $||f + g|| \leq ||f|| + ||g||$ for all $f, g \in \mathcal{F}$.

If all these assumptions hold for all elements of $\mathcal{F}$, then $\mathcal{F}$ is called a *vector space*. □

Definition A.3.2. A complex-valued function

$$\langle \cdot, \cdot \rangle : \mathcal{F} \times \mathcal{F} \to \mathbb{C}$$

is called a *(complex) inner product* on a vector space $\mathcal{F}$ if it satisfies

- $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$ for all $f, g, h \in \mathcal{F}$
- $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle$ for all $\lambda \in \mathbb{R}$ and $f, g \in \mathcal{F}$
- $\langle f, g \rangle = \overline{\langle g, f \rangle}$ for all $f, g \in \mathcal{F}$
- $\langle f, f \rangle > 0$ if $f \neq 0$, while $\langle f, f \rangle = 0$ for $f = 0$.

If all these assumptions hold for all elements of $\mathcal{F}$, then $\mathcal{F}$ is called an *inner product space*. □

Note that every inner product space is an normed space as well; by taking the norm of an element as the square root of the inner product of the element with itself:

$$||\cdot|| = \sqrt{\langle \cdot, \cdot \rangle}.$$

Definition A.3.3. The L^2-space is a vector space consisting of all functions $f : [a, b] \to \mathbb{R}$ for certain a and $b \in \mathbb{R}$. A function $f : [a, b] \to \mathbb{R}$ is an element of $L^2[a, b]$ if

$$\int_a^b |f(t)|^2 \, \mathrm{d}t < \infty.$$

If additionally a and b equal $-\infty$ and ∞ respectively then the space is denoted as $L^2(\mathbb{R})$.
The standard L^2-norm of $f \in L^2[a, b]$ is the mapping $||\cdot||_{L^2} : L^2[a, b] \to \mathbb{R}$ defined as

$$||f||_{L^2} := \sqrt{\int_a^b |f(t)|^2 \, \mathrm{d}t}.$$

□

Definition A.3.4. Another frequently used norm in signal theory is the L^∞-norm.
A function $f : \mathbb{R} \to \mathbb{R}$ is an element of $L^\infty(\mathbb{R})$ if

$$\operatorname*{ess\,sup}_{t \in \mathbb{R}} |f(t)| < \infty.$$

The standard L^∞-norm of $f \in L^\infty(\mathbb{R})$ is defined as

$$||f||_{L^\infty} := \operatorname*{ess\,sup}_{t \in \mathbb{R}} |f(t)|.$$

□

Definition A.3.5. Let $f(t)$ be an absolutely integrable function. Then the following Fourier integral:

$$F(\mathrm{i}\omega) = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t$$

exists for all $\omega \in \mathbb{R}$ and its inverse is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\mathrm{i}\omega) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega.$$

□

The Fourier transform of $f(t)$ is always denoted by a capital: $F(\mathrm{i}\omega)$.

Theorem A.3.6. (Parseval) *Let $f(t)$ be a square integrable function, then*

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\mathrm{i}\omega)|^2 \, \mathrm{d}\omega. \tag{A.3}$$

□

Definition A.3.7. The principal logarithm of a square matrix A without eigenvalues on the branch cut (the negative real axis, including zero) is defined as the unique matrix $B := \log(A)$ for which $e^B = A$ and whose spectrum lies in the open horizontal strip $\{z \in \mathbb{C} | -\pi < \operatorname{Im}(z) < \pi\}$ of the complex plane [2, 5]. □

Definition A.3.8. A matrix A is called *stable* if it is square and all its eigenvalues lie in the left open half of the complex plane $\mathbb{C}$:

$$\operatorname{Re}(\lambda_i) < 0 \quad \forall i.$$

□

A.4 Lifting theory

Definition A.4.1. For any signal $f : \mathbb{R} \rightarrow \mathbb{C}^{n_f}$, the lifting $\breve{f} : \mathbb{Z} \rightarrow \{[0, h) \rightarrow \mathbb{C}^{n_f}\}$ is the sequence of functions $\{\breve{f}[k]\}$ defined as [3]

$$\breve{f}[k](\tau) = f(kh + \tau), \quad k \in \mathbb{Z}, \tau \in [0, h).$$

□

Definition A.4.2. The (lifted) Fourier transform $\mathfrak{F}\{\breve{f}\}$ of a lifted function $\breve{f}$ is defined as

$$\mathfrak{F}\{\breve{f}\} = \breve{f}(\mathrm{e}^{\mathrm{i}\omega h}; \tau) := \sum_{k \in \mathbb{Z}} \breve{f}\,[k]\,(\tau)\mathrm{e}^{-\mathrm{i}\omega k h}$$

where the frequency ω is 2π-periodic [3]. □

Theorem A.4.3. (Key lifting formula) *Let $f \in L^2(\mathbb{R})$. Then the lifted Fourier transform can be determined from the classic Fourier transform via:*

$$\breve{f}(\mathrm{e}^{\mathrm{i}\omega h}; \tau) = \frac{1}{h} \sum_{k \in \mathbb{Z}} F(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\omega_k \tau}$$

for all $\tau \in [0, h)$, where $\omega_k = \omega + 2\omega_N k$. And its inverse is given by [3]

$$F(\mathrm{i}\omega_k) = \int_0^h \breve{f}(\mathrm{e}^{\mathrm{i}\omega h}; \tau)\mathrm{e}^{-\mathrm{i}\omega_k \tau} \mathrm{d}\tau.$$

□

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