# 1 $T+B$ is never invertible for $\|B\|$ sufficiently small

## 1.1 Lay-out

In this example we construct a bounded operator $T$ such that $T+B$ is not invertible for all bounded $B$'s with norm sufficiently small. More theory on this example can be found in Section IV.13 of Taylor and Lay [1].

## 1.2 Example

As Hilbert space we choose $H = \ell^2(\mathbb{N})$, and we define

$$(Tx)(n) = x(n+1), \qquad n \in \mathbb{N} \tag{1}$$

which is the left shift on $H$. The following observations are easy:

1. $Tx = 0$ if and only if $x = \alpha\phi_1$ for same scalar $\alpha$, where $\phi_1 = (1, 0, 0, \dots,)$.
2. $T^*$ is the right shift, and
$$TT^* = I. \tag{2}$$
3. $\|T\| = \|T^*\| = 1$.

Let $B$ be a bounded operator on $H$ with norm less than one, then we know that $I + T^*B$ is boundedly invertible (use the last observation). We show that $y_B := (I + T^*B)^{-1}\phi_1$ is an eigenvector of $T + B$.

$$y_B = (I + T^*B)^{-1}\phi_1 \iff (I + T^*B)y_B = \phi_1.$$

Multiplying both sides by $T$ gives

$$0 = T\phi_1 = T(I + T^*B)y_B = (T + TT^*B)y_B = (T + B)y_B, \tag{3}$$

where we have used equation (2). From equation (3) we conclude that $y_b$ lies in the kernel of $T + B$, and hence $T + B$ is not invertible.

# References


[1] A.E. Taylor and D.C. Lay, *Introduction to Functional Analysis*, second edition, John Wiley and Sons, New York, 1980.