

1 stability of Contraction semigroups

Question Let A be the generator of an exponentially stable, contraction semigroup on the Hilbert space H . Furthermore, assume that Q is bounded, and dissipative, i.e., $Q + Q^* \leq 0$. Is the semigroup generated by $A + Q$ exponentially stable?

Let us first prove some simple things. If A satisfies

$$A + A^* \leq -\varepsilon I$$

for some $\varepsilon > 0$, then the answer is yes. This follows from the fact that

$$(A + Q) + (A + Q)^* \leq -\varepsilon I.$$

Standard Lyapunov theory gives that $A + Q$ generates an exponentially stable semigroup.

In general the answer is no. We shall build an example showing this.

Let n be an element in \mathbb{N} . On \mathbb{R}^{2n} we define A_n as

$$A_n = \begin{bmatrix} -1 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & 0 \\ \cdots & & & & & & \\ \cdots & & & & & & \\ 0 & \cdots & & & & 0 & -1 \end{bmatrix} \quad (1)$$

Hence n blocks of $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$. If I am not mistaken this satisfies $A_n + A_n^* \leq 0$. Furthermore, since the n blocks are un-coupled, we have that

$$\|e^{A_n t}\| \leq M e^{-t} \quad (2)$$

for some M independent of n . Finally, note that A_n is bounded uniformly with respect to n .

Now we construct Q_n similarly.

$$Q_n = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 \\ \cdots & & & & & & & \\ \cdots & & & & & & & \\ 0 & \cdots & & & & & 0 & -1 \end{bmatrix} \quad (3)$$

Hence it is A_n but the first and last row are different. The norm of Q_n can be bounded independently of n . (same holds for A_n).

Now $A_n + Q_n$ is

$$A_n + Q_n = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -2 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 & \cdots & 0 \\ \cdots & & & & & & & \\ \cdots & & & & & & & \\ 0 & \cdots & & & & & 0 & -2 \end{bmatrix} \quad (4)$$

This is a Jordan block.

Note that this Jordan block is twice the Jordan block as used in the famous Datko paper to show that the growth bound and the spectral bound can be different. It is not hard to show that for any $t > 0$ and any $\delta \in (-1, 0)$, there exists an $n \in \mathbb{N}$ such that

$$\|e^{(A_n+Q_n)t}\| \geq e^{\delta t}. \quad (5)$$

Now we build our A and Q . As H we take ℓ^2 , and

$$A = \text{diag}(A_n), \quad Q = \text{diag}(Q_n). \quad (6)$$

By (2) we know that A generates an exponential stable semigroup. Furthermore, this is a contraction. Note that the same holds for Q . By (5) we see that $A + Q$ cannot be exponentially stable.