A boundedly invertible, Q bounded does not imply AQ densely defined.

Here we give an example of closed, boundedly invertible operator A and a bounded operator Q for which AQ on its natural domain is not densely defined.

Example
Take on the Hilbert space $L^2(0,1)$ the operator A

$$Af = \frac{df}{d\zeta}$$

on the domain

$$D(A) = \{f \in L^2(0,1) \mid f \text{ is absolutely continuous}, \frac{df}{d\zeta} \in L^2(0,1) \text{ and } f(1) = 0\}.$$

It is easy to see that A is densely defined, and $A - \lambda I$ is boundedly invertible for all $\lambda \in \mathbb{C}$. Thus in particular, A is closed.

As Q we choose

$$(Qf)(\zeta) = \int_0^\zeta f(\tau)d\tau.$$

Then it is easy to see that

$$\begin{aligned}
D(AQ) &:= \{f \in L^2(0,1) \mid Qf \in D(A)\} \\
&= \{f \in L^2(0,1) \mid \int_0^\zeta f(\tau)d\tau \text{ is absolutely continuous}, \\
&\qquad \frac{d}{d\zeta}\left[\int_0^\zeta f(\tau)d\tau\right] \in L^2(0,1) \text{ and } \int_0^1 f(\tau)d\tau = 0\} \\
&= \{f \in L^2(0,1) \mid \int_0^1 f(\tau)d\tau = 0\}.
\end{aligned}$$

Thus every $f \in D(AQ)$ is orthogonal to the constant function, and so we see that this domain is not dense.