Hamiltonian-based numerical methods for forced-dissipative climate prediction

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Abstract

In absence of forcing and dissipation, the atmosphere can be described within the framework of Hamiltonian dynamics. We introduce the parcel formulation for such a atmosphere, which forms the basis of our Hamiltonian numerical method.

Introduction

Although our climate is ultimately driven by (nonuniform) solar heating, many aspects of the flow can be understood qualitatively from forcing-free and frictionless dynamics. In this limit of zero forcing and dissipation, our weather system falls under the realm of ***Hamiltonian fluid dynamics*** and the flow preserves several conservation laws such as the conservation of energy and phase space structure. One would like to have a numerical forecasting scheme that can reproduce the correct flow structure for the limit of zero forcing and dissipation, as does the exact system. Many present long-term weather forecast models fail at this point. But the question remains, however:

Question:

Is it advantageous to use numerical schemes with a Hamiltonian core for realistic climate modeling?

The primitive equations

1. Model assumptions:

- fluid on rotating plane
- reversible thermodynamics ⇔ entropy conserved per fluid parcel
- hydrostatic balance
- statically stable atmosphere

2. Representation

- State of fluid given by the velocity, density and entropy. Pressure eliminated as dependent variable.
- Rotating frame of reference ⇔ presence of Coriolis force
- Isentropic frame of reference: $(x, y, z) \to (x, y, s)$.
  - The entropy s is used as a vertical coordinate. Since entropy is materially conserved, the flow becomes horizontal.
  - Density given by $\sigma \equiv \rho \frac{\partial Z}{\partial s}$.

Why take isentropic coordinates?
→ Resulting quasi 2D flow structure simplifies Hamiltonian discretization

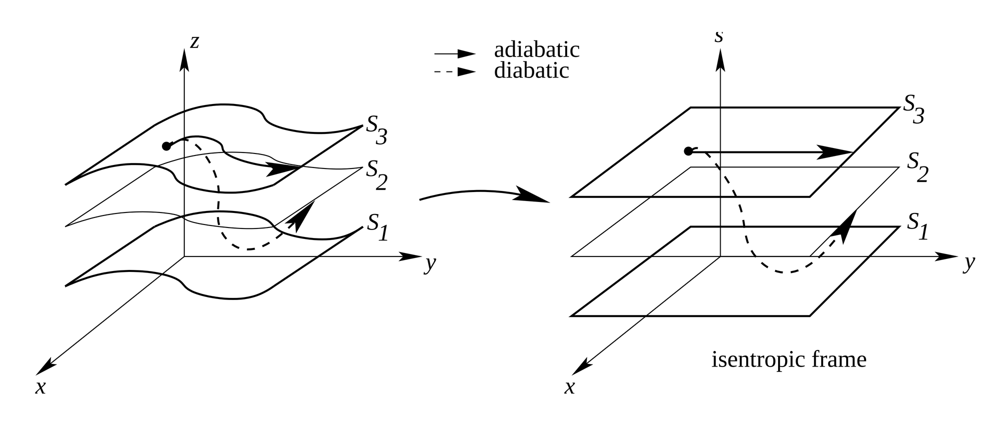

Figure 1: The conserved dynamics in isentropic coordinates is quasi 2D.

3. Equations in isentropic coordinates

- all variables are functions of (x, y, s, t)
- horizontal flow
  $\to \boldsymbol{v} = (u, v, 0)^T$, $\boldsymbol{v}^\perp = (-v, u, 0)^T$, $\boldsymbol{\nabla} \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)^T$

$$\textbf{momentum balance} \qquad \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + f \boldsymbol{v}^\perp + \boldsymbol{\nabla} M = 0 \qquad (1)$$

$$\textbf{mass balance} \qquad \frac{\partial \sigma}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \sigma + \sigma \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \qquad (2)$$

The term $f\boldsymbol{v}^\perp$ in (1) is the Coriolis force. M in (1) is the Montgomery potential, given by $M = gZ(s) + E$ with E the enthalpy (per unit mass).
Closure of system by

$$\textbf{energy balance} \qquad \sigma = -\frac{p_0}{g} \frac{\partial}{\partial s} \left\{ \left(\frac{1}{T_0} \frac{\partial M}{\partial s} \right)^{\frac{c_p}{R}} e^{-(s-s_0)/R} \right\} \qquad (3)$$

Hamiltonian dynamics

The theory of Hamiltonian systems provides methods to elucidate the underlying conservation properties of our reversible atmosphere. A basic example of a ***Hamiltonian system*** is Newton's Second Law with the force arising as the gradient of some potential. Such a system is characterized by some energy function (the Hamiltonian) conserved along the flow. In general, Hamiltonian systems are completely specified by i) the Hamiltonian and, ii) the so-called (generalized) ***Poisson bracket***.
In [1], a rather special Hamiltonian description is introduced: the parcel formulation. We apply the formulation to our atmospheric model.

Why using parcel formulation?
→ The parcel formulation is an alternative continuum description partially Lagrangian and partially Eulerian. This special viewpoint forms the basis of our numerical method.

Parcel formulation

In the parcel formulation, we follow a specific parcel labelled by its initial position $\boldsymbol{A} = (A, B, S)$. Remark: S is the entropy of that specific parcel. $\boldsymbol{\chi}^t$ maps this fluid parcel to its later Eulerian position $(\boldsymbol{X}(t), S) \equiv \boldsymbol{\chi}^t(A, B, S)$, having velocity $(\boldsymbol{V}(t), 0) \equiv \boldsymbol{\chi}^t(A, B, S)$. Then

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{\nabla}_{\boldsymbol{V}} H = V \qquad (4a)$$

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -f\, \boldsymbol{\nabla}_{\boldsymbol{V}}^{\perp} H - \boldsymbol{\nabla}_{\boldsymbol{X}} H = -f\, \boldsymbol{V}^\perp - \boldsymbol{\nabla}_{\boldsymbol{X}} M \qquad (4b)$$

Here, $(\boldsymbol{\nabla}_{\boldsymbol{X}}, \boldsymbol{\nabla}_{\boldsymbol{V}}) \equiv (\frac{\partial}{\partial \boldsymbol{X}}, \frac{\partial}{\partial \boldsymbol{V}})$. The parcel's Hamiltonian reads

$$H(\boldsymbol{X}, S, \boldsymbol{V}, t) = \frac{1}{2} \boldsymbol{V}^2 + M(\boldsymbol{X}, S, t) \qquad (5)$$

Crucial point: potential M regarded as Eulerian function evaluated at parcel position! Write $M(x, y, S, t)|_{(x,y)=\boldsymbol{X}} \equiv M(\boldsymbol{X}, S, t)$.

Procedure repeated for *all* parcels that constitute the fluid. Then the density of our specific parcel follows directly from

$$\sigma(\boldsymbol{X}, S, t) = \int \sigma(\boldsymbol{a}, 0)\, \delta((\boldsymbol{X}, S) - \boldsymbol{\chi}^t(\boldsymbol{a}))\, \mathrm{d}\boldsymbol{a} \qquad (6)$$

Since (6) holds for *all* parcels, we recover σ as function of (x, y, s, t). This can be used to solve (3) for M.

⇓

Parcel Poisson system:

For each parcel we have $\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\}$ for arbitrary F,

Poisson bracket $\{F, H\} = f\, \boldsymbol{\nabla}_{\boldsymbol{V}}^{\perp} F \cdot \boldsymbol{\nabla}_{\boldsymbol{V}} H + \boldsymbol{\nabla}_{\boldsymbol{X}} F \cdot \boldsymbol{\nabla}_{\boldsymbol{V}} H - \boldsymbol{\nabla}_{\boldsymbol{X}} H \cdot \boldsymbol{\nabla}_{\boldsymbol{V}} F$

Note that with $F = (\boldsymbol{X}, \boldsymbol{V})$ we obtain (4).

Note:
The full Eulerian Hamiltonian description can be found from the parcel formulation by integrating out label dependencies. As a result, ODEs ⇒ PDE and functions ⇒ functional. This is in fact a much easier procedure than adopting *reduction techniques* commonly used to convert continuum Lagrangian descriptions to Eulerian ones.

Numerical scheme: HPM

HPM stands for 'Hamiltonian Particle-Mesh Method'. It is a numerical method based on the parcel formulation.
Conserved are mass, energy and potential vorticity $q = \frac{f + \boldsymbol{\nabla}^\perp \cdot \boldsymbol{v}}{\sigma}$.
It includes a *symplectic* time integrator assuring conservation of phase space volume.

Why using particle-based discretization?
→ There does not exist a general method for finding Eulerian discretizations that are still Hamiltonian.

Algorithm of numerical method for atmospheric system:

i) Fluid decomposed in K isentropic layers. On each isentrope s_k we have N_k particles, each with fixed mass. Label particles on isentrope s_k with l_{s^k}.

ii) Lagrangian step:

(a) Apply symplectic time integrator to move the particles on each isentrope s_k.

$$\frac{\Delta \boldsymbol{X}_{l_{s^k}}}{\Delta t} = \boldsymbol{V}_{l_{s^k}} \qquad (7a)$$

$$\frac{\Delta \boldsymbol{V}_{l_{s^k}}}{\Delta t} = -f \boldsymbol{V}_{l_{s^k}}^\perp - \boldsymbol{\nabla}_{(x,y)} \bar{M}_{s^k}(x, y, t)\big|_{(x,y)=\boldsymbol{X}_{l_{s^k}}} \qquad (7b)$$

iii) Eulerian steps:

(a) Fixed grid (x_i, y_j, s_k).

(b) Redistribution of particles induces new density field:

$$\bar{\sigma}_{s^k}(x, y, t) = \sum_{i,j} \sigma_{s^k ij}(t) \cdot \psi((x_i, y_j) - (x, y))$$

Density on the grid found from discretization of (6):

$$\sigma_{s^k ij}(t) = \sum_{l=1}^{N_k} \frac{\sigma(\boldsymbol{a}_{l_{s^k}}, 0)}{\Delta x \Delta y} \cdot \psi((x_i, y_j) - (\boldsymbol{X}_{l_{s^k}}(t)))\, |\Delta \boldsymbol{a}_l|$$

with ψ suitable interpolation function.

(c) Use new σ values on grid to solve energy balance (3) for potential M numerically. This is an ODE for each horizontal grid point!

(d) Interpolate grid values of $M_{s^k}(t)$ back to $\bar{M}_{s^k}(x, y, t)$. Use this updated potential field for new Lagrangian step.

Outlook

HPM has proven to give excellent results for the (spherical) shallow water system, see for example [2]. Our implementation for the atmospheric model is still under construction.

Extensions:

- account for unresolved but important dynamics, like gravity waves
- apply appropriate smoothing to avoid nonlinear interaction with unimportant subgrid flow structures
- method to deal with overturning of isentropes (instability)
- weather fronts, akin to shocks
- spherical model
- **Forcing & friction**

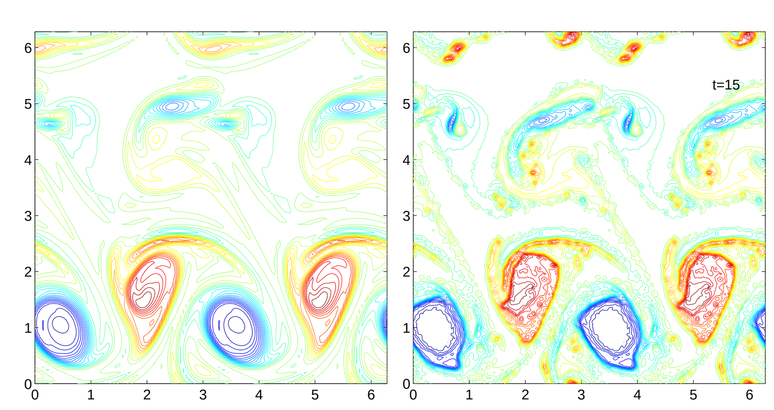

Figure 2: The potential vorticity field for the shallow water system, as computed by a high order 'conventional' method as a reference (left) and by HPM (right), see [2].

References

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