Appendix B.5

Corrections and additions to Chapter 5

In this chapter we collect corrections, extensions, etc. regarding Chapter 5.

B.5.1 Corrections and additions to Section 5.1

It turns out that the result as presented in Theorem 5.1.3 has an easy extension. We need this extension in Chapter 11.

Theorem B.5.1.1 *If A is the infinitesimal generator of the C_0-semigroup $T(t)$ on a Hilbert space Z, f absolutely continuous in the interval* $[0, \tau)$ *and* $z_0 \in \boldsymbol{D}(A)$*, then (5.2) is continuously differentiable on* $[0, \tau)$ *and it is the unique classical solution of (5.1).*

Proof *Uniqueness*: If z_1 and z_2 are two different solutions, then their difference $\Delta(t) = z_1(t) - z_2(t)$ satisfies the differential equation

$$\frac{d\Delta}{dt} = A\Delta, \qquad \Delta(0) = 0$$

and so we need to show that its only solution is $\Delta(t) \equiv 0$. To do this, define $y(s) = T(t-s)\Delta(s)$ for a fixed t and $0 \leq s \leq t$. Clearly, $\frac{dy}{ds} = 0$ and so $y(s) =$ constant $= T(t)\Delta(0) = 0$. However, $y(t) = \Delta(t)$ shows that $\Delta(t) = 0$.

Existence: Clearly, all we need to show now is that $v(t) = \int_0^t T(t-s)f(s)ds$ is an element of $\boldsymbol{C}^1([0, \tau); Z)$, takes values in $\boldsymbol{D}(A)$, and satisfies differential equation (5.1). Now since f is absolutely continuous on $[0, \tau)$ for $s \in [0, t]$ with $t < \tau$, $f(s)$ can be written as $f(s) = f(0) + \int_0^s \dot{f}(\alpha)d\alpha$ with $\dot{f} \in \boldsymbol{L}_1((0, t); Z)$. Thus

$$v(t) = \int_0^t T(t-s)[f(0) + \int_0^s \dot{f}(\alpha)d\alpha]ds$$
$$= \int_0^t T(t-s)f(0)ds + \int_0^t \int_\alpha^t T(t-s)\dot{f}(\alpha)dsd\alpha,$$

where we have used Fubini's Theorem A.5.27. From Theorem 2.1.13.e, it follows that

$$T(t-\alpha)z - z = A\int_\alpha^t T(t-s)zds \qquad \text{for all } z \in Z.$$

Hence $v(t) \in \boldsymbol{D}(A)$, and $\int_0^t \|A\int_\alpha^t T(t-s)\dot{f}(\alpha)ds\|d\alpha = \int_0^t \|T(t-\alpha)\dot{f}(\alpha) - \dot{f}(\alpha)\|d\alpha < \infty$, where we used that $\dot{f} \in \boldsymbol{L}_1((0,t);Z)$ for $t < \tau$. Thus, since A is closed, by Theorem A.5.28 we have that

$$Av(t) = [T(t) - I]f(0) + \int_0^t [T(t-\alpha) - I]\dot{f}(\alpha)d\alpha$$
$$= T(t)f(0) + \int_0^t T(t-\alpha)\dot{f}(\alpha)d\alpha - f(t). \tag{B.5.1}$$

Now for $r \in [0,\tau)$ we introduce the function

$$q(r) = T(r)f(0) + \int_0^r T(r-\alpha)\dot{f}(\alpha)d\alpha$$

Using Lemma 5.1.5 it is easy to show that q is continuous. Furthermore, for $t < \tau$

$$\int_0^t q(r)dr = \int_0^t T(r)f(0)dr + \int_0^t \int_0^r T(r-\alpha)\dot{f}(\alpha)d\alpha dr$$
$$= \int_0^t T(t-s)f(0)ds + \int_0^t \int_\alpha^t T(r-\alpha)\dot{f}(\alpha)drd\alpha$$

where we used $s = t - r$ and Fubini's Theorem A.5.27

$$= \int_0^t T(t-s)f(0)ds + \int_0^t \int_\alpha^t T(t-s)\dot{f}(\alpha)dsd\alpha$$

where we used $s = t + \alpha - r$ in the second integral

$$= \int_0^t T(t-s)f(0)ds + \int_0^t \int_0^s T(t-s)\dot{f}(\alpha)d\alpha ds$$

by Fubini's Theorem A.5.27

$$= \int_0^t T(t-s)\left[f(0) + \int_0^s \dot{f}(\alpha)d\alpha\right]ds = \int_0^t T(t-s)f(s)ds = v(t).$$

So $v(t)$ is an element of $\boldsymbol{C}^1([0,\tau);Z)$, and

$$\frac{dv}{dt}(t) = q(t) = T(t)f(0) + \int_0^t T(t-s)\dot{f}(s)ds.$$

Combining this with (B.5.1), we see that

$$\frac{dv}{dt}(t) = Av(t) + f(t).$$

and so we have proved the assertion. ■

This result can also be found in Cazenave and Haraux [43, Proposition 4.1.6] and Zheng [294, Corollary 2.4.2].