Appendix B.11

Corrections and additions to Chapter 11

In this chapter we collect corrections, extensions, etc. regarding Chapter 11.

B.11.1 Corrections and additions to Section 11.1

It turns out that the proof of part b. in Theorem 11.1.5 is wrong. Below we give a correct proof.

Theorem B.11.1.1 *Let A be the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space Z and consider the following semilinear differential equation*

$$\dot{z}(t) = Az(t) + f(z(t)), \quad t \geq 0 \qquad z(0) = z_0. \tag{B.11.1}$$

If $f : Z \mapsto Z$ is locally Lipschitz continuous, then there exists a $t_{\max} > 0$ such that the differential equation (11.7) has a unique mild solution on $[0, t_{\max})$ with the following properties:

a. For $0 \leq t < t_{\max}$ the solution depends continuously on the initial condition, uniformly on any bounded interval $[0, \tau] \subset [0, t_{\max})$.

b. If $z_0 \in \boldsymbol{D}(A)$, then the mild solution is actually a classical solution on $[0, t_{\max})$.

Moreover, if $t_{\max} < \infty$, then

$$\lim_{t \uparrow t_{\max}} \|z(t)\| = \infty.$$

If the mapping f is uniformly Lipschitz continuous, then $t_{\max} = \infty$.

Proof *a.* This part can be found in the book.

b. It remains to show that for $z_0 \in \boldsymbol{D}(A)$ the mild solution is in fact a classical solution. Let t_1, t_2 be two time instances such that $0 \leq t_1 < t_2 < t_{\max}$, and let $z(t)$

be the (unique) mild solution corresponding to $z(0) = z_0 \in Z$. Then from part a. the following holds

$$z(t_2) = T(t_1)z(t_2 - t_1) + \int_0^{t_1} T(t_1 - s)f(z(s + t_2 - t_1))ds.$$

Thus the mild solution of (11.1) at time t_2 equals the mild solution of

$$\dot{v}(t) = Av(t) + f(v(t)),\ t \geq 0, \qquad v(0) = z(t_2 - t_1)$$

at time t_1. Combining this with (11.13), we find that for $t_2 - t_1$ sufficiently small

$$\|z(t_2) - z(t_1)\| \leq M_1 e^{\omega_1 t_1} \|z(t_2 - t_1) - z_0\|. \tag{B.11.2}$$

It remains to estimate the right-hand side of this inequality. Using (11.3), we deduce

$$\begin{aligned}
\frac{z(t) - z_0}{t} &= \frac{T(t)z_0 - z_0}{t} + \frac{1}{t}\int_0^t T(t-s)f(z(s))ds \\
&= \frac{T(t)z_0 - z_0}{t} + \frac{1}{t}\int_0^t T(t-s)f(z_0)ds + \\
&\quad \frac{1}{t}\int_0^t T(t-s)[f(z(s)) - f(z_0)]ds \\
&= \frac{T(t)z_0 - z_0}{t} + \frac{1}{t}\int_0^t T(q)f(z_0)dq + \\
&\quad \frac{1}{t}\int_0^t T(t-s)[f(z(s)) - f(z_0)]ds.
\end{aligned}$$

Since z_0 is in the domain of A, the first term converges to Az_0 as t converges to zero. The second term converges to $f(z_0)$ since $T(t)$ is strongly continuous. It remains to show that the last term converges to zero.

Since z and f are continuous, we can for every $\varepsilon > 0$ find a $t_\varepsilon > 0$ such that $\|f(z(s)) - f(z_0)\| \leq \varepsilon$ for $s \in [0, t_\varepsilon]$. Hence for $t \in [0, t_\varepsilon]$ there holds

$$\begin{aligned}
\left\|\frac{1}{t}\int_0^t T(t-s)[f(z(s)) - f(z_0)]ds\right\| &\leq \frac{1}{t}\int_0^t \|T(t-s)[f(z(s)) - f(z_0)]\|ds \\
&\leq M_1\varepsilon,
\end{aligned}$$

where M_1 is the maximum of the semigroup over e.g. $[0, 1]$. This can be done for any positive ε, and so we have that the right-derivative of $z(t)$ at $t = 0$ exists. Combining this with (B.11.2) we see that $z(t)$ is Lipschitz continuous.

Next choose $\tau < t_{\max}$. Since $z(t)$ is continuous on $[0, \tau]$, this function is bounded. Combining this with the Lipschitz continuity of f, we see that $f(z(t))$ is Lipschitz continuous on $[0, \tau]$. Here we have used the Lipschitz continuity of z. Since every Lipschitz continuous function is absolutely continuous, we conclude by Theorem

B.5.1.1 that

$$z(t) = T(t)z_0 + \int_0^t T(t-s)f(z(s))ds$$

is the classical solution of (B.11.1) on $[0,\tau]$. Since $\tau < t_{\max}$ was chosen arbitrarily , we have shown the assertion. ■

This result can also be found in Cazenave and Haraux [43, Proposition 4.3.9] and Zheng [294, Corollary 2.5.2]. In the later reference f is assumed to be globally Lipschitz continuous, but as can be seen from our proof this is not needed.