Towards the Infinite Horizon LQ problem

To prepare for the infinite horizon LQ problem (considered in the next section) we analyze in this section what happens with the solution of the LQ problem as $T \to \infty$. To make the dependence on T explicit we add a subscript T to the solution of the RDE (4.22):

$$\dot{P}_T(t) = -P_T(t)A - A^{\mathrm{T}}P_T(t) - Q + P_T(t)BR^{-1}B^{\mathrm{T}}P_T(t), \qquad P_T(T) = S. \tag{4.28}$$

Example 4.4.5. Consider again the integrator system and cost,

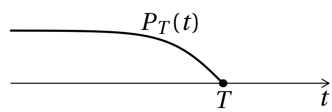
$$\dot{\mathrm{x}}(t) = \mathrm{u}(t), \qquad \mathrm{x}(0) = x_0, \qquad J_{[0,T]}(x_0, \mathrm{u}) = \int_0^T \mathrm{x}^2(t) + \mathrm{u}^2(t)\,\mathrm{d}t.$$

The RDE (4.28) in this case becomes

$$\dot{P}_T(t) = P_T^2(t) - 1, \qquad P_T(T) = 0,$$

and its solution was derived in Example 4.4.4,

$$P_T(t) = \tanh(T-t) = \frac{\mathrm{e}^{T-t} - \mathrm{e}^{-(T-t)}}{\mathrm{e}^{T-t} + \mathrm{e}^{-(T-t)}}.$$



Clearly, as T goes to infinity, the solution $P_T(t)$ converges to

$$P := 1,$$

and, in particular, it no longer depends on t. It is now tempting to conclude that the constant state feedback

$$\mathrm{u}_*(t) := -R^{-1}B^{\mathrm{T}}P\mathrm{x}(t) = -\mathrm{x}(t)$$

is the solution of the *infinite* horizon LQ problem. It is, as we shall see in the next section. □

The example suggests that $P_T(t)$ converges to a constant P as the horizon T goes to ∞. It also suggests that $\lim_{T\to\infty} \dot{P}_T(t) = 0$, which in turn suggests that the Riccati *differential* equation in the limit reduces to an *algebraic* equation,

$$0 = A^{\mathrm{T}}P + PA + Q - PBR^{-1}B^{\mathrm{T}}P. \tag{4.29}$$

This is correct, provided that for each x_0 there exists an input that renders the cost $J_{[0,\infty)}(x_0, \mathrm{u})$ finite:

Theorem 4.4.6 (Solution of the RDE as $T \to \infty$). Consider $\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0$, and suppose $Q \geq 0, R > 0$, and $S = 0$, and that for every x_0 an input exists that renders the cost (4.30) finite. Then the solution $P_T(t)$ of (4.28) converges to a matrix independent of t as the final time T goes to infinity. That is, a constant matrix P exists such that

$$\lim_{T\to\infty} P_T(t) = P \qquad \forall t > 0.$$

This P is symmetric, positive semi-definite, and it satisfies (4.29).

Proof. For every fixed x_0 the expression $x_0^{\mathrm{T}} P_T(t) x_0$ is nondecreasing with T because the longer the horizon the higher the cost. Indeed, for every $\epsilon > 0$ and initial $x(t) = z$ we have

$$\begin{aligned} z^{\mathrm{T}} P_{T+\epsilon}(t) z &= \int_t^{T+\epsilon} x_*^{\mathrm{T}}(\tau) Q x_*(\tau) + u_*^{\mathrm{T}}(\tau) R u_*(\tau) \, \mathrm{d}\tau \\ &\geq \int_t^{T} x_*^{\mathrm{T}}(t) Q x_*(t) + u_*^{\mathrm{T}}(\tau) R u_*(\tau) \, \mathrm{d}\tau \geq z^{\mathrm{T}} P_T(t) z. \end{aligned}$$

Besides being nondecreasing, it is, for any given z, also bounded from above because by assumption for at least one input u_z the infinite horizon cost is finite, so that

$$z^{\mathrm{T}} P_T(t) z \leq J_{[t,T]}(z, u_z) \leq J_{[t,\infty)}(z, u_z) < \infty.$$

Bounded and nondecreasing implies that $z^{\mathrm{T}} P_T(t) z$ converges as $T \to \infty$. Next we prove that in fact the entire matrix $P_T(t)$ converges as $T \to \infty$. Let e_i be the i-th unit vector in $\mathbb{R}^n$, so $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{T}}$, with a 1 on the i-th position. The preceding discussion shows that for each $z = e_i$, the limit

$$p_{ii} := \lim_{T\to\infty} e_i^{\mathrm{T}} P_T(t) e_i$$

exists. The diagonal entries of $P_T(t)$ hence converge. For the off-diagonal entries we use that

$$\begin{aligned} \lim_{T\to\infty} (e_i + e_j)^{\mathrm{T}} P_T(t)(e_i + e_j) &= \lim_{T\to\infty} e_i^{\mathrm{T}} P_T(t) e_i + e_j^{\mathrm{T}} P_T(t) e_j + 2e_i^{\mathrm{T}} P_T(t) e_j \\ &= p_{ii} + p_{jj} + \lim_{T\to\infty} 2e_i^{\mathrm{T}} P_T(t) e_j. \end{aligned}$$

The limit on the left-hand side exists, so the limit $p_{ij} := \lim_{T\to\infty} e_i^{\mathrm{T}} P_T(t) e_j$ exists as well. Therefore all entries of $P_T(t)$ converge as $T \to \infty$. The limit is independent of t because $P_T(t) = P_{T-t}(0)$.

Clearly, $P \geq 0$ because it is the limit of $P_T(t) \geq 0$.

Since $P_T(t)$ converges to a constant matrix, also $\dot{P}_T(t) = -P_T(t)A - A^{\mathrm{T}} P_T(t) + P_T(t) B R^{-1} B^{\mathrm{T}} P_T(t) - Q$ converges to a constant matrix as $T \to \infty$. This constant matrix must be zero because $\int_t^{t+1} \dot{P}_T(\tau) \, \mathrm{d}\tau = P_T(t+1) - P_T(t) \to 0$ as $T \to \infty$. ∎

4.5 Infinite Horizon LQ with Stability

Now we turn to the *infinite horizon* LQ problem. This is the problem of minimizing

$$J_{[0,\infty)}(x_0, \mathrm{u}) := \int_0^\infty \mathrm{x}^{\mathrm{T}}(t)Q\mathrm{x}(t) + \mathrm{u}^{\mathrm{T}}(t)R\mathrm{u}(t)\,\mathrm{d}t \tag{4.30}$$

over all $\mathrm{u}\colon [0,\infty) \to \mathbb{R}^m$ under the dynamical constraint

$$\dot{\mathrm{x}}(t) = A\mathrm{x}(t) + B\mathrm{u}(t), \qquad \mathrm{x}(0) = x_0.$$

As before, we assume that R is positive definite and that Q is positive semi-definite. The terminal cost $\mathrm{x}^{\mathrm{T}}(\infty)S\mathrm{x}(\infty)$ is absent. (For the problems we have in mind the state converges to zero so the terminal cost would not contribute anyway.) Obviously, there are links with the *finite* horizon case, but the theory that we present in this section is self-contained, that is, can be understood independently of the theory presented so far in this chapter.

The classic infinite horizon LQ problem does not consider asymptotic stability of the closed-loop system. For instance, if we choose as cost $\int_0^\infty \mathrm{u}^2(t)\,\mathrm{d}t$ then optimal is to take $\mathrm{u}_*(t) = 0$, even if it would render the closed-loop system unstable, such as when $\dot{\mathrm{x}}(t) = \mathrm{x}(t) + \mathrm{u}(t)$. In applications closed-loop asymptotic stability is crucial. Classically, closed-loop asymptotic stability is incorporated in LQ by imposing conditions on Q. For example, if $Q = I$ then the cost contains a term $\int_0^\infty \mathrm{x}^{\mathrm{T}}(t)\mathrm{x}(t)\,\mathrm{d}t$, and then the optimal control turns out to necessarily stabilize the system. An alternative approach is to include asymptotic stability in the problem definition. This is the version that we explore:

Definition 4.5.1 (Infinite horizon LQ problem with stability). Suppose $Q \geq 0, R > 0$, and consider the linear system with given initial state, $\dot{\mathrm{x}}(t) = A\mathrm{x}(t) + B\mathrm{u}(t), \mathrm{x}(0) = x_0$. The (infinite horizon) *LQ problem with stability* is to minimize (4.30) over all *stabilizing* inputs, meaning inputs that achieve $\lim_{t\to\infty} \mathrm{x}(t) = 0$. □

The next two examples reveal that in some cases the LQ problem with stability has an easy solution.

Example 4.5.2 (LQ with stability). Consider the problem of Example 4.4.5:

$$\dot{\mathrm{x}}(t) = \mathrm{u}(t), \qquad \mathrm{x}(0) = x_0, \qquad J_{[0,\infty)}(x_0, \mathrm{u}) = \int_0^\infty \mathrm{x}^2(t) + \mathrm{u}^2(t)\,\mathrm{d}t.$$

The running cost, $\mathrm{x}^2 + \mathrm{u}^2$, can also be written as

$$\mathrm{x}^2 + \mathrm{u}^2 = (\mathrm{x} + \mathrm{u})^2 - 2\mathrm{x}\mathrm{u} = (\mathrm{x} + \mathrm{u})^2 - 2\mathrm{x}\dot{\mathrm{x}}.$$

Interestingly, the term $-2\mathrm{x}\dot{\mathrm{x}}$ has an explicit antiderivative, namely $-\mathrm{x}^2$, so

$$\mathrm{x}^2 + \mathrm{u}^2 = \tfrac{\mathrm{d}}{\mathrm{d}t}(-\mathrm{x}^2) + (\mathrm{x} + \mathrm{u})^2.$$

Integrating this over $t \in [0,\infty)$ we see that the cost for stabilizing inputs equals

$$J_{[0,\infty)}(x_0,\mathrm{u}) = x_0^2 + \int_0^\infty (\mathrm{x}(t)+\mathrm{u}(t))^2\,\mathrm{d}t. \tag{4.31}$$

It is immediate from (4.31) that the cost for every stabilizing input is at least x_0^2, and that it equals x_0^2 iff

$$\mathrm{u} = -\mathrm{x}.$$

Since the state feedback $\mathrm{u}_* := -\mathrm{x}$ indeed stabilizes (because the closed-loop system becomes $\dot{\mathrm{x}} = -\mathrm{x}$) we conclude that this state feedback is the optimal control, and that the optimal (minimal) cost is

$$J_{[0,\infty)}(x_0,\mathrm{u}_*) = x_0^2.$$

Done! □

Also for systems without input the solution is easily determined:

Example 4.5.3 (Systems without input). If $B = 0$ then the system reduces to $\dot{\mathrm{x}}(t) = A\mathrm{x}(t), \mathrm{x}(0) = x_0$. Obviously, the input does not affect the state in this case, so if this problem is to have a stabilizing input for every x_0 then A needs to be asymptotically stable. The optimal input in that case is $\mathrm{u}_* = 0$, and the optimal cost is

$$\begin{aligned} J_{[0,\infty)}(x_0,\mathrm{u}_*) &= \int_0^\infty \mathrm{x}^{\mathrm{T}}(t)Q\mathrm{x}(t)\,\mathrm{d}t \\ &= \int_0^\infty x_0^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}t} Q \mathrm{e}^{At} x_0\,\mathrm{d}t \\ &= x_0^{\mathrm{T}}\Big(\int_0^\infty \mathrm{e}^{A^{\mathrm{T}}t} Q \mathrm{e}^{At}\,\mathrm{d}t\Big)x_0. \end{aligned}$$

This cost is quadratic in the initial state, and we write it as $x_0^{\mathrm{T}} P x_0$ with

$$P := \int_0^\infty \mathrm{e}^{A^{\mathrm{T}}t} Q \mathrm{e}^{At}\,\mathrm{d}t. \tag{4.32}$$

The previous section suggests that P satisfies the quadratic equation (4.29), which for our case simplifies to the linear equation

$$A^{\mathrm{T}}P + PA = -Q. \tag{4.33}$$

This equation is called the *Lyapunov equation*. The P defined in (4.32) indeed satisfies the Lyapunov equation because

$$\begin{aligned} A^{\mathrm{T}}P + PA &= \int_0^\infty A^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}t} Q \mathrm{e}^{At} + \mathrm{e}^{A^{\mathrm{T}}t} Q \mathrm{e}^{At} A\,\mathrm{d}t \\ &= \big[\mathrm{e}^{A^{\mathrm{T}}t} Q \mathrm{e}^{At}\big]_0^\infty \\ &= 0 - Q = -Q. \end{aligned}$$

Here we used that A is asymptotically stable and that $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{A^{\mathrm{T}}t} Q \mathrm{e}^{At})$ equals $A^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}t} Q \mathrm{e}^{At} + \mathrm{e}^{A^{\mathrm{T}}t} Q \mathrm{e}^{At} A$. □

The matrix P defined in (4.32) is in fact uniquely determined by the Lyapunov equation (4.33) because of asymptotic stability of A:

Lemma 4.5.4 (Lyapunov equation). Suppose $A \in \mathbb{R}^{n\times n}$ is asymptotically stable. Then for every $Q \in \mathbb{R}^{n\times n}$ (not necessarily symmetric) there is a unique solution $P \in \mathbb{R}^{n\times n}$ of (4.33). In particular, $A^{\mathrm{T}}P + PA = 0$ iff $P = 0$.

Proof. For every $Q \in \mathbb{R}^{n\times n}$ the matrix P as defined in (4.32) is a solution. To say it differently, the linear mapping $\mathfrak{L}: \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined as $\mathfrak{L}(P) = A^{\mathrm{T}}P + PA$ is surjective. By the rank-nullity theorem it hence is injective as well. That is, for every Q the solution P of (4.33) exists and is unique. ∎

In the previous two examples we found that the optimal cost is quadratic in the initial state, and that the optimal input can be implemented as a state feedback. Inspired by this we conjecture that every infinite horizon LQ problem has these properties. That is, we conjecture that the optimal cost is of the form

$$x_0^{\mathrm{T}} P x_0$$

for some matrix P, and that the optimal input equals

$$\mathrm{u}_*(t) := -F\mathrm{x}(t)$$

for some matrix F. With that in mind we define $\mathrm{v} = \mathrm{u} + F\mathrm{x}$. (If our hunch is correct then optimal means $\mathrm{v} = 0$.) Next we write $\mathrm{x}^{\mathrm{T}}Q\mathrm{x} + \mathrm{u}^{\mathrm{T}}R\mathrm{u}$ and $-\mathrm{v}^{\mathrm{T}}R\mathrm{v}$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{x}^{\mathrm{T}}P\mathrm{x})$ as quadratic expressions in (x,u):

$$\mathrm{x}^{\mathrm{T}}Q\mathrm{x} + \mathrm{u}^{\mathrm{T}}R\mathrm{u} = \begin{bmatrix}\mathrm{x}^{\mathrm{T}} & \mathrm{u}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}Q & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}\mathrm{x}\\ \mathrm{u}\end{bmatrix},$$

$$\begin{aligned}-\mathrm{v}^{\mathrm{T}}R\mathrm{v} &= -(\mathrm{u}^{\mathrm{T}} + \mathrm{x}^{\mathrm{T}}F^{\mathrm{T}})R(\mathrm{u} + F\mathrm{x})\\ &= \begin{bmatrix}\mathrm{x}^{\mathrm{T}} & \mathrm{u}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}-F^{\mathrm{T}}RF & -F^{\mathrm{T}}R\\ -RF & -R\end{bmatrix}\begin{bmatrix}\mathrm{x}\\ \mathrm{u}\end{bmatrix},\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{x}^{\mathrm{T}}P\mathrm{x}) &= \dot{\mathrm{x}}^{\mathrm{T}}P\mathrm{x} + \mathrm{x}^{\mathrm{T}}P\dot{\mathrm{x}}\\ &= (\mathrm{x}^{\mathrm{T}}A^{\mathrm{T}} + \mathrm{u}^{\mathrm{T}}B^{\mathrm{T}})P\mathrm{x} + \mathrm{x}^{\mathrm{T}}P(A\mathrm{x} + B\mathrm{u})\\ &= \begin{bmatrix}\mathrm{x}^{\mathrm{T}} & \mathrm{u}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}A^{\mathrm{T}}P + PA & PB\\ B^{\mathrm{T}}P & 0\end{bmatrix}\begin{bmatrix}\mathrm{x}\\ \mathrm{u}\end{bmatrix}.\end{aligned}$$

Adding them all up, we find

$$\begin{aligned}&\mathrm{x}^{\mathrm{T}}Q\mathrm{x} + \mathrm{u}^{\mathrm{T}}R\mathrm{u} - \mathrm{v}^{\mathrm{T}}R\mathrm{v} + \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{x}^{\mathrm{T}}P\mathrm{x})\\ &\qquad = \begin{bmatrix}\mathrm{x} & \mathrm{u}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}A^{\mathrm{T}}P + PA + Q - F^{\mathrm{T}}RF & PB - F^{\mathrm{T}}R\\ B^{\mathrm{T}}P - RF & 0\end{bmatrix}\begin{bmatrix}\mathrm{x}\\ \mathrm{u}\end{bmatrix}.\end{aligned}$$

The matrix on the right-hand side is the zero matrix if

- P is symmetric,
- $F = R^{-1}B^{\mathrm{T}}P$,
- $A^{\mathrm{T}}P + PA + Q - PBR^{-1}B^{\mathrm{T}}P = 0$.

So then

$$\mathtt{x}^{\mathrm{T}}Q\mathtt{x} + \mathtt{u}^{\mathrm{T}}R\mathtt{u} = -\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathtt{x}^{\mathrm{T}}P\mathtt{x}) + \mathtt{v}^{\mathrm{T}}R\mathtt{v}.$$

The cost (4.30) can thus be expressed as

$$J_{[0,\infty)}(x_0, \mathtt{u}) = x_0^{\mathrm{T}}Px_0 + \int_0^\infty \mathtt{v}(t)^{\mathrm{T}}R\mathtt{v}(t)\,\mathrm{d}t, \tag{4.34}$$

whenever the input stabilizes the system. From (4.34) it follows that the optimal cost is $x_0^{\mathrm{T}}Px_0$ provided that $\mathtt{v} = 0$ corresponds to a stabilizing input. We defined $\mathtt{v}$ as $\mathtt{v} = \mathtt{u} + F\mathtt{x} = \mathtt{u} + R^{-1}B^{\mathrm{T}}P\mathtt{x}$, so we have $\mathtt{v} = 0$ iff $\mathtt{u} = -F\mathtt{x} = -R^{-1}B^{\mathrm{T}}P\mathtt{x}$, and then the closed-loop system becomes $\dot{\mathtt{x}} = (A - BR^{-1}B^{\mathrm{T}}P)\mathtt{x}$. This leads to the following result.

Theorem 4.5.5 (Solution of the LQ problem with stability). Suppose $P \in \mathbb{R}^{n\times n}$ satisfies

$$A^{\mathrm{T}}P + PA + Q - PBR^{-1}B^{\mathrm{T}}P = 0 \tag{4.35}$$

with the property that

$$A - BR^{-1}B^{\mathrm{T}}P \quad \text{is asymptotically stable.} \tag{4.36}$$

Then P is symmetric, and the linear state feedback

$$\mathtt{u}_*(t) := -R^{-1}B^{\mathrm{T}}P\mathtt{x}(t)$$

is the solution of the LQ problem with stability, and $x_0^{\mathrm{T}}Px_0$ is the optimal cost. Moreover, there is at most one P that satisfies both (4.35) and (4.36).

Proof. If P satisfies (4.35) then

$$(A - BR^{-1}B^{\mathrm{T}}P)^{\mathrm{T}}(P - P^{\mathrm{T}}) + (P - P^{\mathrm{T}})(A - BR^{-1}B^{\mathrm{T}}P) = -Q + Q^{\mathrm{T}} = 0.$$

Since $A - BR^{-1}B^{\mathrm{T}}P$ is asymptotically stable, Lemma 4.5.3 guarantees that $P - P^{\mathrm{T}}$ equals zero. That is, P is symmetric. Above we showed that then the cost equals (4.34) for $\mathtt{v} := \mathtt{u} + R^{-1}B^{\mathrm{T}}P\mathtt{x}$. Obviously, $\mathtt{v} = 0$ holds iff $\mathtt{u} = -R^{-1}B^{\mathrm{T}}P\mathtt{x}$, and this input, by assumption on P, stabilizes the system. Hence this $\mathtt{u}$ solves the LQ problem with stability, and, $x_0^{\mathrm{T}}Px_0$ is the optimal cost.

There is at most one P that satisfies both (4.35) and (4.36) because every such P is symmetric, and the optimal cost $x_0^{\mathrm{T}}Px_0$ is unique. ∎

Equation (4.35) is known as the *(LQ) algebraic Riccati equation* (or *ARE* for short), and we say that P is a *stabilizing solution* of the ARE if it satisfies the ARE and $A - BR^{-1}B^{\mathrm{T}}P$ is asymptotically stable.

The theorem does *not* say that the ARE has a stabilizing solution. It only says that *if* a stabilizing solution P exists, then it is unique and symmetric, and then the LQ problem with stability is solved, with $\mathrm{u}_*(t) := -R^{-1}B^{\mathrm{T}}P\mathrm{x}(t)$ being the optimal control. It is not yet clear under what conditions there *exists* a stabilizing solution P of the ARE (4.35). In Lemma 4.4.3 we managed to express the solution $P(t)$ of the RDE in terms of the Hamiltonian matrix $\mathscr{H}$. For the infinite horizon case there is such a connection as well, and it provides necessary and sufficient conditions under which a stabilizing solution of the ARE exists. Starting point is to rewrite the ARE: a matrix P satisfies the ARE (4.35) iff $-Q - A^{\mathrm{T}}P = P(A - BR^{-1}B^{\mathrm{T}}P)$, or, equivalently, iff

$$\underbrace{\begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix}}_{\mathscr{H}} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} (A - BR^{-1}B^{\mathrm{T}}P). \tag{4.37}$$

(This defines the Hamiltonian matrix $\mathscr{H}$.) This is an interesting form because in the case that all matrices here are numbers (and $\mathscr{H}$ hence a 2×2 matrix) then it says that $\left[\begin{smallmatrix} I \\ P \end{smallmatrix}\right]$ is an *eigenvector* of $\mathscr{H}$, and that $A - BR^{-1}B^{\mathrm{T}}P$ is its *eigenvalue*. This connection between P and eigenvectors/eigenvalues of the Hamiltonian matrix $\mathscr{H}$ is the key to most numerical routines for computation of P. This central result is formulated in the following theorem. The subsequent examples show how the result can be used to find P concretely.

Theorem 4.5.6 (Computation of P). Define $\mathscr{H} \in \mathbb{R}^{(2n)\times(2n)}$ as in (4.37), and assume that $Q \geq 0, R > 0$. A stabilizing solution of the ARE exists iff (A, B) is stabilizable and $\left[\begin{smallmatrix} A-\lambda I \\ Q \end{smallmatrix}\right]$ has rank n for all $\lambda \in i\mathbb{R}$. In that case

1. $\mathscr{H}$ has no imaginary eigenvalues, and it has n asymptotically stable eigenvalues and n unstable eigenvalues. Also, λ is an eigenvalue of $\mathscr{H}$ iff so is $-\lambda$,
2. matrices $V \in \mathbb{R}^{(2n)\times n}$ of rank n exist that satisfy $\mathscr{H}V = V\Lambda$ for some asymptotically stable $\Lambda \in \mathbb{R}^{n\times n}$,
3. for any such $V \in \mathbb{R}^{(2n)\times n}$, if we partition V as $V = \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right]$ with $V_1, V_2 \in \mathbb{R}^{n\times n}$, then V_1 is invertible,
4. the ARE (4.35) has a unique stabilizing solution P. In fact

   $$P := V_2 V_1^{-1},$$

   is the unique stabilizing solution (and, hence, it is also symmetric).

Proof. This proof is involved. We assume familiarity with stabilizability as explained in Appendix A.6. The proof again exploits the remarkable property that solutions of the associated Hamiltonian system (now with initial conditions, possibly complex-valued),

$$\begin{bmatrix} \dot{\mathrm{x}}(t) \\ \dot{\mathrm{p}}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathrm{x}(t) \\ \mathrm{p}(t) \end{bmatrix}, \qquad \begin{bmatrix} \mathrm{x}(0) \\ \mathrm{p}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ p_0 \end{bmatrix} \in \mathbb{C}^{2n} \tag{4.38}$$

satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{p}^*\mathrm{x}) = -(\mathrm{x}^*Q\mathrm{x} + \mathrm{p}^*BR^{-1}B^{\mathrm{T}}\mathrm{p}), \tag{4.39}$$

(see the proof of Lemma 4.2.2). Note that we consider the system of differential equations over $\mathbb{C}^{2n}$, instead of over $\mathbb{R}^{2n}$, and here p^* means the *complex conjugate* transpose of p. The reason is that eigenvalues and eigenvectors may be complex-valued. Integrating (4.39) over $t \in [0,\infty)$ tells us that

$$\int_0^\infty \mathrm{x}^*(t)Q\mathrm{x}(t) + \mathrm{p}^*(t)BR^{-1}B^{\mathrm{T}}\mathrm{p}(t)\,\mathrm{d}t = p_0^*x_0 - \lim_{t\to\infty}\mathrm{p}^*(t)\mathrm{x}(t), \tag{4.40}$$

provided the limit exists. In what follows we denote by $\left[\begin{smallmatrix} \mathrm{x}(t) \\ \mathrm{p}(t) \end{smallmatrix}\right]$ the solution of (4.38). We first assume that (A,B) is stabilizable and that $\left[\begin{smallmatrix} A-\lambda I \\ Q \end{smallmatrix}\right]$ has rank n for all $\lambda \in \mathrm{i}\mathbb{R}$.

1. Suppose $\left[\begin{smallmatrix} x_0 \\ p_0 \end{smallmatrix}\right]$ is an eigenvector of $\mathscr{H}$ with imaginary eigenvalue λ. Then $\left[\begin{smallmatrix} \mathrm{x}(t) \\ \mathrm{p}(t) \end{smallmatrix}\right] = \mathrm{e}^{\lambda t}\left[\begin{smallmatrix} x_0 \\ p_0 \end{smallmatrix}\right]$. Now $\mathrm{p}^*(t)\mathrm{x}(t)$ is constant, hence both sides of (4.39) are zero for all time. So both $\mathrm{x}^*(t)Q\mathrm{x}(t)$ and $B^{\mathrm{T}}\mathrm{p}(t)$ are zero for all time. Inserting this into (4.38) shows that $\lambda x_0 = Ax_0$ and $\lambda p_0 = -A^{\mathrm{T}}p_0$. Thus $\left[\begin{smallmatrix} A-\lambda I \\ Q \end{smallmatrix}\right]x_0 = 0$ and $p_0^*\left[\begin{smallmatrix} A+\lambda I & B \end{smallmatrix}\right] = 0$. Stabilizability and the fact that $\left[\begin{smallmatrix} A-\lambda I \\ Q \end{smallmatrix}\right]$ has rank n, implies that then $x_0 = 0, p_0 = 0$, but $\left[\begin{smallmatrix} x_0 \\ p_0 \end{smallmatrix}\right]$ is an eigenvector, so nonzero. Contradiction, hence $\mathscr{H}$ has no imaginary eigenvalues.

   Exercise 4.19 shows that $r(\lambda) := \det(\lambda I - \mathscr{H})$ equals $r(-\lambda)$. So $\mathscr{H}$ has as many (asymptotically) stable eigenvalues as unstable eigenvalues.

2. Since $\mathscr{H}$ has no imaginary eigenvalues and has n asymptotically stable eigenvalues, linear algebra tells us that a $(2n)\times n$ matrix V exists of rank n such that $\mathscr{H}V = V\Lambda$ with Λ asymptotically stable. (If all n asymptotically stable eigenvalues are distinct then we can simply take $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ where $v_1,\dots,v_n$ are eigenvectors corresponding to the asymptotically stable eigenvalues $\lambda_1,\dots,\lambda_n$ of $\mathscr{H}$, and then Λ is the diagonal matrix with these eigenvalues on the diagonal. If some eigenvalues coincide then one might need a Jordan normal form and use generalized eigenvectors.)

3. Suppose, to obtain a contradiction, that V has rank n but that V_1 is singular. Then the subspace spanned by the columns of $\left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right]$ contains an $\left[\begin{smallmatrix} x_0 \\ p_0 \end{smallmatrix}\right]$ with $x_0 = 0, p_0 \neq 0$. The solution $\left[\begin{smallmatrix} \mathrm{x}(t) \\ \mathrm{p}(t) \end{smallmatrix}\right]$ for this initial condition converges

to zero[1]. Hence the integral in (4.40) equals $p_0^* x_0 = 0$. That can only be if $Q\mathrm{x}(t)$ and $B^{\mathrm{T}}\mathrm{p}(t)$ are zero for all time. Equation (4.38) then implies that $\dot{\mathrm{p}}(t) = -A^{\mathrm{T}}\mathrm{p}(t), \mathrm{p}(0) = p_0$. We claim that this contradicts stabilizability. Indeed, since $B^{\mathrm{T}}\mathrm{p}(t) = 0$ for all time, we have

$$\dot{\mathrm{p}}(t) = -(A^{\mathrm{T}} - LB^{\mathrm{T}})\mathrm{p}(t), \qquad \mathrm{p}(0) = p_0 \neq 0 \tag{4.41}$$

for every L. By stabilizability there is an L such that $A - BL^{\mathrm{T}}$ is asymptotically stable. Then all eigenvalues of $-(A^{\mathrm{T}} - LB^{\mathrm{T}})$ are anti-stable, and thus the solution $\mathrm{p}(t)$ of (4.41) diverges. But we know that $\lim_{t\to\infty}\mathrm{p}(t) = 0$. Contradiction, so the assumption that V_1 is singular is wrong.

4. Let $P = V_2 V_1^{-1}$. Since $\mathscr{H}V = V\Lambda$ we have that $\mathscr{H}\left[\begin{smallmatrix} I \\ P \end{smallmatrix}\right] = \left[\begin{smallmatrix} I \\ P \end{smallmatrix}\right] V_1 \Lambda V_1^{-1}$. Also $V_1 \Lambda V_1^{-1}$ is asymptotically stable because it has the same eigenvalues as Λ (assumed asymptotically stable). Hence

$$\begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} \hat{\Lambda} \tag{4.42}$$

for some asymptotically stable $\hat{\Lambda} \in \mathbb{R}^{n\times n}$. Premultiplying (4.42) from the left with $\begin{bmatrix} -P & I \end{bmatrix}$ shows that

$$\begin{bmatrix} -P & I \end{bmatrix} \begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = 0.$$

This equation is nothing else than the ARE (verify this for yourself). And P is a stabilizing solution because $A - BR^{-1}B^{\mathrm{T}}P = \hat{\Lambda}$ is asymptotically stable. Uniqueness and symmetry of P we showed earlier (Theorem 4.5.5).

Conversely, suppose P is a stabilizing solution. Clearly, (A, B) must then be stabilizable. Also, since $\left[\begin{smallmatrix} I & 0 \\ -P & I \end{smallmatrix}\right]\mathscr{H}\left[\begin{smallmatrix} I & 0 \\ P & I \end{smallmatrix}\right] = \left[\begin{smallmatrix} \Lambda & -BR^{-1}B^{\mathrm{T}}P \\ 0 & -\Lambda^{\mathrm{T}} \end{smallmatrix}\right]$ (for $\Lambda := A - BR^{-1}B^{\mathrm{T}}P$) we have that $\mathscr{H}$ cannot have imaginary eigenvalues. But every imaginary λ for which $\left[\begin{smallmatrix} A-\lambda I \\ Q \end{smallmatrix}\right]$ loses rank is an eigenvalue of $\mathscr{H}$. So $\left[\begin{smallmatrix} A-\lambda I \\ Q \end{smallmatrix}\right]$ must have rank n for all imaginary λ if a stabilizing solution P is to exist. ■

Realize that *any* $V \in \mathbb{R}^{(2n)\times n}$ of rank n for which $\mathscr{H}V = V\Lambda$ does the job if Λ is asymptotically stable. That is, even though there are many such V, we always have that V_1 is invertible and that P follows uniquely as $P = V_2 V_1^{-1}$. As already mentioned in the above proof, in case $\mathscr{H}$ has n *distinct* asymptotically stable eigenvalues $\lambda_1, \dots, \lambda_n$, with eigenvectors $v_1, \dots, v_n$, then we can take

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

[1]If $\left[\begin{smallmatrix} x_0 \\ p_0 \end{smallmatrix}\right] = Vz_0$ for some z_0 then $\left[\begin{smallmatrix} \mathrm{x}(t) \\ \mathrm{p}(t) \end{smallmatrix}\right] = V\mathrm{z}(t)$ where $\mathrm{z}(t)$ is the solution of $\dot{\mathrm{z}}(t) = \Lambda\mathrm{z}(t), \mathrm{z}(0) = z_0$. If Λ is asymptotically stable then $\mathrm{z}(t) \to 0$ as $t \to \infty$.

for then Λ is diagonal with

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix},$$

and this matrix clearly is asymptotically stable.

Example 4.5.7 ($n = 1$). Consider once more the integrator system $\dot{\mathrm{x}}(t) = \mathrm{u}(t)$ and cost $\int_0^\infty \mathrm{x}^2(t) + \mathrm{u}^2(t)\,\mathrm{d}t$. That is, $A = 0, B = Q = R = 1$. The Hamiltonian matrix for this case is

$$\mathscr{H} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Its characteristic polynomials is $\lambda^2 - 1$, and the eigenvalues are $\lambda_{1,2} = \pm 1$. Its asymptotically stable eigenvalue is $\lambda_{\mathrm{as}} = -1$, and it is easy to verify that v is an eigenvector corresponding to this asymptotically stable eigenvalue iff

$$v := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} c, \qquad c \neq 0.$$

According to Lemma 4.5.6 the stabilizing solution P of the ARE is

$$P = v_2 v_1^{-1} = \frac{v_2}{v_1} = \frac{c}{c} = 1.$$

As predicted, P does not depend on the choice of eigenvector (the choice of c). Also, the (eigen)value of $A - BR^{-1}B^{\mathrm{T}}P = -1$ as predicted equals the asymptotically stable eigenvalue of the Hamiltonian matrix, $\lambda_{\mathrm{as}} = -1$. The optimal control is $\mathrm{u}_* = -R^{-1}B^{\mathrm{T}}P\mathrm{x} = -\mathrm{x}$. This agrees with what we found in Example 4.5.2. □

Example 4.5.8 ($n = 2$). Consider the stabilizable system

$$\dot{\mathrm{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathrm{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{u}(t),$$

with standard cost

$$\int_0^\infty \mathrm{x}_1^2(t) + \mathrm{x}_2^2(t) + \mathrm{u}^2(t)\,\mathrm{d}t.$$

The associated Hamiltonian matrix is (verify this yourself)

$$\mathscr{H} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right].$$

Its characteristic polynomial is $\lambda^4 - \lambda^2 + 1$, and the four eigenvalues turn out to be

$$\lambda_{1,2} = -\tfrac{1}{2}\sqrt{3} \pm \tfrac{1}{2}\mathrm{i}, \qquad \lambda_{3,4} = +\tfrac{1}{2}\sqrt{3} \pm \tfrac{1}{2}\mathrm{i}.$$

The first two eigenvalues, $\lambda_{1,2}$, are asymptotically stable so we need eigenvectors corresponding to these two. Not very enlightening manipulation shows that we can take

$$v_{1,2} = \begin{bmatrix} -\lambda_{1,2} \\ -\lambda_{1,2}^2 \\ 1 \\ \lambda_{1,2}^3 \end{bmatrix}.$$

Now $V \in \mathbb{C}^{4\times 2}$ defined as

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & -\lambda_2 \\ -\lambda_1^2 & -\lambda_2^2 \\ 1 & 1 \\ \lambda_1^3 & \lambda_2^3 \end{bmatrix}$$

is the V we need. (Note that this matrix is complex; this is not a problem.) With V known, it is easy to compute the stabilizing solution of the ARE,

$$P = V_2 V_1^{-1} = \begin{bmatrix} 1 & 1 \\ \lambda_1^3 & \lambda_2^3 \end{bmatrix} \begin{bmatrix} -\lambda_1 & -\lambda_2 \\ -\lambda_1^2 & -\lambda_2^2 \end{bmatrix}^{-1} = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

The optimal input is $\mathtt{u}_* = -R^{-1}B^{\mathrm{T}}P\mathtt{x} = -p_{21}\mathtt{x}_1 - p_{22}\mathtt{x}_2 = -\mathtt{x}_1 - \sqrt{3}\mathtt{x}_2$. The LQ-optimal closed-loop system is described by

$$\dot{\mathtt{x}}_*(t) = (A - BR^{-1}B^{\mathrm{T}}P)\mathtt{x}_*(t) = \begin{bmatrix} 0 & 1 \\ -1 & \sqrt{3} \end{bmatrix} \mathtt{x}_*(t),$$

and its eigenvalues are $\lambda_{1,2} = -\frac{1}{2}\sqrt{3} \pm \frac{1}{2}\mathrm{i}$ (which, as predicted, are the asymptotically stable eigenvalues of $\mathscr{H}$). □

In the above example the characteristic polynomial $\lambda^4 - \lambda^2 + 1$ is of degree 4, but by letting $\mu = \lambda^2$ it reduces to the polynomial $\mu^2 - \mu + 1$ of degree 2. This works for every Hamiltonian matrix, see Exercise 4.19.

Example 4.5.9. In Example 4.5.8 we found the stabilizing solution

$$P = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}$$

via the eigenvectors of the Hamiltonian. This solution must be positive semi-definite because optimal costs, $x_0^{\mathrm{T}} P x_0$, obviously are nonnegative. Clearly, P is symmetric, and since $p_{1,1} = p_{2,2} = \sqrt{3} > 0$ and $\det(P) = 2 > 0$ it is positive semi-definite (in fact, positive definite, see Lemma A.1.1). □

Theorem 4.5.6 establishes that a stabilizing solution of the ARE exists iff (A,B) is stabilizable and $\left[\begin{smallmatrix} A-\lambda I \\ Q \end{smallmatrix}\right]$ has rank n for all $\lambda \in \mathrm{i}\mathbb{R}$. If we replace the latter condition with the slightly stronger condition that (Q,A) is detectable then we can characterize P in a couple of other ways:

Theorem 4.5.10 (Three ways to solve the LQ problem with stability). If (A,B) is stabilizable and (Q,A) detectable, then the stabilizing solution P of the ARE can be characterized in the following three equivalent ways:

1. $P = \lim_{T\to\infty} P_T(t)$ where $P_T(t)$ is the solution of RDE (4.28) for $S=0$,
2. P is the unique symmetric, positive semi-definite solution of ARE (4.35),
3. P is the unique stabilizing solution of ARE (4.35).

Proof. We first show that the three P's are the same. Uniqueness is commented on afterwards.

($1 \implies 2$). Since (A,B) is stabilizable, there is a state feedback $\mathrm{u} = -F\mathrm{x}$ that steers the state to zero exponentially fast for every x_0, and, so, renders the cost finite. Therefore the conditions of Theorem 4.4.6 are met. That is, $P := \lim_{T\to\infty} P_T(t)$ exists and it satisfies the ARE, and it is positive semi-definite.

($2 \implies 3$). Assume P is a positive semi-definite solution of the ARE, and let x be an eigenvector of $A - BR^{-1}B^{\mathrm{T}}P$ with eigenvalue λ. We show that $\mathrm{Re}(\lambda) < 0$. The trick is to rewrite the ARE as

$$(A - BR^{-1}B^{\mathrm{T}}P)^{\mathrm{T}}P + P(A - BR^{-1}B^{\mathrm{T}}P) + Q + PBR^{-1}B^{\mathrm{T}}P = 0.$$

Next, postmultiply this equation with the eigenvector x, and premultiply with its complex conjugate transpose x^*:

$$x^*\big((A - BR^{-1}B^{\mathrm{T}}P)^{\mathrm{T}}P + P(A - BR^{-1}B^{\mathrm{T}}P) + Q + PBR^{-1}B^{\mathrm{T}}P\big)x = 0.$$

Since x is an eigenvector of $A - BR^{-1}B^{\mathrm{T}}P$ the above simplifies to a sum of three terms, the last two of which are nonnegative,

$$(\lambda^* + \lambda)(x^*Px) + x^*Qx + x^*PBR^{-1}B^{\mathrm{T}}Px = 0.$$

If $\mathrm{Re}(\lambda) \geq 0$ then $(\lambda^* + \lambda)x^*Px \geq 0$, implying that all the above three terms are in fact zero: $(\lambda^* + \lambda)x^*Px = 0$, $Qx = 0$, and $B^{\mathrm{T}}Px = 0$ (and, consequently, $Ax = \lambda x$). This contradicts detectability. So it cannot be that $\mathrm{Re}(\lambda) \geq 0$. It must be that $A - BR^{-1}B^{\mathrm{T}}P$ is asymptotically stable.

($3 \implies 1$ & uniqueness). Theorem 4.5.6 shows that the stabilizing solution P of the ARE exists and is unique. This equals $P := \lim_{T\to\infty} P_T(t)$ because 1. $\implies$ 2. $\implies$ 3.. Since every symmetric, positive definite P is stabilizing, and stabilizing solutions are unique, also the symmetric, positive semi-definite solution is unique. ∎

Theorem 4.5.10 shows that we have several ways to determine the solution P that solves the LQ problem with stability, namely (a) $\lim_{T\to\infty} P_T(t)$, (b) the unique symmetric positive semi-definite solution of the ARE, and (c) the unique stabilizing solution of the ARE.

Example 4.5.11 (LQ problem with stability of the integrator system solved in three ways). Consider again the integrator system

$$\dot{x}(t) = u(t), \qquad x(0) = x_0$$

and cost

$$J_{[0,\infty)}(x_0, u) = \int_0^\infty x^2(t) + u^2(t)\,dt.$$

This system is stabilizable, and $(Q, A) = (1, 0)$ is detectable. We determine the LQ solution P in the three different ways as explained in Theorem 4.5.10:

1. In Example 4.4.5 we handled the finite horizon case of this problem, and we found that $P := \lim_{T\to\infty} P_T(t) = 1$.
2. We could have gone as well for the unique symmetric, positive semi-definite solution of the ARE. The ARE in this case is

   $$-P^2 + 1 = 0,$$

   and, clearly, the only (symmetric) positive semi-definite solution is $P = 1$.
3. The ARE has two solutions, $P = \pm 1$, and Theorem 4.5.10 guarantees that precisely one of them is stabilizing. The solution P is stabilizing if $A - BR^{-1}B^{T}P = -P$ is less than zero. Clearly this, again, gives $P = 1$.

□

While for low-order systems the 2nd option (that P is positive semi-definite) is often the easiest way to determine P, general numerical recipes usually exploit the 3rd option.

4.6 Controller Design with LQ Optimal Control

In five examples we explore the use of infinite horizon LQ theory for the design of controllers. The first two examples discuss the effect of tuning parameters on the control and cost. The final three examples are about control of cars.

Example 4.6.1 (Tuning the controller). Consider the system with output,

$$\dot{x}(t) = u(t), \qquad x(0) = x_0 = 1,$$
$$y(t) = 2x(t),$$