Exercise 3 - Solutions

a. The number 4132 starts with a 4 and is above average because $2 \cdot 3 \geq 4 + 1$ and $2 \cdot 2 \geq 1 + 3$.

b. Suppose that $a4bc$ is a 4-digit number that is above average, where a, b, and c are the digits 1, 2 and 3 (possibly in a different order). Then $2 \cdot b \geq a + 4 \geq 5$, because a has to be 1 or bigger. This implies that $b \geq 3$. Similarly, we find that $2 \cdot c \geq 4 + b \geq 7$, hence $c \geq 4$. However, this is impossible because c was at most 3.

c. The numbers $1243756, 1234576$ and 1234567 are above average and have digit 7 in the fifth, sixth and seventh position, respectively.

   Digit 7 cannot be in the first position. Suppose that $7abcdef$ would be above average. Then $2 \cdot b \geq 7 + a \geq 8$, hence $b \geq 4$. Then, we must have $2 \cdot \geq a + b \geq 5$, hence $c \geq 3$. Now we find (in turn) that also $d, e, f \geq 4$. It follows that both digit 1 and digit 2 must be in the position of a, which is impossible.

   Digit 7 cannot be in the second or third position. Indeed, otherwise the digit following 7 must be at least 4, which implies that also the digits following it must be at least 4. Digits $1, 2$, and 3 must therefore all be in the first two positions, which is impossible.

   Finally, digit 7 cannot be in the fourth position. Digit 1 cannot be in the third position since $2 \cdot 1 < 2 + 3$.Because the digit in the third position must be at least 2, the digit in the fifth position must be at least 5. The next digit must therefore be at least 6, as must be the digit following it. The digits 1 to 4 must therefore all be in the first three positions, which is impossible.