Exercise 6 - Solutions

a. We will calculate $(32 \oplus 8)$ with the help of the rules stated in the exercise:

$$
\begin{aligned}
32 \oplus 8 &= (2 \cdot 16) \oplus 8 \\
&= \frac{1}{2} + (16 \oplus 8) && \textbf{(rule 1)} \\
&= \frac{1}{2} + ((2 \cdot 8) \oplus 8) \\
&= \frac{1}{2} + (\frac{1}{2} + (8 \oplus 8) && \textbf{(rule 1)} \\
&= \frac{2}{2} + ((2 \cdot 4) \oplus 8) \\
&= \frac{2}{2} + (\frac{1}{2} + (4 \oplus 8)) && \textbf{(rule 1)} \\
&= \frac{3}{2} + (4 \oplus 8) \\
&= \frac{3}{2} + (2^2 \oplus 8) \\
&= \frac{3}{2} + (64 \oplus 2) && \textbf{(rule 2)} \\
&= \frac{4}{2} + (32 \oplus 2) && \textbf{(rule 1)} \\
&= \frac{5}{2} + (16 \oplus 2) && \textbf{(rule 1)} \\
&= \frac{6}{2} + (8 \oplus 2) && \textbf{(rule 1)} \\
&= \frac{7}{2} + (4 \oplus 2) && \textbf{(rule 1)} \\
&= \frac{8}{2} + (2 \oplus 2) && \textbf{(rule 1)} \\
&= \frac{8}{2} + \frac{3}{2} && \textbf{(rule 3)} \\
&= \frac{11}{2}
\end{aligned}
$$

b. To calculate $128 \oplus (16 \oplus 64)$, we first have to calculate $(16 \oplus 64)$:

$$\begin{aligned}
(16 \oplus 64) &= \frac{1}{2} + (8 \oplus 64) && \textbf{(rule 1)} \\
&= \frac{2}{2} + (4 \oplus 64) && \textbf{(rule 1)} \\
&= \frac{2}{2} + (4096 \oplus 2) && \textbf{(rule 2)} \\
&= \frac{3}{2} + (2048 \oplus 2) && \textbf{(rule 1)} \\
&= \ldots && \textbf{(rule 1)} \\
&= \frac{13}{2} + (2 \oplus 2) && \textbf{(rule 1)} \\
&= \frac{13}{2} + \frac{3}{2} && \textbf{(rule 3)} \\
&= \frac{16}{2} = 8
\end{aligned}$$

Now, we can calculate $128 \oplus (16 \oplus 64) = 128 \oplus 8$:

$$\begin{aligned}
(128 \oplus 8) &= \frac{1}{2} + (64 \oplus 8) && \textbf{(rule 1)} \\
&= \ldots && \textbf{(rule 1)} \\
&= \frac{5}{2} + (4 \oplus 8) && \textbf{(rule 1)} \\
&= \frac{5}{2} + (64 \oplus 2) && \textbf{(rule 2)} \\
&= \frac{6}{2} + (32 \oplus 2) && \textbf{(rule 1)} \\
&= \ldots && \textbf{(rule 1)} \\
&= \frac{10}{2} + (2 \oplus 2) && \textbf{(rule 1)} \\
&= \frac{10}{2} + \frac{3}{2} && \textbf{(rule 3)} \\
&= \frac{13}{2}
\end{aligned}$$

So we can conclude: $128 \oplus (16 \oplus 64) = \frac{13}{2}$.