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Kidney Exchange Programmes

A guided tour from ~~2004~~ 2007 to 2023

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Facts and Figures

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- Chronic Kidney Disease (CKD) is the 11th most common cause of death in the World.
- 10% of the Portuguese population suffers from CKD.
- The number of deaths resulting from CKD has almost doubled globally since 1990.
- Has high economic impact on national health services (NHS)
  - in 2010, in the UK, the cost of treating the last stage of CKD – End Stage Kidney Disease (ESKD) – was estimated to be 1–2% of the total NHS budget, although ESKD patients comprise of only 0.05% of the total population.

Treatment options

Two treatment options:

- Dialysis
- Transplantation

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- Transplantation
  - Deceased donors: long waiting list.
  - Living donors (spouse, sibling, ...)
    - More effective: yields better patient and graft survival.
    - Yields better quality of life.
    - But several potential transplants cannot be performed due to incompatibility between patient and donor.

More figures

In Europe (2020)[1]:

[1] Global Observatory on Donation and Transplantation

[2] Biró, P. et al (2019). Building kidney exchange programmes in Europe—an overview of exchange practice and activities. Transplantation, 103(7), 1514.

More figures

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- Spain lead in number of Kidney transplants (deceased + living).

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- The Netherlands ranked 8th.
- More than 18% of the kidney transplants performed were from living donors (30% worldwile).
- “Depending on the country, 40% or more of recipients are incompatible with their intended donors.”[2]

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Incompatibility

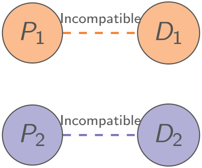
- Blood incompatibility:

Donor	Patient			
	O	A	B	AB
O	V	V	V	V
A	X	V	X	V
B	X	X	V	V
AB	X	X	X	V

- Immunological incompatibility (crossmatch test).

Living donor kidney transplants

The past



Transplants cannot proceed.

Kidney exchange programmes

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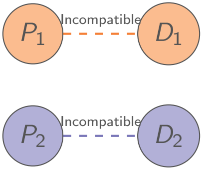

Figure: 2-way kidney exchange

Two transplants are now possible.

Kidney exchange programmes

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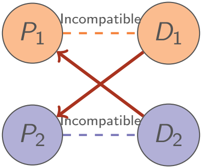

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Kidney exchange programmes

3-way exchange

The idea can be extended to more pairs.

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3-way exchange

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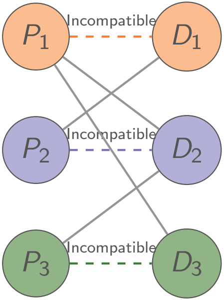

Kidney exchange programmes

3-way exchange

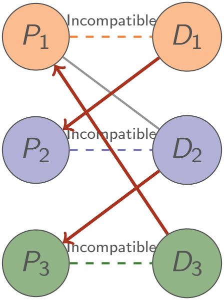

Figure: 3-way kidney exchange

Kidney exchange programmes

3-way exchange

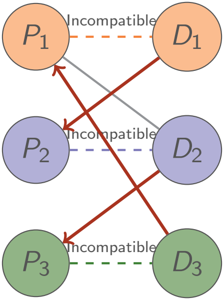

Figure: 3-way kidney exchange

Three transplants are possible if we allow three pairs in an exchange.

Kidney exchange programmes

A need for bounded cycle size

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Because all transplants in a cycle must be done simultaneously, in practice the maximum number of pairs involved in an exchange must be bounded.

- Logistic/personnel issues constrain the number of such simultaneous operations.

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A need for bounded cycle size

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- Logistic/personnel issues constrain the number of such simultaneous operations.

Besides, final compatibility tests may detect new incompatibilities

- if pairs X and Y in a cycle are found to be incompatible, in general, all transplants in the cycle involving X and Y have to be cancelled: the bigger the cycle the more pairs are affected.

- Nondirected donors (leads to the concept of *chain*)
- Multiple donors associated to one patient
- Compatible pairs

Kidney exchange programmes

Facts & figures

[3] Rapaport, F. T. (1986, June). The case for a living emotionally related international kidney donor exchange registry. In Transplantation proceedings (Vol. 18, No. 3) Suppl. 2, pp. 5-9)

Kidney exchange programmes

Facts & figures

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- The first transplants under a KEP took place in South Korea in 1991.
- The Dutch KEP (first in Europe) was established in 2004.
- Currently, the largest KEPs in Europe are: The Netherlands, Spain, and The United Kingdom

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Kidney exchange programmes

Optimising

- Programmes are now set in several countries, e.g.:
  - Portugal*, South Korea, USA*, Switzerland, Turkey, The Netherlands, UK*, Canada, Australia, New Zealand, Czech Republic, Austria, Spain*[4].

[4] Testing a new platform.

Kidney exchange programmes

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Some countries use Integer Programming models to solve the underlying optimisation problem.

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Kidney exchange programmes
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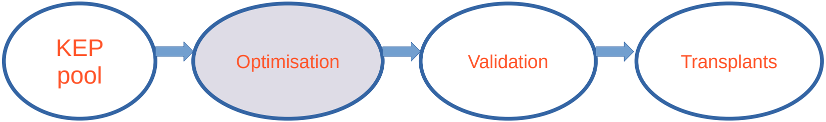

[4] Testing a new platform.

Kidney exchange programmes

Objectives

	Belgium	Czech Rep. (& Austria)	Netherlands	Poland	Portugal	Spain (& Italy)	Sweden	United Kingdom
max size of solution	1	1	1	1	1	1	1	2
min lengths of the cycles	-	-	4	-	-	-	-	-
max # cycles selected	-	2	-	-	-	2	-	3
max # back-arcs	-	-	-	-	-	3	-	4
max # 2-cycles and 3-cycles with embedded 2-cycles	-	-	-	-	-	-	-	1
min# desensitisations	-	w	-	-	-	-	3	-
max HLA-matching	-	w	-	w	-	-	-	w
max DR-antigen matching in particular	-	w	-	-	-	-	-	-
min age-differences between the donors and patients	5	-	-	w	-	w	-	-
priority for paediatric patient	-	-	-	-	-	w	-	-
priority for patients not yet on dialysis	-	-	-	w	-	-	-	-
priority for highly sensitive patients	-	-	-	w	-	4	-	w
priority for O patients	-	-	-	w	-	-	-	-
priority for hard-to-match patients	3	-	3	w	w	w	2	-
priority for waiting time in KEP	-	-	-	-	-	w	-	w
priority for time on dialysis	4	-	6	-	w	w	-	-
priority for same blood-group transplants	2	-	2	-	w	w	-	-
priority for pairs with AB-donors	-	-	-	-	-	w	-	-
max # of transplant centres in (long) cycles	-	-	5	-	-	-	-	-
priority for donor-patients in the same region	-	-	-	-	-	w	-	-
min the donor-donor age differences	-	-	-	w	w	-	-	w
Constraints on length of exchanges	no	no	4	3	no	3	2	3
(Longest exchange already conducted)	3	7	4	3	3	3	n.a.	3
Constraints on length of chains	n.a.	no	4	n.a.	n.a.	no	n.a.	3
(Longest chain already conducted)	n.a.	6	4	n.a	n.a.	6	n.a.	3
providing strictly better donors for compatible pairs	n.a.	yes	yes	n.a.	n.a.	yes	yes	yes
providing strictly better donors for half-compatible pairs	n.a.	yes	n.a.	n.a.	n.a.	n.a.	yes	n.a.
altruistic chain ends in the same region where started	-	-	yes	-	-	yes	-	-

Figure: *Extracted from:* P. Biró et al. Modelling and optimisation in European Kidney Exchange Programmes, European Journal of Operational Research, Vol. 291 (2), 2021, pp. 447–456.

(Base) Optimisation problem at hand

Optimisation problem at hand

Given a pool of N incompatible Patient-Donor pairs, find the maximum number of kidney exchanges (transplants) that involve cycles of size at most K.

Kidney exchange problem

Pre-processing: – transform the bipartite graph of compatibilities into a directed graph in which vertices represent incompatible patient-donor pairs and arcs between vertices represent compatibilities.

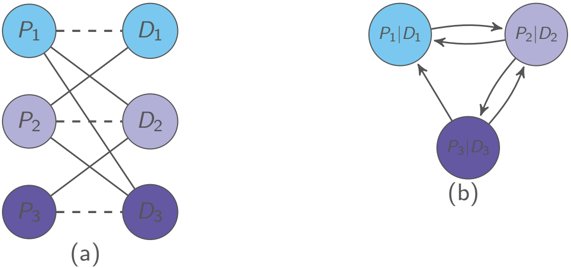

Figure: Bipartite versus Directed graph

A cycle with k nodes in the directed graph corresponds to a k-exchange.

Kidney exchange problem

Problem statement

Let $G(V, A)$ be a directed graph with:

- V – the set of vertices consisting of all incompatible patient-donor pairs;
- A – the set of arcs for designating compatibilities between the vertices.

Two vertices $i, j \in V$ are connected by arc (i, j) if the patient in pair j is compatible with the donor in pair i [5].

[5] If the objective is other than maximising total number of transplants (e.g., maximise weighted exchange) to each arc can be associated a weight w_{ij}.

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Definition: The Kidney Exchange Problem can be defined as follows:

Find a maximum weight packing of vertex-disjoint cycles having length at most k.

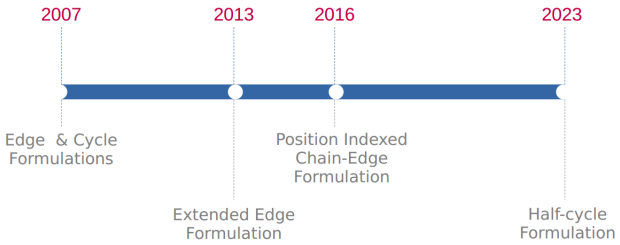
[5] If the objective is other than maximising total number of transplants (e.g., maximise weighted exchange) to each arc can be associated a weight w_{ij}.

For:

- k $= 2$ – the problem reduces to finding a maximum matching which can be solved efficiently (Edmonds 1965);
- k $= \infty$ – the problem can be formulated as an assignment problem and solved efficiently by the Hungarian algorithm;
- k ≥ 3 – *NP*-hard.

Integer Programming Formulations

Integer Programming Formulations
Timeline



Integer Programming Formulations

2007

Two Integer Programming models were presented in (Abraham et al, 2007) and (Roth et al, 2007):

Integer Programming Formulations

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Two Integer Programming models were presented in (Abraham et al, 2007) and (Roth et al, 2007):

- Edge formulation
- Cycle formulation

Proceedings of the 8th ACM conference on Electronic commerce,
June 13-16 2007

Clearing Algorithms for Barter Exchange Markets: Enabling Nationwide Kidney Exchanges

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Efficient Kidney Exchange: Coincidence of Wants in a Markets with Compatibility-Based Preferences

Citation

Roth, Alvin E., Tayfun Sönmez and M. Utku Ünver. 2007. Efficient Kidney Exchange: Coincidence of Wants in a Markets with Compatibility-Based Preferences. American Economic Review 97(3): 828-851.

Edge and cycle formulations

Edge formulation
(2007)

Edge formulation

(2007)

Variable x_{ij} is associated with each arc (i,j) $\in$ A in the graph and defined as:

$$x_{ij} = \begin{cases} 1 & \text{if patient } j \text{ gets a kidney from donor } i, \\ 0 & \text{otherwise.} \end{cases}$$

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(2007)

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Maximize   Number of transplants

Subject to: The number of kidneys donated by pair i must equal the number receiv

Each pair cannot donate more than one kidney.

Cycles cannot exceed length k

Edge formulation

(2007)

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(2007)

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$$\begin{aligned}
&\text{Maximize} && \sum_{(i,j)\in A} w_{ij} x_{ij} && && \text{(1a)} \\
&\text{Subject to:} && \sum_{j:(j,i)\in A} x_{ji} = \sum_{j:(i,j)\in A} x_{ij} && \forall i \in V && \text{(1b)} \\
& && \sum_{j:(i,j)\in A} x_{ij} \leqslant 1 && \forall i \in V && \text{(1c)} \\
& && \sum_{1\leqslant p\leqslant k} x_{i_p i_{p+1}} \leqslant k-1 && \forall \text{paths } (i_1, i_2, \cdots, i_k, i_{k+1}) && \text{(1d)} \\
& && x_{ij} \in \{0,1\} && \forall (i,j) \in A. && \text{(1e)}
\end{aligned}$$

Cycle formulation

(2007)

Cycle formulation

(2007)

Variables z_c are associated to each cycle c of length less or equal to k in the graph and defined as:

$$z_c = \begin{cases} 1 & \text{if cycle } c \text{ is selected for the exchange,} \\ 0 & \text{otherwise.} \end{cases}$$

Cycle formulation

(2007)

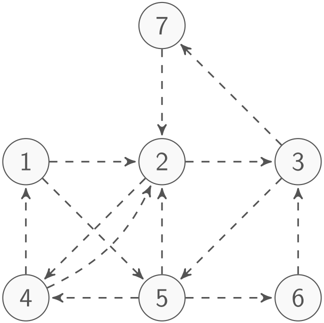

Figure: Base graph

Cycle formulation

(2007)

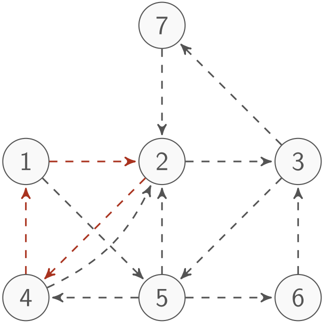

Figure: Base graph

c_1

Cycle formulation

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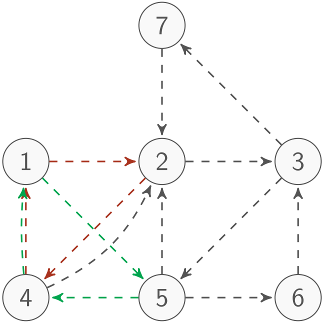

Figure: Base graph

c_1, c_2

Cycle formulation
(2007)

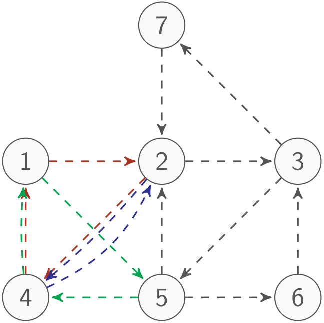

Figure: Base graph

c_1, c_2, c_3

Cycle formulation
(2007)

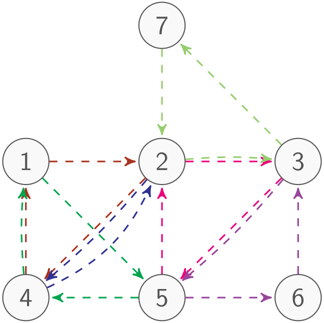

Figure: Base graph

c_1, c_2, c_3, c_4, c_5, c_6

Cycle formulation

(2007)

Cycle formulation

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For each cycle c of length less or equal to k in the graph:

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$$\text{Maximize} \quad \sum_{c \in \mathcal{C}(k)} w_c z_c \tag{2a}$$

$$\text{Subject to:} \quad \sum_{c : i \in c} z_c \leq 1 \qquad \forall i \in V \tag{2b}$$

$$z_c \in \{0, 1\} \qquad \forall c \in \mathcal{C}(k). \tag{2c}$$

- $w_c = \sum_{(i,j) \in c} w_{ij}$;
- constraints (2b): every vertex is in at most one cycle.

Cycle formulation

(2007)

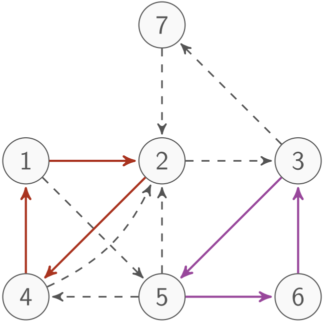

Figure: (One) optimal solution

Edge and Cycle formulations
(2007)

- Cycle formulation
  Exponential number of variables – enumeration of all cycles with length at most k.
- Edge formulation
  Exponential number of constraints – enumeration of all paths of length at most k.

Extended edge formulation

Extended edge formulation
(2013)

Extended edge formulation

(2013)

- Neither the edge nor the cycle formulations are compact: the number of constraints or variables grows exponentially with k or N.

Extended edge formulation
(2013)

- Neither the edge nor the cycle formulations are compact: the number of constraints or variables grows exponentially with k or N.
- (Constantino et al, 2013) propose an Integer Programming formulation whose number of variables and constraints does not depend on k and grows polynomially with N.

European Journal of Operational Research 231 (2013) 57–68
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journal homepage: www.elsevier.com/locate/ejor
ELSEVIER
Discrete Optimization

New insights on integer-programming models for the kidney exchange problem

Miguel Constantino[b,*,1,2], Xenia Klimentova[a,1], Ana Viana[a,c,1], Abdur Rais[d,1]

Extended edge formulation
(2013)

Main idea:

- Since there are at most $|V|$ cycles in the solution, we consider $|V|$ copies of the graph, where at most k edges are selected.
- To avoid multiplicity of solutions, in copy l only cycles where the lowest index of the vertices in that cycle is l are accepted.

Extended edge formulation

(2013)

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- Since there are at most $|V|$ cycles in the solution, we consider $|V|$ copies of the graph, where at most k edges are selected.
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With this reasoning one avoids the "path constraints" from the edge formulation.

Extended edge formulation

(2013)

[6] If a copy l of the graph provides a cycle for some solution, then node l must be in this cycle and all other nodes must have indices larger than l.

Extended edge formulation
(2013)

Maximize   Number of transplants

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Extended edge formulation

(2013)

Maximize Number of transplants

Subject to: In each copy l of the graph, the number of kidneys donated by pair i must equal the number received.

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Extended edge formulation

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A node can be selected in at most one copy of the graph.

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(2013)

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	A node can be selected in at most one copy of the graph.
	At most k edges can be used from each copy of the graph.
	Symmetry constraints[6].

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Extended edge formulation

(2013)

$$x_{ij}^{l} = \begin{cases} 1 & \text{if patient j gets a kidney from donor i \ in copy } l \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{maximize} \quad \sum_{l} \sum_{(i,j)\in A} w_{ij} x_{ij}^{l} \qquad \text{(3a)}$$

$$\text{subject to} \quad \sum_{j:(j,i)\in A} x_{ji}^{l} = \sum_{j:(i,j)\in A} x_{ij}^{l} \qquad \forall i \in V, \forall l \in \{1, ...|V|\} \qquad \text{(3b)}$$

$$\sum_{l} \sum_{i:(i,j)\in A} x_{ij}^{l} \leqslant 1 \qquad \forall j \in V \qquad \text{(3c)}$$

$$\sum_{(i,j)\in A} x_{ij}^{l} \leqslant k \qquad \forall l \in \{1, ...|V|\} \qquad \text{(3d)}$$

$$\sum_{j} x_{ij}^{l} \leqslant \sum_{j} x_{lj}^{l} \qquad \forall i > l \qquad \text{(3e)}$$

$$x_{ij}^{l} \in \{0, 1\} \qquad \forall (i,j) \in A, \forall l \in \{1, ...|V|\}$$

Extended edge formulation

(2013)

Variable elimination:

- If there is no cycle of size at most k containing both node l and an arc (i,j) with $i \geq l, j \geq l$, then variable x_{ij}^{l} can be eliminated from the model.

Extended edge formulation
(2013)

Variable elimination:

- If there is no cycle of size at most k containing both node l and an arc (i,j) with $i \geq l, j \geq l$, then variable x_{ij}^{l} can be eliminated from the model.

Other considerations:

- The cycle formulation dominates the extended edge formulation.
- The (non-compact) cycle formulation is very efficient for low density graphs with small values of K.
- For larger values of K and especially if graphs are denser the extended edge formulation provides better results.

Other compact formulations

(2015, 2016)

Alternative compact formulations followed the Extended Edge Formulation:

Other compact formulations

(2015, 2016)

Alternative compact formulations followed the Extended Edge Formulation:

1. EE-MTZ formulation (Mak-Hau, 2017).

J Comb Optim (2017) 33:35–59
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On the kidney exchange problem: cardinality constrained cycle and chain problems on directed graphs: a survey of integer programming approaches

Vicky Mak-Hau[1]

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2. Position-indexed edge formulation (PIEF) (Dickerson et al, 2016)

In Proceedings of the 2016 ACM Conference on Economics and Computation

Position-Indexed Formulations for Kidney Exchange

JOHN P. DICKERSON, Carnegie Mellon University
DAVID F. MANLOVE, University of Glasgow
BENJAMIN PLAUT, Carnegie Mellon University
TUOMAS SANDHOLM, Carnegie Mellon University
JAMES TRIMBLE, University of Glasgow

Position indexed chain-edge formulation (PICEF)

(2016)

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(2016)

- One type of variable for arcs in chains (polynomial) and another for each cycle (exponential on K).

Position indexed chain-edge formulation (PICEF)
(2016)

- One type of variable for arcs in chains (polynomial) and another for each cycle (exponential on K).
- Innovates by using position indices on arc variables.

Position indexed chain-edge formulation
(2016)

$\mathcal{K}'(i,j)$ – set of possible positions at which arc (i,j) may occur in a chain in G.

Position indexed chain-edge formulation

(2016)

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For $i, j \in V$ such that $(i,j) \in A$:

$$\mathcal{K}'(i,j) = \begin{cases} \{1\} & \text{if i is a NDD} \\ \{2, K\} & \text{otherwise.} \end{cases}$$

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For each $(i,j) \in A$ and each $k \in \mathcal{K}'(i,j)$:

$$y_{ijk} = \begin{cases} 1 & \text{if arc (i, j) is selected at position k of some chain,} \\ 0 & \text{otherwise.} \end{cases}$$

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For each cycle c of length less or equal to K in the graph:

$$z_c = \begin{cases} 1 & \text{if cycle } c \text{ is selected for the exchange,} \\ 0 & \text{otherwise.} \end{cases}$$

Position indexed chain-edge formulation (PICEF)

(2016)

Position indexed chain-edge formulation (PICEF)

(2016)

Maximize Number of transplants

Position indexed chain-edge formulation (PICEF) (2016)

Maximize	Number of transplants
Subject to:	Each patient is involved in at most one chosen cycle or incoming arc of a chain.

Position indexed chain-edge formulation (PICEF)

(2016)

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Subject to:	Each patient is involved in at most one chosen cycle or incoming arc of a chain.
	Each NDD vertex is involved in at most one outgoing arc.

Position indexed chain-edge formulation (PICEF)

(2016)

Maximize Number of transplants

Subject to: Each patient is involved in at most one chosen cycle or incoming arc of a chain.

Each NDD vertex is involved in at most one outgoing arc.

Pair i has an outgoing arc at position $k+1$ of a selected chain only if i has an incoming arc at position k.

Position indexed chain-edge formulation
(2016)

$$\text{Max.} \quad \sum_{(i,j)\in A} \sum_{k\in\mathcal{K}'(i,j)} w_{ij} y_{ijk} + \sum_{c\in\mathcal{C}} w_c z_c \tag{4a}$$

$$\text{Subj. to:} \quad \sum_{j:(j,i)\in A} \sum_{k\in\mathcal{K}'(j,i)} y_{jik} + \sum_{c\in\mathcal{C}:i\in c} z_c \leq 1 \qquad \forall i \in P \tag{4b}$$

$$\sum_{j:(i,j)\in A} y_{ij1} \leq 1 \qquad \forall i \in NDD \tag{4c}$$

$$\sum_{j:(j,i)\in A \wedge k\in\mathcal{K}'(j,i)} y_{jik} \geq \sum_{j:(i,j)\in A} y_{i,j,k+1} \qquad \forall i \in P, k \in \{1, 2, \ldots, K-1\} \tag{4d}$$

$$y_{ijk} \in \{0, 1\} \qquad (i,j) \in A, k \in \mathcal{K}'(i,j) \tag{4e}$$

$$z_c \in \{0, 1\} \qquad c \in \mathcal{C} \tag{4f}$$

Half-cycle formulation

(2023)

Operations Research Letters 51 (2023) 234–241

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Half-cycle: A new formulation for modelling kidney exchange problems

Maxence Delorme[a,*], David Manlove[b], Tom Smeets[a]

[a] Department of Econometrics and Operations Research, Tilburg University, the Netherlands
[b] School of Computing Science, University of Glasgow, United Kingdom

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[b] School of Computing Science, University of Glasgow, United Kingdom

- Based on the cycle formulation.
- A cycle is represented by two compatible half-cycles.
- For every pair of nodes i, j in V, a half-cycle starting in i and ending in j is selected iff another half-cycle starting in j and ending in i is selected.

Half-cycle formulation

(2023)

Half-cycle formulation

(2023)

$\mathcal{H}$ – set of half cycles of size up to $1 + \lceil \frac{K}{2} \rceil$

Half-cycle formulation

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For each half-cycle h in $\mathcal{H}$:

$$z_h = \begin{cases} 1 & \text{if half-cycle } h \text{ is selected for the exchange,} \\ 0 & \text{otherwise.} \end{cases}$$

Half-cycle formulation
(2023)

$\mathcal{H}$ – set of half cycles of size up to $1 + \lceil \frac{K}{2} \rceil$

For each half-cycle h in $\mathcal{H}$:

$$z_h = \begin{cases} 1 & \text{if half-cycle } h \text{ is selected for the exchange,} \\ 0 & \text{otherwise.} \end{cases}$$

Additional notation:

V_s^h: starting vertex of half-cycle h.
V_e^h: ending vertex of half-cycle h.
V_m^h: other vertices of h.

Position indexed chain-edge formulation (PICEF)

(2016)

[7]the starting and ending nodes of selected half-cycles count 0.5 as they both appear in two half-cycles

Position indexed chain-edge formulation (PICEF) (2016)

Maximize   Number of transplants[7]

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Maximize Number of transplants[7]

Subject to: Each pair is no more than once in the middle or more than twice at the start/end of the selected half-cycles.

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Position indexed chain-edge formulation (PICEF)

(2016)

Maximize Number of transplants[7]

Subject to: Each pair is no more than once in the middle or more than twice at the start/end of the selected half-cycles.

Every selected half-cycle must be matched with another selected half-cycle to form a complete cycle.

[7] the starting and ending nodes of selected half-cycles count 0.5 as they both appear in two half-cycles

Half-cycle formulation[8]

(2023)

$$\text{Max.} \quad \sum_{h\in\mathcal{H}} (|V^m(h)| + 1) z_h \tag{5a}$$

$$\text{Subj. to:} \quad \sum_{h\in\mathcal{H}:v\in V^s(h)\cup V^e(h)} 0.5 z_h + \sum_{h\in\mathcal{H}:v\in V^m(h)} z_h \leq 1 \qquad \forall v \in V \tag{5b}$$

$$\sum_{h\in\mathcal{H}:i\in V^s(h),j\in V^e(h)} z_h = \sum_{h\in\mathcal{H}:i\in V^e(h),j\in V^s(h)} z_h \qquad \forall i,j \in V : j > i \tag{5c}$$

$$z_h \in \{0,1\} \qquad \forall h \in \mathcal{H} \tag{5d}$$

[8] If K is odd the following additional constraint is needed to forbid two half-cycles with size $1 + \lceil \frac{K}{2} \rceil$ from being merged: $z_h = 0, \forall h \in \mathcal{H} : V^s(h) > V^e(h) \text{and} V^m(h) = (K-1)/2$.

Half-cycle formulation

(2023)

Variable elimination:

- Eliminate every half-cycle in which the vertex with the lowest index is neither located at the beginning nor at the end of the half-cycle.

Half-cycle formulation

(2023)

Variable elimination:

- Eliminate every half-cycle in which the vertex with the lowest index is neither located at the beginning nor at the end of the half-cycle.

Other considerations:

- With some enhancements (on variable fixing) presents better results than the cycle formulation, for $k \geq 5$.
- Did not compare with the extended edge formulation for $k \geq 5$.

Some other topics deserving attention

- Data uncertainty
- Transnational programmes

Handling data uncertainty

The models that are currently used in practice consider, in general, that data is certain, which is not true:

Kidney exchange programmes

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- Incompatibilities may be detected between pair matching and actual transplantation (arc failure).

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- Incompatibilities may be detected between pair matching and actual transplantation (arc failure).
- Pairs may dropout of the program (node failure).
- Patients and/or donors may be physically unfit when the operation is scheduled (node failure).
- ...

This means that:

Kidney exchange programmes

Handling data uncertainty

This means that:

- Some optimal solutions obtained under a certain scenario can be very bad at the time of implementation (i.e. a big reduction in terms of the actual number of transplants).

Kidney exchange programmes

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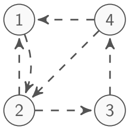
- Some sub-optimal solutions obtained under a certain scenario can be much better than the optimal solution, at the time of implementation.

This means that:

- Some optimal solutions obtained under a certain scenario can be very bad at the time of implementation (i.e. a big reduction in terms of the actual number of transplants).
- Some sub-optimal solutions obtained under a certain scenario can be much better than the optimal solution, at the time of implementation.
- Alternative optimal solutions can lead to very different outcomes.

Kidney exchange programmes

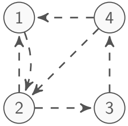
Arc/node failure: an example



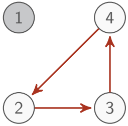
(a) Base graph

Kidney exchange programmes

Arc/node failure: an example



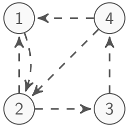
(d) Base graph



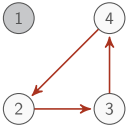
(e) Optimal

Kidney exchange programmes

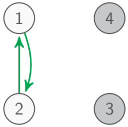
Arc/node failure: an example



(g) Base graph



(h) Optimal



(i) Sub-optimal

Kidney exchange programmes

Current state of research

How has data uncertainty been handled so far?

Kidney exchange programmes

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- Assigning probabilities to node and arc failure
  - Maximise expected number of transplants.

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- Robust optimisation
  - Maximise the number of pairs selected in both the initial and the final solution, given a specific scenario.

Kidney exchange programmes

Current state of research

How has data uncertainty been handled so far?

- Assigning probabilities to node and arc failure
  - Maximise expected number of transplants.
- Robust optimisation
  - Maximise the number of pairs selected in both the initial and the final solution, given a specific scenario.

Plus:

- Recourse policies.
  - Reconstruction policies that can be implemented in a solution if one/some of its nodes and/or arcs fail.

Kidney exchange programmes

Recourse policies

- No recourse:

Kidney exchange programmes

Recourse policies

- No recourse:
  - Maximise the expected number of transplants.
  - Robust optimisation: maximise the number of pairs selected in both the initial and the final solution, in the worst case.

Kidney exchange programmes

Recourse policies

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- Backarcs recourse

Kidney exchange programmes

Recourse policies

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- Subset-recourse (S.O.)

Kidney exchange programmes

Recourse policies

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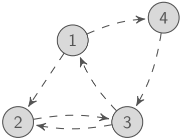
- Full-recourse (R.O.)

Kidney exchange programmes

Examples of recourse policies

- Backarcs recourse



Kidney exchange programmes

Examples of recourse policies

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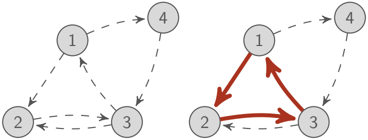

Kidney exchange programmes

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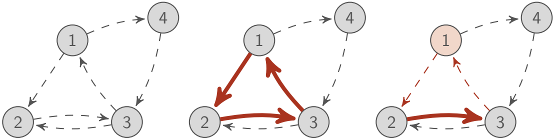

Kidney exchange programmes

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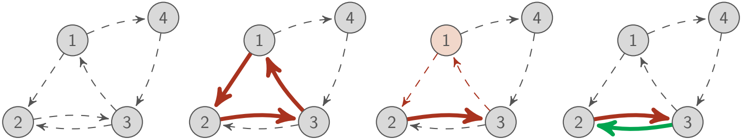
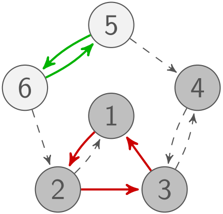


Kidney exchange programmes

Examples of recourse policies

- Subset recourse
  - considers the possibility of involving in the rearrangement vertices not enclosed in the cycle.



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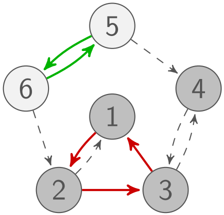

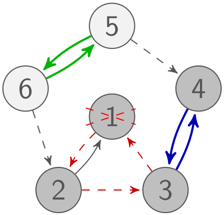

Kidney exchange programmes

Data uncertainty

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Cooperation between multiple agents (mKEP)

mKEP

International collaborations have already been established in Europe:

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- KEP–SAT (2017): Portugal, Spain, Italy.

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mKEP

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  - Merged pools: there is a single pool with all the pairs of all participating countries.

mKEP

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  - Merged pools: there is a single pool with all the pairs of all participating countries.
- (*at a hospital level*) Czechia – Austria (2016) – Israel (2019) .
  - Merged pool.

mKEP

A COST Action willing to address challenges foreseen in international collaborations started in 2016.

[9] https:www.kepsoft-cost.eu

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ENCKEP: European Network for Collaboration o Kidney Exchange Plans (Sept.2016 – Mar. 2021)

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- 28 countries
- Multidisciplinary group: policy makers, clinicians, optimisation experts, ...

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- 28 countries
- Multidisciplinary group: policy makers, clinicians, optimisation experts, ...

ENCKEP was followed by COST Innovators Grant:

- KEPSoft: Software for Transnational Kidney Exchange Programmes (Nov.2021 – Oct. 2022)[9].

[9] https:www.kepsoft-cost.eu

A main deliverable of the two projects is KEPSoft, a software product to assist national and international KEPs:

- Insert pairs and NDDs in a Database.
- Construct a compatibility graph.
- Select from a multitude of objectives (hierarchical optimisation).
- Optimise.

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- Optimise.

KEPSoft led to the KEPSoft Community. (under construction!)

KEPSoft Community.

- A not-for-profit social enterprise aiming at providing products (KEPSoft) and services to the transplantation community.

[10] https:www.gla.ac.ukmyglasgowrisipcommercialisationinnovatingthefuturecasestudieskepsoft

KEPSoft Community.

- A not-for-profit social enterprise aiming at providing products (KEPSoft) and services to the transplantation community.
- KEPSoft Community is now searching for a Commercial Champion to lead the business-focused aspects of setting up the company[10].

[10] https:www.gla.ac.ukmyglasgowrisipcommercialisationinnovatingthefuturecasestudieskepsoft

mKEP

KEPSoft Community

A video by the chair of ENCKEP and KEPSoft projects on the KEPSoft Community[11]

[11] https:gla-my.sharepoint.compersonaldavid_manlove_glasgow_ac_uk_layouts15stream.aspx?id=%2Fpersonal%2Fdavid%5Fmanlove%5Fglas

mKEP

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ORAHS International Seminar Series
University of Twente, The Netherlands, 27 June 2023

Kidney Exchange Programmes

A guided tour from ~~2004~~ 2007 to 2023

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