SOLVING DETERMINISTIC PROBLEMS BY PROBABILISTIC METHODS

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ABSTRACT

Today's deterministic problems have grown too large, too intricate and too difficult for solution by conventional deterministic methods. Moreover, an increased awareness of the fuzzyness inherent to many deterministic problems has spurred interest in alternative solution methods.
Here we focus on probabilistic methods that attempt to solve these deterministic problems by way of a sophisticated sampling procedure. Such methods result in probability statements about the solution of the problem.

We illustrate the discussion on the probabilistic methods with examples taken from large scale linear programming. A probabilistic method to identify redundant constraints is described in detail.

Keywords: Probabilistic Methods, Linear Programming, Redundancy

1. INTRODUCTION

Deterministic problems (problems of which all aspects are known with certainty) generally are easier to solve than their probabilistic counterparts, in which uncertainty plays a role. It is the sheer fact of uncertainty that accounts for this difference. So how could we be interested in introducing probabilistic arguments in the solution process of deterministic problems?

In this paper we argue that today's deterministic problems have grown too large, too intricate and too difficult for solution by conventional deterministic methods. And even if we ignore these facts we must admit that many deterministic problems are not as deterministic as we thought them to be. An increased awareness of the latter fact has spurred interest in the fuzzyness inherent to many problems and appropriate methods to handle that.

For these reasons we believe there is a definite need for alternative solution methods of which a probabilistic approach seems to be a prominent and promising one.

In the next section we will expand on this reasoning and discuss the difficulties arising in deterministic problems in more detail.
In that discussion we will use the well-known linear programming (LP) problem as a generic example.

Section 3 contains a rather general exposition of the probabilistic approach. Basically the probabilistic methods try to solve a problem by way of a sophisticated sampling procedure. This results in probability statements on the solution of the problem. Again the LP problem is used to illustrate this approach.

We conclude the paper with a description of a probabilistic method for the (deterministic) problem of the identification of (non-) redundant constraints in linear systems.

The usefulness of this approach is illustrated by some results from experimental applications.

2. DETERMINISTIC PROBLEMS

The sheer absence of uncertainty in deterministic problems causes them to be more easily solved than their probabilistic counterparts. And indeed for many deterministic problems solution methods have been developed; the simplex method for linear programming (LP) is a prime example of a very succesful and widely used method for a deterministic problem.

However the development of methods did not stay in line with the recognition and formulation of "new" Operations Research (O.R.) problems in practice. And, more importantly, the size of practical instances of known and well-solved OR problems grew even faster than the increasing efficiency and power of (deterministic) methods and computing machinery. In the early fifties LP problems of 20 constraints and 50 variables (20 x 50) were considered to be large, whereas nowadays LP problems of the order 3000 x 10000 are solved routinely. But now also LP problems of 10,000 x 100,000 are being formulated and they are beyond the present limitations with respect to numerical accuracy, computing time etc. Moreover the above characterictics tend to appear simultaneously and increase (worsen) each other: in large LP problems quite frequently a reinversion is followed by some phase I steps to restore feasibility, thus worsening the objective function value, requiring more solution time etc.

Apart from the increasing size and difficulty of the problems and the insufficient development of deterministic methods a new aspect has been recognized recently: deterministic problems (and especially the larger ones) are not as deterministic as they seem to be.
They tend to incorporate elements that are "fuzzy" in nature, i.e. elements that are not known precisely (which is essentially different from randomness, see e.g Bellman and Zadeh (1970)). In LP problems for instance the presence or absence of constraints may not be known exactly (or be disputed).

Conventional methods offer no ways to handle the fuzzyness in these (otherwise deterministic) problems.

Now we have established the shortcomings of existing methods for today's deterministic problems, we sketch a few ways out of this situation:

(a) <u>Suboptimal solutions</u>

If solving the problems (to optimality) is not feasible one has to resort to intermediate or suboptimal solutions , if available. The availability of these solutions depends on the specific solution method used and its relative power on the problem.
To give an example: for the LP problem the ellipsoidal method does not provide a feasible solution until the very last step, whereas the simplex method usally does; but even the simplex method may "get stuck" in phase I (before producing a feasible solution).

(b) <u>Heuristics</u>

A heuristic method is based on an assumption that is not necessarily true and therefore has to be validated afterwards. An LP heuristic would be to take into account only part of the variables and constraints and claim the resulting solution to be optimal (clearly the assumption is not necessarily true).
Formalized techniques like aggregation and curtaining belong to this class.
The power of a heuristic is determined by the relative value of the solution, i.e. is it close to the "real" solution? (which is usually hard to determine) is it useful (feasible) at all?, etc.

An important drawback of heuristic methods is that they generally do not give any indication at all about the relative value of the solution they produce.
And it is exactly in this aspect that the third alternative we propose (the probabilistic approach) is a powerful instrument.

3. PROBABILISTIC METHODS

Basically a probabilistic approach consists of (1) a sophisticated sampling procedure, (2) a set of rules to derive (partial) results from the sample and (3) a probability statement expressing the degree to which the partial results approximate the complete result.

The sampling procedure has to be a sophisticated one indeed since the probability statement (3) is based upon the specifics of the sampling procedure. Generally the sampling procedure should induce a certain distribution of sampled points within the region of interest. Then this distribution can be used to derive statistical functions (e.g. order statistics) describing the solution to the problem.

Such an approach is by no means new. In global (nonconvex) optimization it is a quite well-known procedure and a very succesful one too (see e.g. Patel and Smith (1980), Boender et al. (1981). Also the probabilistic approach has been used in large scale, highly nonlinear (and nonconvex) problems, in which it proved to be a very practical approach (Boneh and Golan (1979)).

The point we want to stress here is the usefulness of the probabilistic approach for large, well-solved problems as well. For example in LP, solving 10,000 x 100,000 problems with the simplex method generally is not feasible. But with the probabilistic approach it is not prohibitively expensive to derive probability statements about the solution in the following way: (1) generate uniformly distributed points in the feasible region, (2) note the lower bound on the objective function provided by the best of these sample points, (3) derive probability statements about the solution from the sample (see e.g. de Haan (1981)). Of course, the above example is not a very sophisticated implementation of the probabilistic approach; many refinements (e.g. gradient steps etc.) are possible. The authors are currently involved (with C.G.E. Boender and A.H.G. Rinnooy Kan) in a

effort to extend this approach to include Bayesian arguments; results will be reported shortly.
One area in which the probabilistic approach has already outperformed the conventional deterministic approach is the identification of redundancy in systems of linear constraints. We devote the last section to this topic.

4. IDENTIFICATION OF (NON)REDUNDANCY

This section is an illustration of the probabilistic approach taken from Smith and Telgen (1981), to which we refer for more details and proofs. General references on redundancy include Telgen (1979) and Karwan et al. (1981).

The basic observation for this application is the fact that redundant constraints cannot be "hit" from an interior point, i.e. every ray, extended from an interior point has to cross another constraint before it can cross a redundant constraint. Now the procedure is to generate interior points, extend rays from all these points and label the constraints "hit" as nonredundant.
After many iterations the non-"hit" constraints are labelled redundant (possibly with an error).
To apply this method, which was first proposed in Boneh and Golan (1979), we need a method to generate interior points. Boneh and Golan (1979) suggested to extend a ray in both directions from an interior point, until there are two "hitpoints".
Then the new interior point is chosen uniformly on the line between the two hitpoints.
Independently Smith (1981) developed the same procedure and showed that the distribution of the interior points is asymptotically uniform over the feasible region.
Finally Smith and Telgen (1981) showed that the hitpoints are asymptotically uniformly distributed over the boundary of the feasible region. The latter fact ebables one to construct formulas giving the number of hitpoints necessary to identify all nonredundant constraints with a certain probability. This number depends on the relative volume of the facets corresponding to the constraints in the following way:

$$k^*_{1-\alpha} \leqslant \frac{1}{p_{min}} (\ln m - \ln \alpha)$$

in which: $k^*_{1-\alpha}$ = number of hitpoints necessary to identify all noredundant constraints with probability $1-\alpha$

p_{min} = smallest percentage of the total volume belonging to one facet

m = number of constraints.

Alternative stopping criteria based on a Bayesian analysis are provided in Boender et al.(1981 B).
This approach has been tested on linear problems on which great computational savings can be achieved by using coordinate directions instead of general rays to obtain the hitpoints. Some representative results are given in table 1.

PROBLEM SIZE	ACTUAL NO OF NONRED. CONSTRAINTS	BEST DETERM. METHOD SEC. CPU TIME	PROBABILISTIC METHOD			
			NO.OF NONRED. CONSTRAINTS IDENTIFIED IN 1 SEC.	NO.OF HITPOINTS IN 1 SEC.	TIME LAST ONE FOUND	$k^{*}_{.95}$ ($P_{min} = 10^{-2}$)
26 x 12	13	.13	13	696	.08	625
22 x 10	11	.11	11	790	.11	609
29 x 20	29	.20	23 (79%)	620	.16	636
37 x 15	23	.47	20 (87%)	540	.15	660
59 x 23	54	2.26	41 (76%)	345	.33	706
29 x 5	11	.37	11	485	.05	636

Table 1. Performance of the probabilistic method.

should be doubled

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