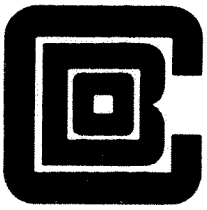

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An ellipsoidal interactive multiple goal programming method

Jaap Spronk and Jan Telgen

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CENTRE FOR RESEARCH IN BUSINESS ECONOMICS
Erasmus University Rotterdam
Burgemeester Oudlaan 50
3062 PA Rotterdam
The Netherlands

Tel.: 010-145511

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AN ELLIPSOIDAL INTERACTIVE MULTIPLE GOAL PROGRAMMING METHOD

by Jaap Spronk and Jan Telgen

CONTENTS

Jaap Spronk
Assistant Professor
Department of Finance
Erasmus University
Rotterdam, The Netherland

Jan Telgen
Visiting Research Associate*
Management Science Program
University of Tennessee
Knoxville, TN 37916, USA

*NATO Science Fellow for the Netherlands Organization for the Advancement of Pure Research (ZWO).

Abstract

A new interactive multiple goal programming method is presented, using the recently developed ellipsoidal algorithm that solves LP - equivalent problems in polynomially bounded time.

At each iteration of the method the decision-maker is presented two items: (1) the current proposed solution and (2) bounds on the possible values of all goals. The decision-maker has to indicate how he wants to improve the proposed solution. Then the method yields the effects of that decision on the bounds on the values of the goals. The decision-maker must choose between the last proposed solution and the "improved" one keeping in mind the change in the range of possible values for all goals.

If the ranges of possible values of all goals are always computed exactly, the resulting method is basically Spronk's IMGP method. However, the use of the ellipsoidal algorithm enables one to determine approximate bounds without additional calculations. Thus if inexact bounds are acceptable, the number of computations per iteration is greatly reduced, which in turn improves the response time for the decision-maker.

Whether exact or inexact bounds are used two more points should be noted about this method: first, nonbinding constraints may be identified with few additional calculations thus giving valuable insight into the model. Second, under the assumption that the decision-maker is satisfied after a polynomial number of iterations, the method is polynomially bounded. To our knowledge this is the first such result for interactive multiple objective programming methods.

1. Introduction

The development of the ellipsoidal algorithm evoked a great deal of excitement among mathematicians and computer scientists. L. G. Khachian [1979] showed that this algorithm is capable of determining in polynomially bounded time whether or not a system of linear constraints is feasible. Since the latter problem is known to be LP-equivalent (see e.g. Telgen [1979]) this result implies that the ellipsoidal algorithm is a formally 'good' algorithm for the linear programming problem. Hence, in the framework of the theory of computational complexity (see e.g. Garey and Johnson [1979]) linear programming is an 'easy' problem i.e. belongs to the class P.

It was known for quite some time that the simplex method (Dantzig [1963]) is not a formally 'good' algorithm for linear programming (Klee and Minty [1972]). Since the theoretical quality of an algorithm is measured according to its worst case behavior, this does not necessarily imply that the simplex method should be replaced by the ellipsoidal algorithm in practical applications. For these purposes the average case performance is usually more important and until now the simplex method seems to be superior in that aspect.

Multiple objective programming methods generally employ a series of linear programming subproblems. The method used to solve these subproblems should have a good average case performance. Therefore it seems unlikely that the simplex method will be replaced by the ellipsoidal algorithm in practical applications of multiple objective programming.

However there is one class of multiple objective programming methods for which the development of the ellipsoidal algorithm opens an entire

spectrum of new possibilities: the class of interactive multiple objective programming methods using a systematic search over different levels of objective values. One of these methods is Interactive Multiple Goal Programming, (IMGP) see Nijkamp and Spronk [1980] and Spronk [1980]. Roughly, it solves a number of subproblems over a feasible region which is continually decreasing in size because of the addition of constraints representing minimal and maximal levels of the goal variables.

Basically, we propose to solve the subproblems by means of the ellipsoidal algorithm. Furthermore we introduce an option in the method: we do not require to solve the subproblems to optimality. This reduces the computational effort required in each iteration of the method and consequently the response time for the decision maker considerably. In the ellipsoidal algorithm approximate information can usually be obtained fairly simple without many computations. Especially in the early stages of the application of the IMGP method this may considerably improve convergence to an optimal solution. However if exact information is required this can also be supplied.

But apart from the practical aspects it can easily be seen that the method only requires a polynomially bounded number of computations per iteration. Therefore, if the decision maker only needs a polynomially bounded number of decisions (iterations), which seems to be a reasonable assumption, then this variant of the IMGP method is polynomially bounded. Hence this interactive multiple goal programming method is likely to be a member of the class P. To our knowledge this is the first such result in the field of interactive multiple programming methods.

The plan for this paper is as follows: first we review the ellipsoidal algorithm and the IMGP method in sections 2 and 3 respectively. Then we present the ellipsoidal interactive multiple goal method in section 4. We conclude the paper with a discussion on the merits of our method.

2. The ellipsoidal algorithm

Consider the system of linear constraints

$$Ax \leq b \tag{2.1}$$

where A is an (m x n) matrix, x is an n- vector and b is an m- vector. Denote the rows of A by α_i^T, i = 1, ... m.

The ellipsoidal algorithm starts from an ellipsoid

$$E_0 = \{ x \in R^n \mid (x - x^0)^t Q_0^{-1} (x - x^0) \leq 1 \}$$

which is so large that it contains a feasible solution to the system (1) if there is one. Usually the center x^0 of the starting ellipsoid is taken to be the origin.

In the k - th step of the algorithm a constraint that is violated by the current center x^k is selected, say $\alpha_\ell^T x \leq b_\ell$. Then that part of E_k that satisfies $\alpha_\ell^T x \leq b_\ell$ is circumscribed by the smallest possible ellipsoid, which is defined as E_{k+1}. This operation is achieved by setting

$$x^{k+1} = x^k - \frac{(1 + n\delta_\ell)}{(n + 1)} \frac{Q_k \alpha_\ell}{\sqrt{(\alpha_\ell^T Q_k \alpha_\ell)}} \tag{2.2}$$

$$Q_{k+1} = \frac{n^2(1-\delta_\ell^2)}{n^2-1} \left[Q_k - \frac{2(1+n\delta_\ell)}{(n+1)(1+\delta_\ell)} \frac{Q_k \alpha_\ell \alpha_\ell^T Q_k}{\alpha_\ell^T Q_k \alpha_\ell} \right] \tag{2.3}$$

where δ_ℓ is the distance from x^k to the hyperplane $\alpha_\ell^T x = b_\ell$ in the metric induced by the ellipsoid i.e. $\delta_\ell = \dfrac{\delta_\ell x_k - b_\ell}{\sqrt{\alpha_\ell^T Q_k \alpha_\ell}}$ (2.4)

(For a derivation of these formulas see Goldfarb and Todd [1980]). Then it can be proved that if the algorithm does not find a feasible solution in $16\, n^2 L$ steps no feasible solution exists (L is the length of the binary encoding of all problem data a_{ij}, b_i, m and n); for a proof see e.g. Gacs and Lovasz [1979] or Padberg and Rao [1979]. Many improvements and refinements on this basic scheme are possible: however these do not basically affect the method and we refer the interested reader to the excellent treatment in Goldfarb and Todd [1980].

One point however should be noted: it is fairly easy to see whether or not $\alpha_i^T x = b_i$ and E_k intersect. Just calculate δ_i according to (2.4) and if $|\delta_i| \geq 1$ they do not intersect. This allows one to draw conclusions about redundancy of the constraint or infeasibility of the system.

Using the same formula upper and lower bounds on the value of an objective function $c^T x$ can be calculated: setting

$$|\delta_0| = \frac{c^T x^k - z}{\sqrt{d^T Q_k c}} < 1 \qquad (2.5)$$

and solving for z yields

$$c^T x - \sqrt{c^T Q_k c} < z < \sqrt{c^T Q_k c} + c^T x^k \qquad (2.6)$$

Of course this upper (lower) bound may be too high (low), but if more exact bounds are required a dichotomy search over the interval (2.6) using the ellipsoidal algorithm may provide this information.

3. Interactive multiple goal programming (IMGP)

Consider the multiple goal programming problem:

$$\begin{aligned} &\max \quad g_i(x) \qquad \forall i \\ &\text{subject to} \quad Ax \leq b \end{aligned} \tag{3.1}$$

In order to deal with the difficulties posed by the multiplicity of the objective functions (goal variables) in (3.1), methods have been devised to solve (3.1) in an interactive process with a decision-maker. One of these methods in interactive multiple goal programming (IMGP) (see Nijkamp and Spronk [1980] and Spronk [1980]).

In IMGP the decision-maker provides information about his preferences on the basis of a solution and a potency matrix presented to him. The potency matrix consists of 2 vectors, representing the pessimistic and the ideal solution. For each of the goal variables separately the pessimistic solution lists a minimum value (usually supplied by the decision-maker), whereas the ideal solution lists the individual maximum values given the pessimistic solution. The decision-maker has to indicate whether or not a solution is satisfactory, and if not, which of the minimum goal values should be raised. Then a new solution is presented to him together with a new potency matrix. The decision-maker has to indicate whether the shifts in the solution are outweighted by the shifts in the potency matrix. If not, a new solution is calculated and so forth.

To simplify the explanation, we describe the method here assuming that at each iteration one and only one element of the solution will change. A generalization to more elements is straightforward.

Step 1 - Maximize successively each of the m goal variables $g_i(x)$ separately and denote the maxima by g_i^* and the m corresponding combinations of the instrumental variables by $x^{i}*$ i=1,...,m. It is not possible to find a feasible value of $g_i(x)$ which exceeds g_i^*. Generally, it is not necessary to accept a value of $g_i(x)$ which is lower than g_i^{min}, defined as:

$$g_i^{min} = \min_j \{g_i(x^j*)\} \tag{3.2}$$

the lowest value of $g_i(x)$ resulting from the successive maximizations of the goal variables. Then, the final solution S* must be found between the 'ideal' (but mostly infeasible) solution I, and the 'pessimistic' solution Q, which are defined respectively as:

$$I = [g_1^*, g_2^*, \ldots, g_m^*] \text{ and}$$

$$Q = [g_1^{min}, g_2^{min}, \ldots, g_m^{min}] \tag{3.3}$$

To facilitate the notation we have included the ideal solution I and the pessimistic solution Q in the (2 x m) 'potency matrix' P.

Step 2 - Define for all j = 1, ...,m

$$\delta_j = g_j^* - g_j^{min} \tag{3.4}$$

Step 3 - Define the initial solution as:

$$S_1 = [\, g_1^{min}, g_2^{min}, \ldots, g_m^{min} \,] \tag{3.5}$$

which is thus equal to the pessimistic solution defined in (3.3). Present this solution together with the potency matrix P_1 to the decision-maker.

Step 4 - Does the proposed solution satisfy the decision-maker? If so, stop; if not, define R_i as the subset of R defined by the goal levels in S_i, and continue with step 5.

Step 5 - The decision-maker has to answer the question: "Given the provisional solution S_i, which goal variable should be improved first?"

Step 6 - Assume that the decision-maker wants to augment the j-th goal variable. Then construct a new trial solution $\hat{S}_{i+1}$, which differs from S_i only in the value of the j-th goal variable (denoted by $g_j(x)_{\hat{S}_{i+1}}$ and $g_j(x)_{S_i}$ respectively).

Define:

$$g_j(x)_{\hat{S}_{i+1}} = g_j(x)_{S_i} + \tfrac{1}{2}.\delta_j \tag{3.6}$$

and introduce the restriction:

$$g_j(x) \geq g_j(x)_{\hat{S}_{i+1}} \tag{3.7}$$

Step 7 - Combine the restriction formulated in step 6 or in step 9 with the set of restrictions describing the feasible region R_i. Calculate a new potency matrix, like in step 2, but subject to the new set of restrictions. Label this potency matrix $\hat{P}_{i+1}$.

Step 8 - Confront the decision-maker with S_i and $\hat{S}_{i+1}$ on the one hand and with P_i and $\hat{P}_{i+1}$ on the other hand. The shifts in the potency matrix can be viewed as a 'sacrifice' for reaching the proposed solution. If the decision-maker judges this sacrifice to be justified, accept the proposed solution by putting $S_{i+1} = \hat{S}_{i+1}$ and $P_{i+1} = \hat{P}_{i+1}$ and put $\delta_j = \tfrac{1}{2}\,\delta_j$. Continue with step 4. If the decision-maker considers the sacrifice unjustified, the proposed value of $g_j(x)$ is obviously too high. Therefore, drop the constraint added in step 7.

Step 9 - We now know that $g_j(x)_{S_i}$ is too low and that $g_j(x)_{\hat{S}_{i+1}}$ is too high in the decision-maker's view. Set δ_j equal to the difference between these two values. A new proposal value $\hat{S}_{i+1}$ is then calculated according to (3.6). As in step 6 we add the restriction that $g_j(x)$ must equal or exceed the new proposal value and go to step 7.

For further details on IMGP (inclusion of aspiration levels etc.) we refer to Nijkamp and Spronk [1980] and Spronk [1980].

4. Interactive ellipsoidal multiple goal programming.

In Spronk's interactive multiple goal programming method (IMGP) the main computational work is done in the determination of the potency matrix in each iteration. If the problem has linear constraints and the goal variables are linear functions of the instrumental variables, basically this amounts to solving m linear programming problems. Obviously the similarity between these LP problems may give way to substantial savings as compared to solving m completely different problems. However the computational effort per iteration can generally not be neglected.

To improve upon the latter aspect we propose to use the ellipsoidal algorithm for the calculation of the potency matrices, or approximations thereof. Implementing the ellipsoidal algorithm in IMGP requires the determination of an ellipsoid which contains the feasible region R entirely. This ellipsoid is updated (actually, decreased in size) most of the times when new constraints on the goal variables are imposed in steps 7 and 9. Then the calculation of the potency matrix in steps 1 and 7 can be done using this ellipsoid as a starting point.

Because the initial ellipsoid may be used several times as a starting point, and since its volume codetermines the accuracy of the generated bounds (see below), it is useful to spend some extra time in determining a small initial ellipsoid which contains all of R.

The big advantage of using the ellipsoidal algorithm in IMGP is the possibility it opens to perform approximate computations. First, the new potency matrix can be calculated by m dichotomy searches but it can also be approximated by using relation (2.6) with $c^T x = g_j(x)$. This approximation actually gives lower bounds on the pessimistic solution. Note that these bounds are directly available without any iterative process.

Second, even if bounds are not sufficient, the decision maker may supply some tolerable accuracy within which the dichotomy search may be halted. (Revision of these accuracy's afterwards can be done very simply.) This is an advantage because exact knowledge of pessimistic and ideal solutions (which will by nature never be used c.g. attained) somewhere in the middle of the iterative process is generally useless. Consequently the determination of these solutions may be done in a cheap and inexact way.

Finally we note that this implementation of IMGP is not restricted to the linear case: it is not so much the linearity that makes the ellipsoidal algorithm work, but the convexity of the feasible region.

5. Conclusion

We concentrated on the computational aspects of "ellipsoidal IMGP" as compared to IMGP in its original form, but equally or more important are the impacts for the decision-maker. If he desires so, he can still

get all the information provided by IMGP in its original form. But if he considers the accuracy of some information to be of secondary importance and is willing to accept bounds, he is rewarded in the amount of calculations to be performed in that iteration and consequently in response time.

So the net effect of the development of the ellipsoidal IMGP version is the addition of an option. From a decision makers point of view it is a fairly common and reasonable option; if he wants more accurate information, he has to sacrifice some computations (computer time).

From a theoretical point of view the development of the ellipsoidal IMGP method in section 4 has a totally different impact. Since the ellipsoidal algorithm in itself requires only a polynomially bounded number of steps to determine the feasibility of a system of linear constraints, the whole procedure might be polynomially bounded. The dichotomy search in this context is shown to be polynomially bounded (Papadimitriou [1979]) and it is easily seen that all other steps in IMGP require a number of computations per iteration that is bounded from above by a polynomial function of the size of the problem.

Therefore, the whole procedure is polynomially bounded if the decision-maker requires only a polynomially bounded number of iterations. This does not only seem to be a reasonable assumption but it can be formalized too. The rather abstract assumption of a polynomially bounded number of iterations may be replaced by the assumptions that the decision-maker is (i) consistent and (ii) indifferent to differences in the goal variable values smaller than $\varepsilon > 0$. To pin down a range of ε

around the optimal value of every goal variable requires $O(\log \frac{R^j}{\varepsilon})$ iterations, where R^j is the range of possible values of $g_j(x)$ in R. Since $R^j < 2^{L+1}$ the number of iterations per goal variable is polynomially bounded for suitable ε (e.g. $\varepsilon = 2^{-m}$). To us this is a rather surprising result which seems to be the first of its kind in relation to interactive multiple objective programming methods.

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