

Tuned Risk Aversion as Interpretation of Non-Expected Utility Preferences*

Berend Roorda[†] Reinoud Joosten[†]

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Abstract

We introduce the notion of Tuned Risk Aversion as a possible interpretation of non-expected utility preferences. It refers to tuning patterns of risk (and ambiguity) aversion to the composition of a lottery (or act) at hand, assuming only an overall ‘budget’ for accumulated risk aversion over its sub-lotteries. This makes the risk aversion level applied to a part intrinsically depending on the whole, in a way that turns out to be in line with frequently observed deviations from the Sure-Thing Principle. This is illustrated by applying the concept to the Allais paradox and to the 50:51 example, related to ambiguity aversion. We give a general justification for applying the method in contexts where the law of one price does not hold, and derive unique updating from a substitution axiom induced by a non-recursive form of consistency. In a

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[†]School of Management and Governance, Department of Industrial Engineering and Business Information Systems, University of Twente, the Netherlands. Email corresponding author: b.roorda@utwente.nl. (submitted version May 21 2014 with a few small corrections)

third example, we propose a solution to a well-known puzzle on consistency of decision making in the Ellsberg paradox.

Keywords: Allais paradox, Ellsberg paradox, ambiguity aversion, dynamic consistency, Sure-Thing Principle, preference updating

1 Introduction

There is an abundance of evidence that risk attitudes towards a compound lottery, or act, cannot be properly understood in terms of risk attitudes towards each of its sub-lotteries separately, contrary to the implications of the von Neumann-Morgenstern expected utility framework. Since the famous example of Allais, decades of empirical and theoretical research have identified systematic aspects in human decision making that clearly violate the Independence Axiom underlying expected utility. This has resulted in several explanations why the risk attitude towards a sub-lottery may depend on payoffs that are no longer relevant once the sub-lottery would realize, in terms of psychological factors as regret, framing, and subjective perception of small probabilities. These perspectives have led to various proposals for modeling so-called non-expected utility preferences. We refer to Machina and Viscusi (2013) for a recent overview on this topic. Despite agreement on the observed facts, there is still some controversy whether these deviations should be interpreted as biases of the human mind, comparable to optical illusions, or biases in the Independence Axiom itself, rendering it as less rational than it seems to be at first sight. Our approach supports the latter view, by pointing at a straightforward explanation in terms of *Tuned Risk Aversion (TRA)*, which is inspired by recent findings in research on non-recursive valuation in the context of nonlinear pricing and risk measures in finance, see Roorda and Schumacher (2013) (henceforth rs13) and the references therein.

The idea of TRA is best explained in the setting of a two-stage lottery, with only three degrees of risk aversion considered per stage: low, medium, high. To all the nine combinations possible we can associate an induced ‘overall’ degree of risk aversion, applying to the lottery as a whole. Let us assume that a moderately risk averse person considers the following three combinations as not too conservative (a) medium in both stages, (b) first low, then high, and (c) the other way around. Variants are possible, of course, but the point is that in general there are level curves of overall risk aversion consisting of different patterns of distributing it. Tuned Risk Aversion, in this simple context, amounts to applying all three possibilities in this *tuning set*, and then selecting the one with minimum outcome. For example, if there is only risk in the second stage, (b) will be chosen, while (c) is most effective if only the first stage is risky. In this way risk aversion is tuned to the compound lottery as a whole, by ‘spending’ it ‘economically’, where it hurts most.

Natural as it seems, this example immediately raises serious concerns about consistency of the induced preferences, certainly for the experts in the field. Non-expected utility itself is already debated critically, and there is an additional issue concerning dynamic consistency. It seems that we allow for a discrepancy between making conditional decisions in a certain state beforehand and when that state actually materializes, so that recursiveness is lost.

We will carefully address these concerns. First we explain why TRA is indeed incompatible with the law of one price, and we show that relaxing this law pulls the sting out of the common ‘make book’ arguments against violations of the Sure-Thing Principle. These considerations lead a revision of the notion of simple acts as building blocks of compound acts. We then propose the notion of *sequential consistency*, introduced in Roorda and Schumacher (2007), as a form of dynamic consistency that fits the idea of TRA much better than the far more restrictive axiom of recursiveness. We show that it is strong enough to induce unique updates of initial

preferences, and formulate a substitution axiom (which we call *c-Substitution*) that produces this update, if it exists. This axiom may be of independent interest.

TRA is illustrated by applications to three well-known paradoxes. Firstly, we show that the observed violation of the independence axiom in the Allais paradox is perfectly in line with TRA, in a way that largely confirms the well-known behavioral explanations in terms of framing and regret. These stylized findings suggest that TRA can be seen as a rational mechanism through which these phenomena have their effect. The effect of combining standard exponential utility with TRA is depicted in a Marschak-Machina triangle.

We then explore TRA in the so-called *50:51 example*, introduced in Machina (2009) in the context of ambiguity aversion, in Section 6. This example reveals some limitations of Choquet Expected Utilities (CEU) preferences in capturing intuitive tradeoffs between ambiguity and risk aversion, due to the so-called tail-separability property inherent in CEU. In Baillon et al. (2011) it is shown that many alternative approaches to non-expected utility suffer from this anomaly as well. We will show that an elementary form of TRA added to CEU preferences already suffices to overcome this problem.

In a third example, we discuss consistent decision making in the Ellsberg paradox, following the two-stage formulation in Hanany and Klibanoff (2007). The analysis confirms consistency of updating by *c-Substitution*, which in this case amounts to Bayesian updating of preferences, if the absence of the law of one price is carefully taken into account.

This paper is organized as follows. The formal definition of TRA is given after the introduction, followed by a section devoted to the justification of the violation of the Sure-Thing Principle that it involves. In Section 4 we describe sequential consistency and the induced unique updating rule by *c-Substitution*. The three subsequent sections contain the three applications, and conclusions follow in Section

8. Regularity conditions and proofs are collected in an appendix.

Placement in the literature and contribution

The setup in this paper is mathematically simple, in that state spaces are assumed to be finite. Its scope is quite general, however, due to the fact that it does not refer to probabilities, and hence no formal distinction has to be made between risk and ambiguity at the outset. In other words, the setup applies to acts, including objective or subjective lotteries as special case. Moreover, we also abstract from specific shapes of preference functions, and only assume that they are continuous, monotone, and strictly monotone on constants. This makes it compatible with a wide variety of non-expected utility frameworks, including Cumulative Prospect Theory (PT) (Tversky and Kahneman, 1992; Wakker and Tversky, 1993), Rank-Dependent Utility (RDU) (Quiggin, 1982), Maxmin and Choquet Expected Utility (MEU, CEU) (Gilboa and Schmeidler, 1989; Schmeidler, 1989), Vector Expected Utility (VEU) (Siniscalchi, 2009), Dynamic Variational Preferences (VP) (Maccheroni et al., 2006), and Expected Uncertain Utility (EUU) (Gul and Pesendorfer, 2014).

The distinctive feature of TRA is that it combines non-recursiveness with a specific form of updating that does not rely on the Sure-Thing Principle. Relaxing recursiveness makes it different from most models in MEU, VP, CEU, and RDU. PT and VEU concentrate on behavioral aspects of updating, while we derive it straightforwardly from a consistency property, without reference to probabilities. This updating rule is compared to notions in Gilboa and Schmeidler (1993); Siniscalchi (2009); Gumen and Savochkin (2013); Maccheroni et al. (2006) in Section 3, and to Hanany and Klibanoff (2007) in Section 7. To our knowledge, the proposed update rule has not been used before in the literature on non-expected utility.¹

¹It is closely related to the conditionally consistent updating rule in (Roorda and Schumacher, 2007, Def. 3.1) and the refinement update introduced in rs13, but extends their scope to preference

Also the refined notion of sub-acts, as elementary building blocks of compound acts, is new.² The idea to keep track of more than one conditional utility level per state is also present in VEU and EUU, but used differently. Further comparison at this point is an interesting point of future research.

Perhaps the main contribution does not lie in the ideas themselves, which are elementary, but rather in their justification. By three classical examples we show that several paradoxes resolve if absence of the law of one price is carefully taken into account. This may give rise to reconsider the degree of rationality of well-observed aspects of human judgment and decision making, and influence directions of future research in this field.

Some potential applications of TRA that fall outside the scope of this paper concern prudence, hyperbolic discounting, and incomplete preferences; their link with TRA is briefly indicated in Section 8. Concerning the empirical aspect, we view the estimation of the shape of non-rectangular tuning sets as an interesting topic for experiments. The fact that dynamic preferences may still be parameterized by just one parameter under TRA may enhance the formulation and testing of hypotheses.

2 Definition of TRA

The definition of TRA applies to compound acts, for which we use the following notation. We assume a finite state space $S = \{s_1, \dots, s_n\}$ for the first stage acts. The act with consequence, or outcome, x_i in state s_i is denoted as $(x_1, s_1; \dots; x_n, s_n)$,

functions that not necessarily satisfy the axiom of translation invariance (which is, together with monotonicity, the defining property of so-called monetary risk measures, see Föllmer and Schied (2011)).

²A similar idea is mentioned in Roorda and Schumacher (2014), as a topic of future research on convex risk measures.

or, in vector notation, as (x, s) . A simple act³ has $x \in \mathbb{R}^n$, a compound act has x_i consisting of another act, called a (second stage) sub-act, on some finite state space S_i .⁴ The set of all compound acts on S is denoted as \mathcal{A} , that of simple acts on S as $\mathcal{A}^{\text{simple}}$. The sub-act of $f \in \mathcal{A}$ in state $s \in S$ is denoted as f_s , which may be a simple or compound act.

An act with the same consequence $c \in \mathbb{R}$ in all states is called a *sure thing*, or the constant (act) c , or a sure amount c . We use the same notation for sure things on different state spaces.

We take our starting point in given (ordinal) preference functions, parameterized by a risk aversion parameter, for the simple acts in the first stage, and for the second stage acts in each state of S . We call $c \in \mathbb{R}$ a certainty equivalent (in the weak sense, as we explain later on) of an act f under preference function V , if f is indifferent to the sure amount c under V . To ensure existence and uniqueness, we assume that all given preference functions are monotone, continuous, and strictly monotone on sure things $c \in \mathbb{R}$.⁵

Under Tuned Risk Aversion we will compare the outcomes of several preference functions, relating to different patterns of risk aversion. Therefore it is convenient to replace the ordinal preference functions by the induced certainty equivalence functions, which represent exactly the same preference ordering, but at the same time have intrinsically defined units, because they are normalized on constants.

So we will assume that the following certainty equivalent functions are given

³This should not be confused with the use of the adjective ‘simple’ to indicate that an act has finite range, as e.g. in Gilboa and Schmeidler (1993).

⁴Probabilistic or deterministic dependencies between states are not incorporated at the outset; they may be incorporated by specifying a joint distribution on all subsets of states in $\cup_{i=1, \dots, n} S_i$.

⁵An a priori restriction of outcomes to a subinterval $X \subset \mathbb{R}$ can be handled without difficulty. Incorporation of state-dependent restrictions for outcomes of second stage acts seems less straightforward.

- $ce_a : \mathcal{A}^{\text{simple}} \rightarrow \mathbb{R}$ for $a \in A$, with $A \subset \mathbb{R}$ denoting a range of risk aversion levels over the first stage.
- $cce_b : \mathcal{A} \rightarrow \mathbb{R}^n$, with $cce_b(f)(s)$ the certainty equivalent of f_s in state $s \in S$ for $b \in B \subset \mathbb{R}^n$, with B specifying given ranges of risk aversion levels for the second stage in each state. The vector $cce_b(f)$ is identified with a simple act.

TRA is defined as taking the worst outcome over different patterns of risk (and/or ambiguity) aversion over the stages.

DEFINITION 2.1 Tuned Risk Aversion, specified by a non-empty set $R \subset A \times B$, corresponds to the preference ordering represented by the certainty equivalent function on compound acts given by

$$(2.1) \quad CE_R(f) = \inf_{(a,b) \in R} ce_a(cce_b(f)).$$

Some remarks are in order here. As we explained in the introduction, the crux of TRA is that R , which we call the *tuning set*, need not be rectangular, but may reflect mutual restrictions between a and b . In that respect it is more general than the recursive utility approach, as described in Epstein and Schneider (2003). In Section 4 we describe the notion of sequential consistency, a form of so-called weak time consistency, which allows for non-rectangular tuning sets.

For lotteries, an obvious choice is to let $a = 0$ and $b = 0$ correspond to taking (conditional) expectations. A standard way to mirror positive to negative levels in nonlinear pricing is by setting $ce_{-a}(f) = -ce_a(-f)$, see e.g. Cherny and Madan (2010); negative levels then pertain to selling rather than buying f , and risk/ambiguity neutrality for acts then means that $ce_0(f) = -ce_0(-f)$.

In the examples we will always have that the infimum that defines CE_R is a minimum, so that for all compound acts an optimally tuned pattern of risk aversion exists.

The notion of an overall level of risk aversion can be formalized by specifying a parameterized family of nested sets $(R_\gamma)_{\gamma \in \Gamma}$ with $\Gamma \subset \mathbb{R}$, and we can write CE_γ for CE_{R_γ} . The definition can then be applied recursively in the obvious way. We come back to this issue in the next section.

We have assumed a single risk aversion parameter for each stage, resp. $a \in A, b \in B$, but we could also incorporate different types of risk aversion in each stage, leading to higher dimensional sets A and B . In fact, the example in Section 6 suggests an extension in this direction.

To streamline the exposition, we collect some technicalities and proofs in the appendix. This includes the standard properties of certainty equivalence functions, labeled (p1-3) in Section 9.1, three regularity conditions (r1-3) which guarantee that the infimum in (2.1) is a minimum, a sensitivity condition (r4) for ce_a , and conditions on the risk aversion parameterization, labeled (p4) and (r5).

3 Consistency issues

Before we describe the notion of consistency that we impose in TRA, we make the general point that there need not be a trace of irrationality in violations of the Sure-Thing Principle. This is crucial for the appreciation of all other results.

For the sake of the argument, we will consider an agent with an initial certainty equivalence function $CE : \mathcal{A} \rightarrow \mathbb{R}$, today at $t = 0$, and conditional certainty equivalence function $cce : \mathcal{A} \rightarrow \mathbb{R}^n$ tomorrow, at $t = 1$.

To avoid confusion in terminology, we first have to disentangle two interpretations of the certainty equivalent of a sub-act: the weak one, as replacement value of outcomes *within* the sublottery, and a strong one, pertaining to the replacement of the sub-act *itself* in the compound act. Notice that cce relates to the weak interpretation, and that the Sure-Thing Principle requires that the initial preference

CE validates the strong interpretation for *cce* as well.

Although there may be valid psychological considerations to explain discrepancies between making conditional decisions in a certain state beforehand (according to CE) and when that state actually realizes (according to *cce*), our justification of TRA is not relying on this. Instead, we will consider a professional agent whose imagination is strong enough to anticipate regret, and who takes the model serious enough to stick to his plan, at least as a thought experiment. Hence, we will not allow for any other certainty equivalent in the strong sense than given by the vector function *cce*. The crucial question is: does it exist?

If it does, this in fact implies that the agent is willing to exchange sub-acts f_s for $cce(f)(s)$ and vice versa. The hidden assumption is that the agent's preferences obey the law of one price: the amount c given for obtaining an act, is the same as the amount asked in return for the act.⁶ This may be a perfectly valid assumption in e.g. a context of a frictionless market. We hardly see any room for TRA if the law of one price holds.

However, this law is an idealization, and even highly competitive markets may exhibit considerable bid-ask spreads, due to market frictions and unhedgeable uncertainty. The very existence of bid-ask spreads is apparently not a matter of irrationality. This implies that two preference orderings coexist, CE^{bid} and CE^{ask} , which may be strongly related, e.g. by the rule $CE^{ask}(f) = -CE^{bid}(-f)$, but do not coincide. Let us assume that CE and *cce* relate to ask prices.

The Sure-Thing Principle now turns its sterner face: it implies that the ask price

⁶The term 'law of one price' as we use it here is most appropriate in a market context. Assuming that sure things c trade for c , hence assuming outcomes in monetary units and ignoring risk-free discounting, it amounts to the assumption that for all acts f there exists price $c \in \mathbb{R}$ such that both $f - c$ and $c - f$ are market opportunities, i.e., indifferent to 0. A term that would better reflect our general setting, not restricted to a market context, would be 'the law of one type of preference', but this is a little long-winded.

of today only depends on conditional ask prices of tomorrow, regardless the spread that will prevail at that day. In other words, recursion in ask and bid prices should be fully separated.

A seemingly strong argument for this separation is that otherwise always a pair of acts f, g can be found with $CE(f) < CE(g)$ yet $cce(f)(s) \geq cce(g)(s)$ in all states $s \in S$. However, as we have shown in (rs13, Example 3.9), such a twist is perfectly explained as an effect of f having lower conditional bid prices than g . Bid and ask prices may be *jointly* recursive, so that at $t = 0$ a sub-act must be represented by two figures, and market frictions effectively invalidate the standard ‘make book’ arguments against it.

Summarizing our point, it is obvious that two different complete preference orderings may coexist in a rational way, and the Sure-Thing Principle overlooks that they can be intertwined without causing anomalies.

The subtle aspect is that, because $t = 0$ has been a future moment in the past, the initial preference itself is partial and complete at the same time: complete in the technical sense of orderings, but partial, in the sense that also other types of initial preferences matter. Even more subtle, in our example the one ordering can be *derived* from the other, by the rule given above, so that CE is still complete as a model specification of initial preferences. Combined with uniqueness of updates, as we address in the next section, it would even implicitly determine cce. So, despite its incompleteness as decision criterion, CE may fully specify all modeling aspects of interest.

Extended simple acts as building blocks in TRA

We conclude this section by translating these observations to an alternative formulation of TRA, that brings back the idea of recursion. From (2.1) it is obvious what the relevant features of a sub-act f_s in state $s \in S$ are under TRA: the outcomes

of $cce_b(f)$ in s for *for the entire range* of risk aversion levels b that occur in R . So under TRA, the standard substitution axiom should be weakened accordingly, by only requiring that preferences do not change if a sub-act is replaced by another one that is equivalent in this strong sense. This gives rise to a different view on what the elementary building blocks of compound acts are. If the interface of sub-act f_s in state $s \in S$ with the compound act f is the entire mapping $\tilde{f}_s : b \mapsto cce_b(f)(s)$, this means that f can be viewed as a simple act on S only with an extended specification of consequences per state, consisting of the entire mapping \tilde{f}_s , rather than one number. So an extended simple act takes the form (\tilde{x}, s) , with extended specification $\tilde{x} = (x_b)_{b \in B}$. The substitution axiom described above now amounts to reduction of compound lotteries f to extended simple acts \tilde{f} , as intermediate outcome in evaluating TRA backward recursively. In a multi-stage recursion, the single-step valuations become conditional mappings from extended simple acts to extended outcomes for a range of risk aversion levels. Let γ denote the parameter of overall risk aversion, specified by a nested series of tuning sets $R_\Gamma = (R_\gamma)_{\gamma \in \Gamma}$, cf. Section 2. With slight abuse of notation, we can apply CE_γ to \tilde{f} in the obvious way, and define $CE_\Gamma := (CE_\gamma)_{\gamma \in \Gamma}$. Dynamic certainty equivalence functions for TRA can be seen as recursively composed of this type of building blocks in each state.

4 Sequential consistency and unique updating

We have argued in the introduction that recursiveness is too restrictive for TRA. So, assuming that initial and conditional preferences are specified by resp. CE and cce, as before, we do not impose the rule

$$(4.1) \quad CE(f) = CE(cce(f)).$$

Inspired by rs13, we propose to replace this by the much weaker condition of sequential consistency, which requires that $CE(f)$ is *in the range of* $cce(f)$. By this

we mean that for all $f \in \mathcal{A}$,

$$(4.2) \quad CE(f) \in [\min cce(f), \max cce(f)],$$

with the boundaries of the interval resp. the minimum and maximum element of the vector $cce(f)$. In other words, certainty equivalents (in the weak sense) do not increase or decrease for sure. Notice that this makes sense for bid- and ask prices, even if their dynamics are intertwined, and that in this way straightforward arbitrage opportunities are ruled out.⁷ The following characterization further underlines the strong intuition of this concept.⁸

PROPOSITION 4.1 *Sequential consistency (4.2) is equivalent to the condition*

$$(4.3) \quad cce(f) = c \quad \Rightarrow \quad CE(f) = c.$$

In a forward looking perspective, cce is called a sequentially consistent *update* of CE if the condition above holds true. Following the line of reasoning in rs13, we derive uniqueness of sequentially consistent updates in TRA, under some mild regularity conditions, and formulate a substitution axiom, which we call *c-Substitution*, that provides the update if it exists. The unique update extends the refinement update described in rs13 to value functions that are not necessarily translation invariant.

We use the following notation. For a given tuning set R , define the vector β_R as the *point-wise* supremum of second-stage levels that occur in R , i.e.,

$$(4.4) \quad \beta_R := \sup R_1 \in \mathbb{R}^n \text{ with } R_1 := \{b \in B \mid (a, b) \in R \text{ for some } a \in A\}.$$

⁷Absence of arbitrage opportunities by dynamic strategies is easily obtained in a context of a market with frictions and possibly perfectly liquid hedging opportunities, cf. rs13, Section 5.

⁸A similar characterization is given in rs13, restricted to $c = 0$. For so-called monetary valuations, which satisfy the axiom of translation invariance, $V(f + c) = V(f) + c$, the criterion (4.3) then follows.

The regularity conditions in the theorem below are described in Section 9.1.

THEOREM 4.2 *The pair $(\text{CE}_R, \text{cce}_\beta)$ is sequentially consistent if β is the maximum element of R_1 defined in (4.4), i.e., (i) $\beta = \beta_R$ and (ii) $\beta_R \in R_1$.*

Under the regularity conditions (r1-5), cce_{β_R} is the unique candidate for a sequentially consistent update of CE_R , and it has this property if and only if (ii) holds for R the maximal tuning set representing CE_R , which is given by (9.2).

The unique candidate, cce_{β_R} , is characterized by the following axiom.

AXIOM 4.3 (*c-Substitution*) *If a subact g in state $s \in S$ has conditional certainty equivalent c , then c is also the certainty equivalent of the compound act $g^c \in \mathcal{A}$ consisting of the subact g in s and the sure amount c in every other state of S .*

Formulated more compactly, in terms of a certainty equivalence function $\text{CE} : \mathcal{A} \rightarrow \mathbb{R}$, and a conditional certainty equivalence function $\text{cce} : \mathcal{A} \rightarrow \mathbb{R}^n$, the axiom requires that for all $f \in \mathcal{A}$, $s \in S$,

$$(4.5) \quad \text{cce}(f)(s) = c \Leftrightarrow \text{CE}(f_s^c) = c,$$

with f_s^c defined as g^c in the definition, with $g = f_s$.

In other words, certainty equivalents in the weak sense must have the strong interpretation as well in the very specific ‘neutral’ context of all other sub-acts being sure things with the same value, so that all risk aversion will be tuned to the second stage.

THEOREM 4.4 *Under assumptions (r1-5), the c-Substitution axiom defines a unique update for CE_R , which coincides with cce_{β_R} .*

So the *c-Substitution* axiom constitutes a universal principle for updating preferences, implied by sequential consistency (4.2) under suitable regularity conditions. This principle hence also applies to recursive preferences (4.1) as a special case.

It may be illuminating to compare this rule to Bayesian updating. In line with Gilboa and Schmeidler (1993), define for $h \in \mathcal{A}$ the (single state) h -Bayesian update for $f \in \mathcal{A}$ in $s \in S$ by

$$\text{cce}(f)(s) \leq \text{cce}(g)(s) \Leftrightarrow \text{CE}(f_s^h) \leq \text{CE}(g_s^h),$$

with f_s^h the result of pasting f_s in h , cf. also (Siniscalchi, 2009, Section 4.1). Consequentialism, or the Sure-Thing Principle, would require independency of h , cf. (Gumen and Savochkin, 2013, Thm. 1). For instance, the Bayes update rule (Maccheroni et al., 2006, (6)) applies to recursive Variational Preferences, which essentially boils down to stepwise Bayesian updating.⁹ In the paper Gilboa and Schmeidler (1993), the case with h the maximum possible value of outcomes is emphasized. We take h the constant act c that is the fixed point of the monotone mapping $c \mapsto \text{CE}(f_s^c)$. Therefore, we propose to call it *fixed-point updating*.

Summarizing, sequential consistency implies a very specific form of ‘insequentialism’ that amounts to h -Bayesian updating in each state $s \in S$, with h the fixed-point sure thing. According to Theorem 4.2, if these can be combined in one conditional update for all $s \in S$ jointly, it is the sequentially consistent one, otherwise sequential updates fail to exist. This existence condition may be interpreted as absence of tuning restrictions among mutually exclusive events, which in fact means that essential violations of consequentialism are avoided. We refer to Roorda and Schumacher (2007, Examples 5.3-4) for elementary examples illustrating when sequentially consistent updates do not exist.

⁹The analogous result in the risk measure literature is (Föllmer and Penner, 2006, Thm. 4.5), on recursive convex risk measures. In a current working paper (Roorda and Schumacher, 2014) we describe the update rules induced by sequential consistency for this class.

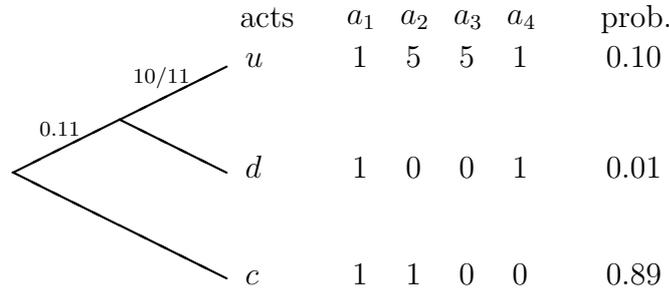


Figure 1: The four lotteries related to the Allais paradox (payoffs in \$million).

5 Tuned Risk Aversion and the Allais paradox

We apply TRA to the lotteries of the Allais paradox, see Figure 1. It has been well documented that many subjects prefer a_1 over a_2 , and a_3 over a_4 , contrary to the Certainty-Independence Axiom that states that preferences should not switch if only c is changed from 0 to 1 while the subplottery is kept the same.

To become concrete, we consider an agent with preference orderings of subplotteries based on exponential utility, $u(x) = -e^{-\beta x}$, for some $\beta > 0$. The corresponding (weak) certainty equivalent (ceq) of the subplottery $g = (u, p; d, 1 - p)$ is then given by

$$(5.1) \quad cce_{\beta}(g) := -\frac{1}{\beta} \log(pe^{-\beta u} + (1 - p)e^{-\beta d}).$$

Under the expected utility framework, the ceq has also the stronger interpretation that the agent is indifferent to replacing g , as subplottery of a compound lottery, by $cce_{\beta}(g)$. It immediately follows that the agent's preference for a_1 and a_4 over resp. a_2 and a_3 depends on whether $cce_{\beta}(g) < 1$ for g the subplottery in a_1 and a_4 ; this is the case when $\beta > 2.4$.

Under TRA this strong interpretation of certainty equivalents no longer holds. Let us, to avoid the introduction of additional parameters, assume that the agent applies the same utility function to each stage separately. However, applying this level of risk aversion in both stages consecutively, is considered as too conservative.

Following TRA in its simplest form, the agent considers to limit the sum of levels of risk aversion, by using the criterion

$$(5.2) \quad CE_\beta(f) = \min\{ce_{\beta_0}(cce_{\beta_1}(f)) \mid \beta_0, \beta_1 \geq 0, \beta_0 + \beta_1 = \beta\}.$$

Notice that this is sequentially consistent, and that the update is given by cce_β .¹⁰

We already saw that we should choose $\beta > 2.4$ in order to induce that a_1 is preferred over a_2 . It turns out that for the other lotteries, a_3 and a_4 , risk aversion is then most effective in the first stage, i.e., the minimum in (5.2) for the other two lotteries is achieved for $\beta_0 = \beta$. Consequently, for the second stage expected values are considered, and hence a_3 is preferred over a_4 for $\beta \geq 2.4$.

Does this really solve the paradox? Nothing has been solved if one still tacitly assumes the law of one price. After all, the Allais paradox has been *designed* to reveal that different values of the sublottery at $t = 1$ play a role in decision making at $t = 0$. Rather than viewing this as inconsistent, we follow the opposite direction, and conclude that the sublottery should be represented in terms of more than one value, if c is not yet known. Moreover, then also at $t = 0$, $CE_\beta(f)$ cannot have been the only relevant value at $t = -1$ for the agent. So, even though it is formally correct to write $f \sim CE_\beta(f)$, and convention dictates to articulate this as ‘to the agent, f is indifferent to the amount $CE_\beta(f)$ ’, one should immediately add a phrase like ‘if it comes to buying’, ‘if it comes to selling’, or ‘from a certain perspective’, to avoid the symmetric interpretation suggested by the law of one price.

In this way the Allais paradox can be interpreted as a rational effect of TRA: in a_2 , all risk aversion is attracted to the sublottery, because there it has the most effect. For the same reason, albeit somewhat less pronounced, in the lotteries a_3

¹⁰Notice that CE_β induces ce_β over the first period, i.e., for lotteries without risk in the second stage, so both stages are treated alike in CE_β . The crux of TRA is hence not explicit time dependency, but the fact that short term properties do not dictate long term features.

and a_4 conservatism is focused on the initial stage, so that the risk in the subplottery is penalized less heavily.

We would like to avoid the impression that this should be seen as an interpretation that is alternative, or even opposite to the classical behavioral explanations in terms of framing and regret. On the contrary, it is very much in line with it. The different framing ($c = 1$ vs. $c = 0$) is precisely the reason for the different patterns of risk aversion between the first and second pair of lotteries, and the fact that a full loss is more painful in a_2 as compared to a_3 , nicely goes along with the fact that under TRA one is indeed more relaxed about that risk in the latter lottery. From this perspective, TRA may be seen as a mechanism through which these psychological factors have their effect. Our analysis seems to indicate that they have a more rational justification than generally believed, although we do not exclude that e.g., strong framing can lead to less efficient tuning of risk aversion.

To conclude this section, we depict the effect of TRA in the so-called Marschak-Machina triangle, introduced in Marschak (1950), for the TRA preference (5.2) with $\beta = 3$, see Figure 2. Notice the fanning out effect in the lower region, inducing the outcome of preference orderings as described. We remark that the counter-intuitive North-West direction of curves in the upper region can be avoided if one minimizes over all three possible definitions of subplotteries in the Allais paradox.

6 Tuning Risk and Ambiguity Aversion: the 50 : 51 example.

In Machina (2009) the so-called 50:51 example has been introduced, as an illustration of limitations in the CEU approach, introduced in Schmeidler (1989), to model the tradeoff between ambiguity and risk aversion. We take our starting point in the formulation of this example in Baillon et al. (2011), depicted in Figure 3.

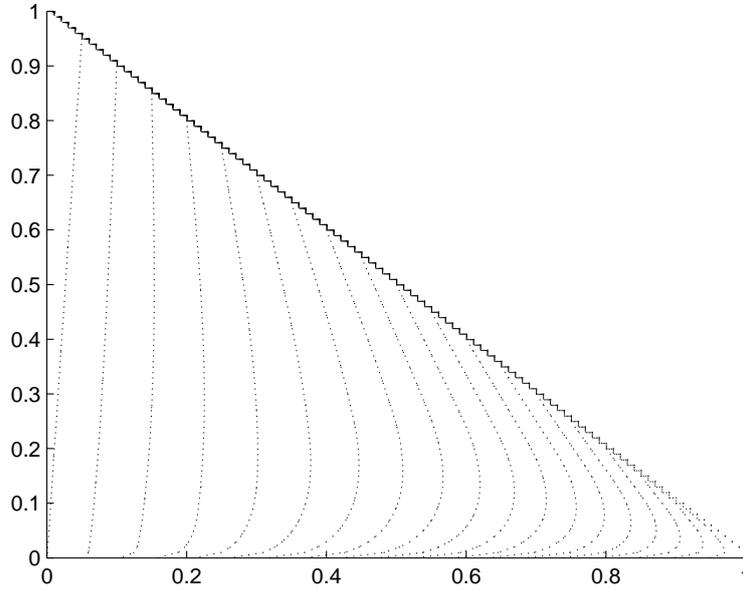


Figure 2: The Marschak-Machina triangle with level curves of the TRA preference (5.2) with $\beta = 3$ for the Allais lotteries. At the horizontal axis is the probability p_1 on outcome 0, on the vertical probability p_3 on outcome 5. The remaining probability $p_2 = 1 - p_1 - p_3$ is assigned to outcome 1. The lotteries of Figure 1 correspond to the following locations: $a_1 : (0, 0)$, $a_2 : (0.01, 0.1)$, $a_3 : (0.9, 0.1)$, $a_4 : (0.89, 0)$.

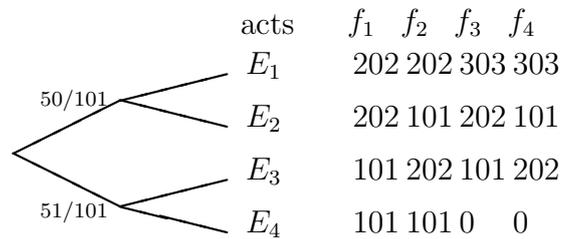


Figure 3: The four acts in the 50:51 example.

CEU implies that f_1 is preferred to f_2 if and only if f_3 is preferred to f_4 . In Baillon et al. (2011) it is shown that the forward implication is induced in most other classes of ambiguity-averse preferences as well. The paradox is that the informational advantage of f_1 with respect to f_2 is much stronger than that of f_3 compared to f_4 , and hence it is natural allow for preferences \prec with $f_1 \succ f_2$ and $f_3 \prec f_4$. We will show that TRA admits such a preference.

To keep things as simple as possible, we consider a preference function that is the minimum of two CEU functions. For ambiguity aversion, we use the MINVAR criterion, introduced in Cherny and Madan (2010) in the context of bid-ask price modeling. For simple, binary acts $g = (u, p; d, (1 - p))$, with $d \leq u$, this amounts to

$$(6.1) \quad V_k(g) = p^k u + (1 - p^k)d.$$

We choose this form because of its strong intuition, its precise functional form is not essential. For k an integer, the intuition is that the expected value is considered of the minimum outcome in k independent trials. We take uniform distribution as reference measure for the sub-acts, and choose $k = 2$. So the ambiguity penalty for the sub-acts amounts to a quarter of the spread of outcomes. There is no ambiguity in the first stage, so we take expected values of the outcome for both sub-acts, and define

$$(6.2) \quad U^{amb} = \frac{50}{101} \frac{E_1 + 3E_2}{4} + \frac{51}{101} \frac{E_3 + 3E_4}{4}.$$

The second CEU, reflecting risk aversion, is again obtained from exponential utility $u(x) = -e^{-\beta x}$. The stepwise application of (5.1), with the same parameter in both periods, is equivalent to applying this utility once to four outcomes, so we take

$$(6.3) \quad U^r = -\frac{1}{\beta} \log\left(\frac{50}{101} \frac{e^{-\beta E_1} + e^{-\beta E_2}}{2} + \frac{51}{101} \frac{e^{-\beta E_3} + e^{-\beta E_4}}{2}\right).$$

These are both certainty equivalence functions, and we take their minimum outcome as final preference function V . It turns out that the values of f_1 and f_2 are

equal for $\beta = 0.02365$. We choose $\beta = 0.015$ so that f_1 is preferred over f_2 . The corresponding values of the acts are given by

	f_1	f_2	f_3	f_4
U^{amb}	151	126.25	125.25	100.5
U^r	133.5	134.0	75.4	75.6
V	133.5	126.25	75.4	75.6

It follows that indeed $f_1 \succ f_2$ and $f_3 \prec f_4$, as desired.

We remark that TRA is certainly not the only way to cope with the 50:51 puzzle. We just showed that adding just one bit of TRA to CEU, nothing more, is enough, indicating that TRA extends scope of CEU in a relevant direction. We refer to Dillenberger and Segal (2012) for an alternative solution to this puzzle in terms of recursive preference functions.

7 Consistent decision making in the Ellsberg paradox

We emphasized the interpretation of a budget of risk aversion in TRA, reflected by mutual restrictions between a and b in the tuning set R . The syntax of Definition 2.1 may also be applied in a context in which such restrictions are based on different considerations. For instance, if $a \in A$ and $b \in B$ correspond to the number of balls of a certain color one supposes to be present in one and the same ambiguous urn, it is a matter of consistency to impose $a = b$ in R . We apply this idea to the two-stage formulation of the Ellsberg paradox in Hanany and Klibanoff (2007), which is essentially the same as the example in (Siniscalchi, 2009, Section 2)

In our notation, this can be represented as follows. Take $S = \{s, s'\}$, $S_s = \{B, R\}$ and $S_{s'} = \{Y\}$ as outcome space for resp. the first stage, the sub-act in s , and the (trivial) sub-act in s' . Here $B/R/Y$ stands for drawing a black / red / yellow

ball from an urn, in which 40 of the 120 balls are known to be blue, and the remaining 80 are red or yellow, in unknown proportion (note that the symbol B has been redefined). To streamline the exposition, we let $(f_B, f_R, f_Y) \in \mathbb{R}^3$ denote the compound act $f \in \mathcal{A}$ with subact $f_s = (f_B, B; f_R, R)$ in state s and act f_Y in state s' leading to Y .

Following Hanany and Klibanoff (2007), we consider the MEU preference function

$$\text{CE} : (f_B, f_R, f_Y) \mapsto \min\{1/3f_B + af_R + (2/3 - a)f_Y \mid a \in [1/4, 5/12]\},$$

which corresponds to taking the worst expected value under the assumption that the proportion of red balls (parameter a) is not lower than $1/3 - 1/12$, and not higher than $1/3 + 1/12$. Notice that CE is translation invariant, i.e., it satisfies $\text{CE}(f + c) = \text{CE}(f) + c$; in fact it belongs to the class of coherent risk measures, introduced in Artzner et al. (2007). Formulated in term of a tuning set, we can take $R = \{(a, b) \mid a, b \in [1/4, 5/12], a = b\}$, ce_a taking expected value with probability $1/3 + a$ assigned to s , and $\text{c}e_b$ taking expected value, conditioned on s , with probability $1/(1 + 3a)$ for B . Then indeed $\text{CE}_R = \text{CE}$. Note that the tuning restriction, $a = b$, now is a matter of intrinsic consistency.

Consider now the following four acts: $f = (1, 0, 0)$, $g = (0, 1, 0)$, $f' = (1, 0, 1)$, $g' = (0, 1, 1)$. Then $\text{CE}(f) = 1/3 > 1/4 = \text{CE}(g)$, while $\text{CE}(f') = 7/12 < 2/3 = \text{CE}(g')$; notice that in both pairs, the ambiguity premium is $1/12$.

We will argue that there is no compelling argument to deviate from the update rule induced by c -Substitution. Illustrating the rule directly, this requires that $\text{c}e(f_s) = c$ with c the solution of $\text{CE}(1, 0, c) = c$, which yields $c = 4/9$. Similarly, $\text{c}e(g_s)$ must be equal to $3/7$. This coincides with Bayesian updating.

In Hanany and Klibanoff (2007) it is argued that this constitutes a dynamic inconsistency in terms of decision making. On the one hand, the preference of g' over f' reflects that at $t = 0$, one prefers to bet on outcome R rather than B , even

if this decision can be postponed to $t = 1$. However, at $t = 1$, this preference is predictably revised, because $f'_s = f_s$ is preferred over $g'_s = g_s$. As we have indicated in the introduction, we also consider this as truly inconsistent (that is, in the scope of this paper, not as a general claim); we hence do not follow the so-called *consistent planning* argument in Siniscalchi (2009). The solution proposed in Hanany and Klibanoff (2007) is to adjust the update rule by ruling out priors that cause the inconsistency.

However, it is not the update rule that is to blame here, or to be adjusted, the pitfall is again ignoring the fact that the law of one price is lost. Let us assume that the decision maker (DM), confronted with the choice between f' and g' , indeed chooses to afford g' , in line with the preference function CE. Then, at $t = 1$, if s realizes, the DM has the option to exchange the bet on R for one on B ; in state s' no further choices have to be made. Now this is a choice between (i) replacing the bet on R by one on B , or (ii) do nothing; (i) corresponds to buying in state s the subact $h = (1, B; -1; R)$, which according to the updated preference has value $-1/9$. So the DM will stick to the bet on R , and already knows this at $t = 0$. Comparing the bid prices of both bets is not the issue. Notice that also in case the DM would have started with f' , the option to switch would not have been exercised. Frictions help to ‘stick to your plan’. This friction effect could shed new light on the topic of timing indifference, as studied in e.g. Strzalecki (2013).

This illustrates once more that despite the fact that preference functions induce a full ordering, this ordering still may be partial in the sense that it reflects just one aspect of valuation.

8 Concluding remarks

We have introduced TRA as a natural aspect of decision making under uncertainty that can be rationally justified if the law of one price does not hold. In our framework, sequential consistency replaces recursiveness as a more flexible principle adjusted to TRA, which induces c -Substitution as general updating rule for preferences on acts. This rule derives a preference ordering on sub-acts from an initial preference function, under a specific choice for the context of sub-acts, intrinsically defined by a fixed point condition, thus coping with the violations of the Sure-Thing Principle admitted under TRA.

TRA has interesting links with several important themes in non-expected utility that are out of the scope of this paper. Incomplete preferences can be defined, in the spirit of Aumann (1962) and Dubra et al. (2004), by combining a family of preferences in one overall preference, defined by the rule that an act f is ‘overall’-preferred to g only when f is preferred to g under each preference in the given family. For instance, one may compare both bid and ask prices, and deem f and g incomparable if f is more attractive to buy, but at the same time more easily sold than g . More generally, incomplete orderings may be defined by the condition $CE_\gamma(f) \leq CE_\gamma(g)$ for all $\gamma \in \Gamma$, cf. Section 3. Inspiration for research in this direction may be derived from the results in Ok et al. (2012) on partial completeness.

Another possible application is prudence, see Eeckhoudt and Schlesinger (2006). Prudence seems inherent in TRA, in fact also if rectangular tuning sets are used for the two-stage lottery in the definition of prudence. Non-rectangular sets, however, may play a role in characterizing higher-order concepts related to the sign of the k -th derivative of utility functions for $k > 3$, such as temperance ($k = 4$) and edginess ($k=5$).

Finally we indicate how TRA may support the analysis of non-recursive time preferences, in particular hyperbolic discounting (Phelps and Pollak (1968); Laib-

son (1997), see also Joosten (2014) for a recent application). Assuming a regular time axis, one may first determine maximum levels of risk aversion for each period separately that induce an appropriate discount rate for a reference set of acts over each time step resembling a given level of risk. The corresponding rectangular tuning set would induce then exponential discounting. In TRA, restrictions can be imposed for lighter discounting over multiple steps, and in particular hyperbolic discount rates can be obtained. The results on compound risk measures in (rs13, Section 6), and the description of all convex risk measures with prescribed stepwise properties in (Roorda and Schumacher, 2014, Section 7) may provide useful starting points for implementing this idea.

More generally, sequential consistency and the induced fixed-point update rule may provide a rational basis for giving long term consequences an appropriate weight in decision making, in a way that may be substantially different from the mechanically derived implications of local properties in a recursive approach.

9 Appendix

9.1 Regularity conditions

We first list the standard properties of the certainty equivalence functions ce_a and cce_b with $a \in A$ and $b \in B$ that are assumed throughout the paper.

(p1) ce_a and cce_b are monotone and continuous in outcomes of acts

(p2) ce_a and cce_b are normalized on constants, i.e., $ce_a(c) = c$, $cce_b(c) = c$

(p3) ce_a and cce_b are non-increasing in resp. a and b .

Without loss of generality we may assume that the parameterizations are injective,

(p4) $ce_a \neq ce_{a'}$ for $a \neq a'$ and $cce_b \neq cce_{b'}$ for $b \neq b'$,

since violations of (p4) can be eliminated by an obvious adjustment of the parameterization. Properties (p1-4) are standing assumptions in TRA.

The following three regularity conditions guarantee that the infimum defining CE_R is a minimum.

(r1) $A = [a_{\min}, a_{\max}]$ with $a_{\min} < a_{\max}$, $B = [b_{\min}, b_{\max}]$ with $b_{\min} \leq b_{\max}$.

(r2) the mapping $(f, a) \mapsto \text{ce}_a(f)$ is continuous on $\mathcal{A}^{\text{simple}} \times A$

(r3) the mapping $(f, b) \mapsto \text{cce}_b(f)$ is continuous on $\mathcal{A} \times B$

LEMMA 9.1 *Under assumptions (r1-3), CE_R is continuous on \mathcal{A} , and*

$$\text{CE}_R(f) = \text{CE}_{\bar{R}}(f) = \min_{(a,b) \in \bar{R}} \text{ce}_a(\text{cce}_b(f)),$$

with \bar{R} the closure of R .

PROOF We start with the second claim. It is easily verified that under the assumptions (r1-3),

(9.1) the mapping $(a, b, f) \mapsto \text{ce}_a(\text{cce}_b(f))$ is continuous.

Due to (r1), R is bounded, and hence \bar{R} is compact. So $\text{CE}_R(f)$ is the minimum of a continuous function over a compact domain, and by the Weierstrass Theorem it follows that the infimum is a minimum. The first equality in the lemma also follows from (9.1).

For the first claim, we have to prove that if $f_n \rightarrow f$, then $\text{CE}_R(f_n) \rightarrow \text{CE}_R(f)$. In view of the first result, we can write $h_n := \text{CE}_R(f_n) = \text{ce}_{a_n^*}(\text{cce}_{b_n^*}(f))$ and $m := \text{CE}_R(f) = \text{ce}_{a^*}(\text{cce}_{b^*}(f))$ for risk aversion levels in \bar{R} . Then $h_n \leq \text{ce}_{a^*}(\text{cce}_{b^*}(f_n)) =: h'_n$, and hence $\limsup h_n \leq \lim h'_n = m$, where the last equality follows from (9.1). On the other hand, $m' := \liminf h_n \geq m$, which can be seen as follows. Let $(h_n)_{n \in I}$ be a converging subsequence, to cluster point k say. Because \bar{R} is compact, there

must be a sub-subsequence $(h_n)_{n \in J}$ with $J \subset I$, also converging to k of course, for which $(a_n, b_n)_{n \in J} \rightarrow (a', b')$, and hence $(a_n, b_n, f_n) \rightarrow (a', b', f)$. From (9.1) it then follows that $k = ce_{a'}(cce_{b'}(f))$, so $k \geq m$. Because m' is the infimum of all clusterpoints k , also $m' \geq m$. \square

Under (r1), the maximal tuning set representing CE_R exists, and is given by

$$(9.2) \quad R^{\max} = \{(a, b) \in A \times B \mid ce_a(cce_b(f)) \geq CE_R(f) \text{ for all } f \in \mathcal{A}\}.$$

It follows from the previous lemma that this is a closed set under (r1-3).

For some results we assume the following sensitivity property of ce_a , for $f \in \mathcal{A}$ bounded by a constant d :

$$(r4) \quad f \lesssim d \Rightarrow ce_a(f) < d \text{ and } f \gtrsim d \Rightarrow ce_a(f) > d.$$

A slightly stricter version of (p4) for second stage acts is needed in the theorems.

$$(r5) \quad \text{For all pairs of vectors } b' \leq b, \text{ with } b'(s) < b(s), \text{ and all } c \in \mathbb{R}, \text{ there exist } f \in \mathcal{A} \text{ such that } cce_b(f)(s) = c < cce_{b'}(f)(s).$$

9.2 Proof of Prop. 4.1

It is clear that the implication in (4.3) is a consequence of (4.2). It remains to show that if (4.2) is violated, (4.3) cannot hold true. So assume that for some f , $CE(f) < \min cce(f) =: m$ (the proof for $CE(f) > \max cce(f)$ is entirely analogous). Then there must exist $g \in \mathcal{A}$, with $g \geq 0$, such that $cce(f - g)(s) = m$ for all $s \in S$ (this follows easily from the standing assumptions (p1-2) for cce , cf. Section 9.1). On the other hand, $CE(f - g) \leq CE(f) < m$, contradicting (4.3).

9.3 Proof of Thm. 4.2

We prove the first claim using (4.3). So consider $f \in \mathcal{A}$ with $cce_\beta(f) = c$. Condition (i) in the theorem implies that $CE_R(f) \geq \inf_{(a,b) \in R} ce_a(cce_\beta(f)) = \inf_{(a,b) \in R} ce_a(c) =$

c . From condition (ii) it follows that there exists $a \in A$ with $(a, \beta) \in R$, and hence $\text{CE}_R(f) \leq \text{ce}_a(\text{cce}_\beta)(f) = \text{ce}_a(c) = c$. So equality must hold, and (4.3) follows.

For the claim on uniqueness, consider a function $\text{cce} : \mathcal{A} \rightarrow \mathbb{R}^n$ that is normalized on constants, and assume $\text{cce} \neq \text{cce}_{\beta_R}$. So there exist $f \in \mathcal{A}$ and $s \in S$ such that

$$c := \text{cce}(f)(s) \neq \text{cce}_{\beta_R}(f)(s) =: c'.$$

Define the act f_s^c as the sure thing c on S with in s the act c replaced by f_s . Observe that $\text{cce}(f_s^c) = c$, in all states in S , because cce is normalized on constants.

If $c' < c$, then $\text{cce}_b(f_s^c) \not\geq c$ for some $b \in B$, and hence $\text{CE}_R(f_s^c) \leq \text{ce}_{a_{\min}}(\text{cce}_b(f_s^c)) < c$, where the strict inequality is due to (r4) (see (r1) above for the notation a_{\min}). So (4.3) is violated for f_s^c . Similarly, for the case $c < c'$, we show that $\text{CE}_R(f_s^c) > c$. Clearly

$$(9.3) \quad \text{CE}_R(f_s^c) \geq \inf_{(a,b) \in R} \text{ce}_a(\inf_{b' \in R_1} \text{cce}_{b'}(f_s^c)),$$

with R_1 the set defined by (4.4). Let $g \in \mathbb{R}^n$ denote the inner (point-wise) infimum. Then $g \not\geq c$, and (r4) implies that $\text{CE}_R(f_s^c) > c$. So also if $c < c'$, (4.3) does not hold, and hence $c' = c$. This proves that cce_{β_R} is indeed the only candidate for a sequentially consistent update of CE_R .

For the last claim, notice that under (r1) the maximal tuning set R^{\max} is indeed given by (9.2), and that R^{\max} is closed under (r1-3). It follows that $R_1^{\max} = \{b \mid (a_{\min}, b) \in R^{\max}\}$, and that for all $s \in S$, $\beta_{R^{\max}}(s) = \max\{b(s) \mid (a_{\min}, b) \in R^{\max}\}$. Obviously $\beta_{R^{\max}} \geq \beta_R$, and we show that equality must hold. Indeed, if $v := \beta_{R^{\max}}(s) = \beta_R(s) + \delta$ for some $s \in S$ and $\delta > 0$, then, due to (p4), there would exist a sub-act f_s in s with $\text{cce}_{\beta_R}(f)(s) =: c > \text{cce}_v(f)(s)$, so that $\text{CE}_R(f_s^c) = c$, while, by (r4) for $\text{ce}_{a_{\min}}$, $\text{ce}_{a_{\min}}(\text{cce}_v(f_s^c)) < c$; a contradiction with (9.2), as $v = b(s)$ for some $(a_{\min}, b) \in R^{\max}$.

So $\beta_R = \beta_{R^{\max}}$, and it follows from the first claim that cce_{β_R} is a sequentially consistent update if R_1^{\max} satisfies condition (ii).

It remains to derive the reverse implication: if cce_{β_R} is sequentially consistent, then $\beta_R \in R_1^{\max}$. If $\beta_R = b_{\min}$, the claim is trivial. Otherwise, the subset $S' := \{s \in S \mid \beta_R(s) - \delta \in B\}$ is non-empty for sufficiently small $\delta > 0$. According to (r5), we can choose $c \in \mathbb{R}$, and find $f \in \mathcal{L}$ such that $cce_{\beta_R}(f) = c$, while $cce_{b'}(f)(s) > c$ whenever $b'(s) \leq \beta_R(s) - \delta$. Now if cce_{β_R} is a sequentially consistent update, $CE_R(f) = c$, and hence $ce_{a^*}(cce_{b^*}(f)) = c$ for some $(a^*, b^*) \in R^{\max}$. In view of (r4) for ce_{a^*} , it must hold that $cce_{b^*}(f) = c$, and hence $b^*(s) \geq \beta_R(s) - \delta$ for all $s \in S$. As δ can be chosen arbitrarily small, and R_1^{\max} is closed, it follows that $\beta_R \in R_1^{\max}$.

9.4 Proof of Thm. 4.4

First show that cce_{β_R} satisfies the c -Substitution axiom. To derive the forward implication in (4.5), take $f \in \mathcal{A}$, $s \in S$, and let $c = cce_{\beta_R}(f)(s)$. Then $CE_R(f_s^c) \leq ce_{a_{\min}}(cce_{\beta_R}(f_s^c)) = ce_{a_{\min}}(c) = c$. The reverse inequality is obvious from the definition of β_R , so $CE_R(f_s^c) = c$. For the backward implication in (4.5), which immediately implies uniqueness of updates, assume $CE_R(f_s^c) = c$. From Lemma 9.1,

$$(9.4) \quad CE_R(f_s^c) = ce_{a^*}(cce_{b^*}(f_s^c)) \text{ for some } (a^*, b^*) \in R.$$

By definition of β_R , $b^*(s) \leq \beta_R(s)$, so $cce_{b^*}(f)(s) \geq c$, and equality follows from (r4). So cce_{β_R} indeed satisfies the c -Substitution axiom.

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